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Brief Reports

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Rectangular type-II superconducting wires carrying axial current

Albert Shadowitz
Fairleigh Dickinson University, Teaneck, New Jersey 07666
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Previous calculations for the distribution of $\vec{J}$ and $\vec{B}$ in a type-II superconducting wire with a square cross section carrying an axial current are here extended to obtain corresponding results for a rectangular cross section. We do this (a) for wires and thick films, (b) for whiskers, and (c) for thin films. The superconducting self-inductance in the first case varies as λ; in the cases of thin films and whiskers the variation is as λ^2. This agrees with reported results. We know of no definitive experimental results for the $\vec{J}$ and $\vec{B}$ distributions for comparison with our theoretical results.

I. INTRODUCTION

In this paper we extend the results previously reported for an axial current through a wire of square cross section[1] to the case of an axial current through a wire of rectangular cross section. The former is simpler theoretically; the latter is much more common in actual practice.

We take the origin at the center of a cross section with $-a \leq x \leq a$ and $-b \leq y \leq b$. Let

$$J_z = J_1 \cosh(x/\lambda) + J_2 \cosh(y/\lambda) \quad . \tag{1}$$

Then

$$I = \int J_z \, dS = \alpha J_1 + \beta J_2 \quad ,$$

where

$$\begin{aligned} \alpha &= 4\lambda b \sinh(a/\lambda) \quad , \\ \beta &= 4\lambda a \sinh(b/\lambda) \quad . \end{aligned} \tag{2a}$$

The energy density, $u = -\frac{1}{2}\vec{J}\cdot\vec{A}$, becomes

$$u = (\mu_0\lambda^2/2)[J_1^2 \cosh^2(x/\lambda) + 2J_1J_2\cosh(x/\lambda)\cosh(y/\lambda) + J_2^2\cosh^2(y/\lambda)] \quad .$$

The energy per length, $U_l = \int u \, dS$ is then found to be

$$U_l/\mu_0\lambda^2 = \gamma J_1^2 + \delta J_1 J_2 + \epsilon J_2^2 \quad , \tag{3}$$

with

$$\begin{aligned} \gamma &= b[a + \lambda \sinh(a/\lambda)\cosh(a/\lambda)] \quad , \\ \delta &= 4\lambda^2 \sinh(a/\lambda)\sinh(b/\lambda) \quad , \\ \epsilon &= a[b + \lambda\sinh(b/\lambda)\cosh(b/\lambda)] \quad . \end{aligned} \tag{2b}$$

II. WIRES, THICK FILMS ($a >> \lambda$, $b >> \lambda$)

Here

$$\begin{aligned} \alpha &= 4\lambda b \sinh(a/\lambda) \quad , \\ \beta &= 4\lambda a \sinh(b/\lambda) \quad , \\ \gamma &= (\lambda b/2)\sinh(2a/\lambda) \quad , \\ \delta &= 4\lambda^2\sinh(a/\lambda)\sinh(b/\lambda) \quad , \\ \epsilon &= (\lambda a/2)\sinh(2b/\lambda) \quad . \end{aligned} \tag{4}$$

Putting $J_2 = (I - \alpha J_1)/\beta$ into the energy expression gives

$$U_l/\mu_0\lambda^2 = [\gamma - (\alpha\delta/\beta) + (\alpha^2\epsilon/\beta^2)]J_1^2 + [(\delta/\beta) - (2\alpha\epsilon/\beta^2)]IJ_1 + (\epsilon/\beta^2)I^2 = 0 \quad .$$

Taking the derivative with respect to J_1 and equating to zero yields

$$J_1 = (\alpha\epsilon - \tfrac{1}{2}\beta\delta)I/(\beta^2\gamma - \alpha\beta\delta + \alpha^2\epsilon) \quad . \tag{5}$$

The second derivative of $U_l/\mu_0\lambda^2$ with respect to J_1 is positive when

$$\left(\frac{a}{\tanh(a/\lambda)}\right) + \left(\frac{b}{\tanh(b/\lambda)}\right) > 4\lambda \quad ;$$

or, here, whenever $(a+b) > 4\lambda$. So for this case Eq. (5) always gives that current density component varying along x which minimizes the tatal energy.

Ignoring the two δ terms in Eq. (5)—they are much smaller than the other pertinent terms—gives $J_1 = I/[4\lambda(a+b)\sinh(a/\lambda)]$. Then

$$J_2 = I/[4\lambda(a+b)\sinh(b/\lambda)]$$

and

$$\vec{J} = \hat{z}\frac{I}{4\lambda(a+b)}\left[\frac{\cosh(x/\lambda)}{\sinh(a/\lambda)} + \frac{\cosh(y/\lambda)}{\sinh(b/\lambda)}\right] . \qquad (6)$$

From $\vec{B} = -\mu_0\lambda^2\vec{\nabla}\times\vec{J}$ the field is

$$\vec{B} = \frac{\mu_0 I}{4(a+b)}\left[-\hat{x}\frac{\sinh(y/\lambda)}{\sinh(b/\lambda)} + \hat{y}\frac{\sinh(x/\lambda)}{\sinh(a/\lambda)}\right] . \qquad (7)$$

The energy expression

$$(U_l/\mu_0\lambda^2)_{\min} \simeq (\gamma J_1^2 + \epsilon J_2^2)_{\min}$$

gives

$$(U_l/\mu_0 I^2)_{\min} = \lambda/[16(a+b)] .$$

We have shown[1] this is $(1/2\mu_0)$ times the kinetic inductance, L_{ks}, of the superconducting wire per length. This inductance equals the self-inductance, L_s, for superconducting wires

$$L_s = \mu_0\lambda/8(a+b) . \qquad (8)$$

The variation $L_s \propto \lambda$ for $a \gg \lambda$, $b \gg \lambda$ has been reported experimentally[2,3] in thick rectangular films where $a \gg b \simeq 10\lambda$.

III. WHISKERS ($2a \lesssim \lambda$, $2b \lesssim \lambda$)

We return to the parameters $\alpha-\epsilon$ listed in Sec. I, Eqs. (2a) and (2b). From $I = \alpha J_1 + \beta J_2$ we obtain, if we set $J_0 = I/4ab$, $m = a/\lambda$, and $n = b/\lambda$, the expression

$$J_0 = J_1(\sinh m)/m + J_2(\sinh n)/n .$$

Putting J_z from this into $U_l/\mu_0\lambda^2$ and taking the derivative with respect to J_1 gives $J_1 = (N/D)J_0$, where

$$N = \frac{(abn^2/m)(\sinh m)}{(\sinh^2 n)} + \frac{\lambda^2 n^2(\sinh m)(\cosh n)}{(\sinh n)} - [2\lambda^2(\sinh m)] ,$$

$$D = (ab) + [n\lambda^2(\sinh m)(\cosh m)] - [(4\lambda^2 n/m)(\sinh^2 m)] + \frac{(abn^2/m^2)(\sinh^2 m)}{(\sinh^2 n)} + \frac{(n^2\lambda^2/m)(\sinh^2 m)(\cosh n)}{(\sinh n)} .$$

Inserting a Taylor series through the fifth power in small quantities gives $N = ab(2n^4/45)$ and $D = ab2(m^4+n^4)/45$. Then

$$J_1 = (I/4ab)[b^4/(a^4+b^4)] ,$$

$$J_2 = (I/4ab)[a^4/(a^4+b^4)] ,$$

and to this order, by expanding the hyperbolic functions in Eq. (1),

$$J_z = \frac{I}{4ab}\left[1 + \frac{b^4}{2(a^4+b^4)}\left(\frac{x}{\lambda}\right)^2 + \frac{a^4}{2(a^4+b^4)}\left(\frac{y}{\lambda}\right)^2\right] . \qquad (9)$$

J has almost the constant value $I/4ab$, but there is a small added elliptical variation.

In

$$U_l/\mu_0\lambda^2 = \gamma J_1^2 + \delta J_1 J_2 + \epsilon J_2^2$$

we now have $\gamma = \epsilon = 2ab$, $\delta = 4ab$ giving

$$U_l/\mu_0 I^2 = \lambda^2/8ab$$

or

$$L_s = \mu_0\lambda^2/4ab . \qquad (10)$$

When $a = b = s$ this gives the value previously obtained for square whiskers.

The field, $\vec{B} = -\mu_0\lambda^2\vec{\nabla}\times\vec{J}$, is

$$\vec{B} = [B_0/(a^4+b^4)][-\hat{x}a^4(y/\lambda) + \hat{y}b^4(x/\lambda)] , \qquad (11)$$

where $B_0 = \mu_0\lambda J_0$. The constant current contours, which are also the field lines, are ellipses.

IV. THIN FILMS ($a \gg \lambda$, $2b \lesssim \lambda$)

Here

$$\begin{aligned} \alpha &= 4\lambda^2 n \sinh m , \\ \beta &= 4\lambda^2 m \sinh n , \\ \gamma &= \lambda^2 n \sinh^2 m , \\ \delta &= 4\lambda^2(\sinh m)\sinh n , \\ \epsilon &= \lambda^2 mn + (\lambda^2 mn/2)\sinh 2n . \end{aligned} \qquad (12)$$

J_1 is again evaluated by minimizing U_l via Eq. (5). $d^2U_l/dJ_1^2 > 0$ for $a > 2\lambda$. The numerator in Eq. (5) is now

$$N = (2\lambda^4 m \sinh m)(2n^2 + n\sinh 2n - 4\sinh^2 n);$$

inserting a Taylor series to the fifth power in n for sinh n gives

$$N = (2\lambda^4 m\sinh m)(4n^6/45) = \tfrac{8}{45}\lambda^4 n^6 m \sinh m .$$

Similarly, the denominator in Eq. (5) becomes

$$D = 16\lambda^6 (m^3 \sinh^2 m)(n \sinh^2 n) \ .$$

Then

$$J_1 = [2n^5/(45 \sinh m)](I/4ab) \ . \qquad (13a)$$

Even for the thickest thin film, $n = \frac{1}{2}$, the quantity in brackets is very small and we would ordinarily write

$$J_1 = 0 \ . \qquad (13b)$$

However, this small but finite J_1 leads to a finite B_y. Measurement of the relative variation of this small perpendicular component has been reported by Rhoderick and Wilson[4]; we will keep this term in order to make a B_y comparison possible, at least to some extent. J_2 then becomes

$$J_2 = \frac{1 - 2(b/\lambda)^5}{45(a/\lambda)}(I/4ab) \qquad (14a)$$

or, for all practical purposes,

$$J_2 = I/4ab \ . \qquad (14b)$$

Thus,

$$\vec{J} = \hat{z}(I/4ab)\left[\frac{\frac{2}{45}(b/\lambda)^5 \cosh(x/\lambda)}{\sinh(a/\lambda)} + \cosh\frac{y}{\lambda}\right] \ . \quad (15)$$

Ignoring the first term, for $2b/\lambda = 1$, J varies about 13% between $y = 0$ and b; but as b gets smaller and smaller the values of J becomes more and more constant.

The field becomes, if $B_0 = \mu_0 \lambda (I/4ab)$,

$$\vec{B} = -\hat{x}B_0 \sinh(y/\lambda) + \hat{y}B_0[2(b/\lambda)^5/45]\frac{\sinh(x/\lambda)}{\sinh(a/\lambda)} \ . \qquad (16)$$

Rhoderick and Wilson[4] reported measurements of the B_y component to check the variation against that resulting from a postulated $\vec{J}$ distribution suggested by W.A. Bowers (unpublished). That $\vec{J}$ is assumed to be constant through the thickness (here $2b$) but varies along the width ($2a$ here but w there) with a equal to a constant $\simeq 1$. $J(x)$ is given by $J_0[1-(2x/w)^2]^{-1/2}$ if x is not too close to $x = w/2$; while, close to $x = w/2$,

$$J(x) \sim \exp\left[\frac{-[d(w/2 - x)]}{a\lambda^2}\right] \ ;$$

and the two solutions are joined at a point distant $a\lambda^2/2d$ from the edge. Rhoderick and Wilson modified the Bowers distribution to make the central variation extend to $2x = w$; I is finite despite the resulting infinity in J at the edge. Their results gave excellent agreement with the variation of B_y calculated from the modified Bowers expression for J. It is interesting to note that the Bowers expression for $\vec{J}$ near the edge ($x = a$ here) gives an exponential behavior; so does the small, x-varying component, the first term of our Eq. (15); and the modified Bowers expression also gives J predominantly near the edges. Other authors[2,5] have used similar expressions; in Ref. 3, p. 5047, the authors write—"the current distribution. . .is predicted to be sharply peaked at the edges of the film." This is the behavior displayed by the first, small, term in Eq. (15); but we know of no experimental measurements of B_x, which would be connected with the $\cosh(y/\lambda)$ term in Eq. (15) when $y \simeq b$. Consequently we have difficulty comparing Eq. (16) with the measured results.

To find the self-inductance we evaluate $U_l/\mu_0 I^2$ from Eq. (3) using the coefficients from Eq. (12) and setting

$$J_1 = [b^3/90\lambda^4 a \sinh(a/\lambda)]I \ ,$$

$$J_2 = I/4ab \ .$$

Then

$$\gamma J_1^2 = \left[\frac{(b/\lambda)^9}{8100}\right]\frac{I^2}{a^2} \ , \qquad \delta J_1 J_2 = \left[\frac{(b/\lambda)^4}{90}\right] I^2/a^2 \ ,$$

$$\epsilon J_2^2 = \left[\frac{(\frac{1}{16})(a/\lambda)}{(b/\lambda)}\right]\frac{I^2}{a^2} \ .$$

So

$$\frac{U_l}{\mu_0 \lambda^2} = \left[\frac{(b/\lambda)^9}{8100} + \frac{(b/\lambda)^4}{90} + \frac{(a/\lambda)}{16}\frac{1}{(b/\lambda)}\right]\frac{I^2}{a^2} \ ,$$

$$\frac{U_l}{\mu_0 I^2} = \frac{1}{m^2}\left[\frac{n^9}{8100} + \frac{n^4}{90} + \frac{m}{16n}\right] \ , \qquad (17)$$

$$L_s = \frac{2\mu_0}{m^2}\left[\frac{n^9}{8100} + \frac{n^4}{90} + \frac{m}{16n}\right] \ .$$

This simplifies, when $(b/\lambda) << (a/\lambda)^{1/5}$, to the last term alone,

$$L_s = \mu_0 \lambda^2/8ab \ . \qquad (18)$$

The λ^2 variation for thin films has been verified experimentally in numerous reports.[2–4] The more complicated behavior in Eq. (17) would apply to the narrow transition region $\lambda \lesssim b/2 \lesssim 10\lambda$. For $20\lambda < b$ one has $L_s \propto \lambda$ while for $b < 2\lambda$ one has $L_s \propto \lambda^2$.

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