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Structure of a compressible superfluid

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A simple model of the spin-polarized hydrogen system in an external field is developed. The thermal and interaction support of the fluid work to prevent its collapse when the Bose-Einstein condensation occurs. The existence of the condensate causes a reduction in the thermal support of the fluid; the condensate is modeled with a local Ginsburg-Landau functional. The effects of temperature, sample size, and external field on the evolution of the fluid are explored.

One of the goals of experimental studies[1] of the spin-polarized atomic hydrogen ($H\downarrow$) system is to see the superfluid transition and to try to elaborate the relationship of this transition to the phenomenon of Bose-Einstein condensation[2] (BEC). The superfluid transition in $H\downarrow$ may be quite different from that in ^{4}He. The ^{4}He system is self-bound (a liquid) so that the superfluidity occurs in a piece of material whose structure (density, profile, etc.) has been established by energetics much larger than that involved in the occurrence of superfluidity. Thus, the occurrence of superfluidity goes almost unnoticed in the gross properties of the helium system; i.e., the density, pressure, etc., show no dramatic signature. The $H\downarrow$ system is not self-bound[3,4] (it is a gas) so that the density of the system is determined by the number of $H\downarrow$ atoms plus the volume of the container in which they reside, and possibly, by the strength of a confining magnetic field. The energetics associated with the occurrence of superfluidity could be comparable to that associated with establishing the structure of the fluid. Thus the occurrence of superfluidity could well change that structure drastically. (The BEC causes an ideal gas confined by a field to "collapse" into the localized ground state determined by the field.) While the $H\downarrow$ system (gas) is not as strongly interacting as the ^{4}He system (liquid), it is not so weakly interacting that the ideal gas model is useful. We have developed a simple Ginsburg-Landau model that we believe correctly incorporates the qualitative features of the important physical events occurring in the interacting Bose fluid.

Consider a Bose fluid at a constant temperature, T, residing in a cylinder of cross-sectional area A. The confinement in the $\hat{x}$ direction is achieved by an external field, which we take, as Walraven and Silvera[1] did, to be harmonic: $B(x) = B_0[1-(x/d)^2]$, where B_0 is the strength of the field at the center of the magnet and d is a constant. (If one wanted to be rigorous, one should choose a solenoidal field, but this would clutter up the formalism. Our treatment yields essentially the correct features of the axial distribution of the densities.) A wall in the plane $x=0$ keeps the fluid in the $x>0$ region. If $n(x)$ is the number of particles per unit volume and $\Gamma = \mu_B B_0/d^2$, where μ_B is the Bohr magneton, the magnetic field contribution to the energy density for particles in the $m_s = -\frac{1}{2}$ electronic spin state is $dE_m = \frac{1}{2} n(x) \times \Gamma x^2 dx$. The fluid is able to support itself because of its thermal energy (and pressure), $dE_T = n(x) k_B T dx$ and because of its interaction energy (and pressure) $dE_I = 2\pi\epsilon_0 a^3 n^2(x)dx$. Here $\epsilon_0 = \hbar^2/ma^2$ and we have taken the first term in an expansion for the hard-sphere model.[4,5] The diameter of the hard spheres is $a = 0.72$ Å. The primary effect of the BEC is to extract the thermal energy from the particles that go into the ground state and to reduce accordingly the thermal pressure available to support the fluid. As this BEC occurs the fluid shrinks because of a reduction in its thermal support. Consequently, the average density of the fluid increases and so does the condensation temperature. This latter event drives more particles into the ground state and further reduces the thermal support for the fluid, etc. Eventually, at suitably low temperature the thermal support gives way to interaction support, and a further reduction in temperature produces little change in the structure of the fluid. The events outlined here are described by the equations of hydrostatic equilibrium for the fluid. If P is the hydrostatic pressure, we write

$$dP(x,\psi) = -\Gamma x n(x) dx \quad . \tag{1}$$

The pressure in the fluid is

$$P(x,\psi) = 2\pi\epsilon_0 a^3 n^2(x) + k_B T[n(x) - \lambda|\psi(x)|^2] \quad , \tag{2}$$

where $|\psi(x)|^2$ is the superfluid density at x and λ is a constant of order 1 ($\lambda \leq 1$) which accounts for the proportion of atoms in the superfluid that actually belong to the condensate.[6] The first term on the right-hand side of Eq. (2) accounts for the *interaction sup-*

port of the fluid and the second term accounts for its *thermal support.* We have introduced the effect of the BEC on the pressure by simply removing the particles which go into the condensate from the thermal support term. Finally, we need an equation for ψ. We write a Ginsburg-Landau energy occurs at

$$E[\psi]=A\int dx\left[\frac{\hbar^2}{2m}|\nabla\psi|^2-\alpha(T,n(x))|\psi|^2+\tfrac{1}{2}\beta(T,n(x))|\psi|^4\right] , \qquad (3)$$

where α and β depend upon the local density and temperature. If $T < T_c(n(x))$, we write

$$\alpha(n(x),T)=\alpha_0(n(x))[1-T/T_c(n(x))]^{\nu} ,$$

$$\beta(n(x),T)=\beta_0(n(x))[1-T/T_c(n(x))]^{\mu} ,$$

where

$$k_B T_c(n(x))=\gamma\epsilon_0 a^2 n(x)^{2/3}$$

(for the ideal gas BEC, $\gamma=3.31/3^{2/3}$). [Note that, strictly speaking, the expansion (3) is valid only when the coherence length $\xi(T)=(\hbar^2/2m\alpha)^{1/2}$ is at least of the order of the interatomic distance, $n^{-1/3}$. If $\nu=1$ as in the usual Landau theory, then $\xi(T)=n^{-1/3}$ when $T_c(x)-T\approx 0.16T_c(x)$.] The constants α_0 and β_0 are fixed by the physical arguments $\lim_{T\to 0}(\alpha_0/\beta_0)\simeq n(x)$, $\lim_{T\to 0}(\alpha_0^2/2\beta_0)\simeq k_B\times T_c(n(x))n(x)$. If we fix the number N of particles in the system, a normalization condition must be satisfied:

$$N=A\int_0^{\infty} n(x)\,dx . \qquad (4)$$

Equations (1)–(4) constitute a closed self-consistent system, which is relatively easy to solve if the spatial variation of ψ is assumed to be slow enough that the curvature term in Eq. (3) can be ignored. (This approximation can be checked self-consistently.) Before going on to look at the solution to Eqs. (1)–(4), it will be useful to discuss their content. Beginning with a fixed number of particles in a sample chamber, at high temperature and low density the fluid is supported against the external field by its thermal energy. As the temperature is lowered, thermal support of the fluid gives way to interaction support. From Eq. (2) this crossover occurs at $n^*(x)=T^*/2\pi$, where we define $n^*(x)=n(x)a^3$ to be the dimensionless density of particles and $T^*=k_BT/\epsilon_0$. Since the most dense fluid is at the center of the system, this crossover occurs first at the center and we have $n^*(0)=T^*/2\pi$. In Fig. 1 we show the separation of the $n^*(0)-T^*$ space into thermal and interaction support regions along the line $n^*(0)=T^*/2\pi$. We also show the transition temperature at the center, $T_c^*(n^*(0))$. The superfluidity onset in a piece of fluid having a density $n^*(x)$ occurs at $T_c^*(n^*(x))=\gamma n^*(x)^{2/3}$.

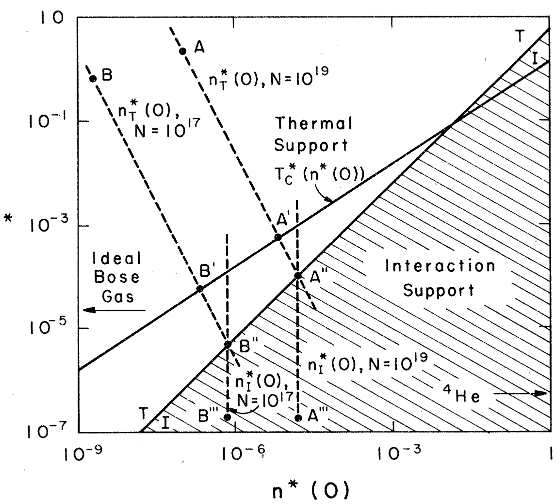

FIG. 1. Schematic evolution of two typical samples ($N=10^{17}$ and 10^{19} atoms) in the $n^*(0)-T^*$ plane. We take $d=5$ cm and $\gamma=3.31/3^{2/3}$.

When the fluid is supported thermally, the density at its center is

$$n_T^*(0)=(Na^3/A)(2\Gamma/\pi\epsilon_0 T^*)^{1/2} . \qquad (5)$$

The curve A → A′ → A″ shows the evolution of a sample of $N=10^{18}$ particles in a field of 10 T. We note that the sample will undergo the BEC at A′, a temperature somewhat above A″ at which thermal support gives way to interaction support. Thus one might expect the profile of the sample to show evidence for the loss of thermal support as T^* evolves from $T_{A'}^*$ to $T_{A''}^*$. Eventually the sample will end up at A‴ supported only by the interactions. The $T=0$ profile of the sample and the location of A‴ are found from the solution of Eqs. (1), (2), and (4) for $T=0$; e.g.,

$$n_I^*(0)=(3Na^3/4A)^{2/3}(\Gamma/2\pi\epsilon_0)^{1/3} .$$

It is clear that as the size of the hydrogen sample is decreased (say, to $N=10^{17}$) the effect of temperature becomes more important (evolution toward the ideal Bose gas behavior). As the size of the hydrogen sample is increased the effect of the interactions becomes increasingly important, and at $N\approx 10^{22}$, the thermal influence on the structure of the sample when the BEC occurs is notably reduced (evolution towards ^{4}He behavior).

Equations (1)–(4) can be transformed and integrated: (a) form dP/dn from Eq. (2) for use in Eq. (1), (b) find $|\psi|^2$ from the local solution to Eq. (3) (with the curvature term set equal to zero), (c) set $\nu=1$, $\mu=0$. Then

$$x^2 = \begin{cases} C(1-y) - Dt\ln y + \frac{3}{2}Et^2(y^{-2/3}-1) \quad (x \leq R_s) & (6) \\ C(1-y) - D't\ln y + \frac{3}{2}D_\lambda t\ln t + \frac{3}{2}Et(1-t) \quad (x \geq R_s) \ , & (7) \end{cases}$$

where

$$y = n^*(x)/n^*(0) \ ,$$

$$t = T/T_c(n^*(0)) \ ,$$

$$C = 8\pi\epsilon_0 n^*(0)/\Gamma \ ,$$

$$D = 2\gamma\epsilon_0 n^*(0)^{2/3}(1-\lambda)/\Gamma \equiv D' - D_\lambda \ ,$$

$$E = 2\gamma\epsilon_0\lambda n^*(0)^{2/3}/3\Gamma \ ,$$

and R_s, the location of the superfluid to normal transition, is (if $t \leq 1$)

$$R_s^2 = C(1-t^{3/2}) - \frac{3}{2}Dt\ln t + \frac{3}{2}Et(1-t) \ . \quad (8)$$

The dimensionless superfluid density is

$$\chi^2(x) \equiv a^3|\psi(x)|^2 = n^*(0)y(1-ty^{-2/3}) \quad (9)$$

$$(x \leq R_s, \ t \leq 1) \ .$$

Equations (6) to (9) are a set of implicit equations, and the normalization condition, Eq. (4), has to be implemented numerically. The asymptotic behavior of the density of particles is given by

$$n^*(x) = n^*(0)\exp(-x^2/D't) \ ,$$

if $x >> (D't)^{1/2}$, and

$$n^*(x) = n^*(0)[1 - x^2/(C + Dt + Et^2)] \ ,$$

if $x \to 0$.

In Fig. 2 we show the profile of a sample of $N = 10^{18}$ atoms in an external field of 10 T. At $t = 1$ a condensate appears at the center; it becomes larger and larger as t decreases. The evolution of the system as a function of the field at $t = 1 [T = T_c(x = 0)]$ is depicted in Fig. 3 for $N = 10^{18}$ atoms. The effect of the field in the $n^*(0)$-T^* plane can be roughly assessed from Eq. (5).

At low densities and high temperatures, the thermal support term in Eq. (2) is much larger than the interaction support term, and we can obtain an explicit solution of the system of Eqs. (1)–(4) in terms of N, t, and Γ. The results are somewhat cumbersome, but some of their features are worth noting: (1) The superfluid fraction N_s/N depends only on t. (2) The functional dependence of the densities is

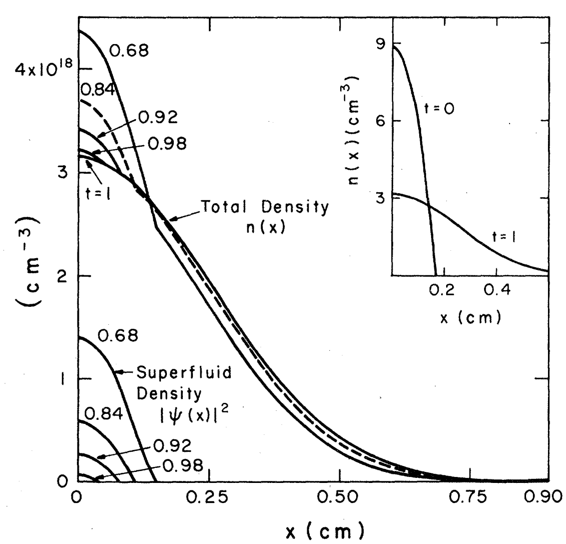

FIG. 2. Total and superfluid density profiles at several temperatures. We have taken $\lambda = 1$ and $\gamma = 3.31/3^{2/3}$. The numbers by the curves are the dimensionless temperatures. Note the change in the slope of $n(x)$ at $x = R_s$ and the spatial effect of the condensation. Although the $t = 0.68$ curves lie outside the supposed range of validity of our theory, we plot them because they emphasize the expected features of the system. In the inset we show the profile of the system at $t = 0$. (The density units are those in the main figure.)

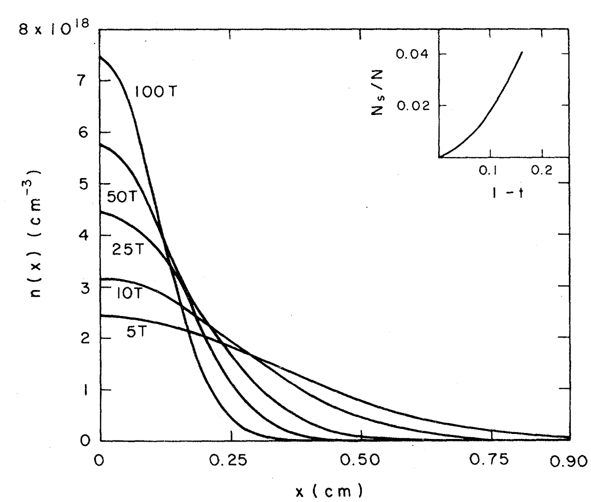

FIG. 3. Change in the density profile as a function of B_0. We keep $d = 5$ cm. In the inset we show the superfluid fraction N_s/N as a function of the temperature if we fix $B_0 = 10$ T, $\lambda = 1$, and $\gamma = 3.31/3^{2/3}$.

$n^*(x) = n^*(0) f_1(t, x^2/N^{1/2})$, $\chi^2(x) = n^*(0) \times f_2(t, x^2/N^{1/2})$. (3) The superfluid density goes linearly to zero when $x \rightarrow R_s$. (4) At $x = R_s$, the derivative of the total density has a jump.

While our model places the important effects operating in the fluid in proper relation to one another, we do not expect it to yield an exact description of the fluid. We do believe that the physical picture this model gives of the behavior of the fluid is qualitatively and semiquantitatively useful.

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