Generalized percolation

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A generalized model of percolation encompassing both the usual model, in which bonds are occupied with probability p and are vacant with probability $(1-p)$, and the model appropriate to the statistics of lattice animals, in which the fugacity for occupied bonds is p and that for unoccupied bonds is unity, is formulated. Within this model we discuss the crossover between the two problems and we study the statistics of large clusters. We determine the scaling form which the distribution function for the number of clusters with a given number of sites n assumes as a function of both n and p. For p near p_c we find that the distribution function depends on percolation exponents for $u = n(p_c - p)^{\Delta_p}$ small, where Δ_p is a crossover exponent, and on exponents appropriate to the lattice-animals problem for large values of u. We thus have displayed the relation between the two limits and show conclusively that the lattice-animals exponents cannot be obtained by any simple scaling arguments from the percolation exponents. We also demonstrate that essential singularities in the cluster distribution functions for $p > p_c$ arise from metastable states of the Potts model.

I. INTRODUCTION

In recent years, considerable effort has been devoted to the study of the statistics of clusters on a lattice.[1–3] Both site and bond clusters can be defined. Site clusters occur in lattices in which sites can be either occupied or vacant and consist of groups of adjacent occupied sites. Bond clusters occur in lattices where bonds are either occupied or vacant and consist of groups of sites connected by occupied bonds. In this paper we will concentrate on the case of bond clusters. The statistics of large clusters depend on the rules for occupying bonds. In the lattice-animal problem occupied bonds are weighted with relative probability p and vacant bonds with relative probability unity. This model also describes the statistics of branched polymer in the dilute limit if p is interpreted as the fugacity for bond formation.[4] In the percolation problem bonds are occupied with probability p and are vacant with probability $(1-p)$.

For lattice animals a statistical quantity of interest is $A(n)$, the total number of clusters (animals) per site containing n bonds. For large n it is known[5] that $A(n)$ is of the form

$$A(n) \sim n^{-\theta_a} \lambda^n \ , \tag{1.1}$$

where λ is a constant and θ_a is the animal-cluster exponent. An exact analysis[6] of the problem for the Cayley tree yields the value $\theta_a = \frac{5}{2}$, which is identified as the mean-field-theory result. Values of θ_a for spatial dimensionality d between 2 and 8 have been obtained using series expansions[2,3,7] and Monte Carlo simulation.[8] These results show that θ_a varies between 1 at $d=2$ and its mean-field value at high dimensionality. The renormalization-group treatment of this problem[4] showed that the critical exponents are given correctly by mean-field theory for $d > 8$, and they were calculated using an ϵ expansion in $8-\epsilon$ dimensions.

For the percolation problem one often considers $B(n,p)$, the average number of clusters per site containing n sites. Near the percolation threshold at $p = p_c$, numerical evidence,[8,9] supported also by field-theoretic and ϵ-expansion studies[10,11] of the one-state Potts model, shows that $B(n,p)$ scales as

$$B(n,p) = n^{-\tau_p} C(n(p-p_c)^{\Delta_p}) \ , \tag{1.2}$$

where τ_p and Δ_p are related to the order-parameter exponent, β_p, and the susceptibility exponent, γ_p, of percolation by

$$\Delta_p = \beta_p + \gamma_p \ , \tag{1.3a}$$

$$\tau_p = 2 + \beta_p/(\beta_p + \gamma_p) \ , \tag{1.3b}$$

and Δ_p is usually called the gap exponent. In the limit $n \to \infty$ Kunz and Souillard[12] have proven that for sufficiently large $|p - p_c|$ one has

$$\ln B(n,p) \sim \begin{cases} -n, & p < p_c \quad (1.4a) \\ -n^{1-d^{-1}}, & p > p_c. \quad (1.4b) \end{cases}$$

Numerical evidence[8] seems to suggest that these results hold even for arbitrarily small values of $|p - p_c|$. Thus one of the forms of $B(n,p)$ that has

been suggested is

$$B(n,p) \sim n^{-\theta}\exp(-An^{\rho}) \ . \tag{1.5}$$

Up to now it has been unclear whether Eq. (1.5) is supposed to be consistent with or whether it is supposed to supercede Eq. (1.2). In addition, it has not been established whether θ is the same as the animals exponent θ_a and also whether or not there are simple scaling relations between the percolation exponents and θ. Indeed, as yet there have been no satisfactory derivations of Eq. (1.2) using the renormalization group. Stephen[13] has attempted such a treatment, but as we shall see, his result for $B(n,p)$ is incorrect.

In this paper we study the various distribution functions for cluster statistics typified by $B(n,p)$. To do this we will show in Sec. II that the generating function for both lattice animals and percolating clusters can be obtained from the one-state Potts model in the presence of an external field and an external anisotropy field. This formulation is similar to that of Giri *et al.*[14] for the site-bond generating function for percolating clusters. In Sec. III we study $A(n)$, $B(n,p)$, and selected other functions in mean-field theory. Though some of the results presented here are not new, they are needed as a basis for later discussion. At the level of mean-field theory τ_p and θ_a are equal, so it is impossible to tell whether θ in Eq. (1.5) is equal to θ_a or to τ_p. In Sec. IV we present a renormalization-group analysis of the cluster-generating function. In particular we are led to consider the one-state Potts model in a negative external field and with external anisotropy. In the absence of these fields we recover the usual results,[10] namely, that the critical exponents are mean-field-like for $d > 6$. In the presence of these fields we find that there is a crossover to the animals behavior. For the animals problem the critical exponents, in particular θ_a, depart from their mean-field values for $d < 8$.[4] A study of this crossover from percolation to animals enables us to construct the scaling function for $B(n,p)$. For $6 < d < 8$ this scaling function is of the form

$$B(n,p) \sim n^{-\tau_p} G[n(p-p_c)^{\Delta_p}, n(p-p_c)^{\mu}] \ , \tag{1.6}$$

in disagreement with Eq. (1.2), where $\mu = \beta_p\Delta_p/\phi_p$, where ϕ_p is the crossover exponent for anisotropy for the one-state Potts model. Thus $B(n,p)$ undergoes a crossover from percolation to animals behavior as a function of $u = n|p-p_c|^{\mu}$. We find that

$$\beta_p = 1 \ , \tag{1.7a}$$

$$\Delta_p = 2 \ , \tag{1.7b}$$

$$\phi_p = (8-d)\nu_p = \tfrac{1}{2}\epsilon \ , \tag{1.7c}$$

$$\mu = 4/\epsilon \ , \tag{1.7d}$$

where $\epsilon = 8-d$. In Sec. V we extend the discussion via a scaling argument to $d < 6$. There one has

$$\phi_p = \beta_p \ , \tag{1.8a}$$

$$\mu = \Delta_p \ . \tag{1.8b}$$

Thus for $d < 6$, C in Eq. (1.2) is in fact a scaling function of a single variable, but it has a singularity at large argument causing a crossover from percolation to animals. For $p \to p_c^-$ and $n \to \infty$ we have the results

$$B(n,p) \sim n^{-\tau_p}\exp[-An(p_c-p)^{\Delta_p}] \ , \qquad u \ll 1 \ , \tag{1.9a}$$

$$B(n,p) \sim n^{-\theta_a}(p_c-p)^{(\tau_p-\theta_a)\Delta_p}\exp[-An(p_c-p)^{\Delta_p}] \ , \qquad u \gg 1 \ . \tag{1.9b}$$

This behavior predicts that $C(u)$ has a maximum at some value of u, say $u_{\max}$, since it must behave as $u^{\tau_p-\theta_a}\exp(-au)$ for large u, and $\tau_p - \theta_a$ is positive. This behavior has been noted and discussed by Stauffer[8] in some detail.

In Sec. VI we show that the dimensionality-dependent essential singularity predicted by Kunz and Souillard[12] arises from instantons of finite action in the Potts model in a field. These states are essentially identical to the metastable Ising clusters in a negative field discussed by Langer.[15] Near p_c we find

$$B(n,p) \sim n^{-\tau'}\exp[-cn(p-p_c)^{\Delta_p}]^{1-1/d} \ , \qquad p > p_c \ , \tag{1.10}$$

where c is a constant and τ' is a critical exponent whose calculation will be presented in a future publication.

II. POTTS MODEL AND CLUSTER STATISTICS

It is now well known that the one-state limit of the s-state Potts model generates the statistics of percolating clusters.[16] We will rederive this result in this section. We will also show how the generating function for lattice animals with a given number of bonds, perimeter bonds, and sites can be obtained from the one-state Potts model.

Consider a lattice with coordination number z and N sites $\vec{x}$. We write the Hamiltonian for the s-state Potts model as

$$\begin{aligned}\mathcal{H} = &-\sum_{\langle \vec{x},\vec{x}'\rangle} K(\delta_{\sigma(\vec{x}),\sigma(\vec{x}')} - s^{-1}) \\ &- \sum_{\langle \vec{x},\vec{x}'\rangle} (K_1 - K)(\delta_{\sigma(\vec{x}),1}\delta_{\sigma(\vec{x}'),1} - s^{-1}) \\ &- \sum_{\vec{x}} H(\delta_{\sigma(\vec{x}),1} - s^{-1}) \ ,\end{aligned} \tag{2.1}$$

where $\langle \vec{x}, \vec{x}' \rangle$ indicates that the sum is over pairs of nearest-neighboring sites, $\sigma(\vec{x})$ is an s-state variable, and δ is the Kronecker δ. The anisotropic coupling represented by the second term in Eq. (2.1) has previously been introduced by Giri *et al.*[14] to discuss cluster statistics. The partition function for this model can be written as

$$Z = Z' \exp[(s-1)(NH + N_B K_1)/s] \ , \tag{2.2}$$

where N is the total number of sites in the lattice, and $N_B = \frac{1}{2}Nz$ is the total number of bonds on the lattice, and

$$Z' = \exp(-N_B K_1) \sum_{\{\sigma(\vec{x})\}} \prod_{\vec{x}} \exp[H(\delta_{\sigma(\vec{x}),1} - 1)] \prod_{\langle \vec{x}, \vec{x}' \rangle} \{1 + v\delta_{\sigma(\vec{x}),\sigma(\vec{x}')} \exp[J(\delta_{\sigma(\vec{x}),1} - 1)]\} \ , \tag{2.3}$$

where

$$1 + v = e^{K_1} \ , \tag{2.4a}$$

$$1 + ve^{-J} = e^{K} \ . \tag{2.4b}$$

We now evaluate Z' by associating each term in the expansion of Eq. (2.3) in powers of v with a graph $\mathcal{G}$. To each factor $v\delta_{\sigma(\vec{x}),\sigma(\vec{x}')}$ we associate an occupied bond connecting sites $\vec{x}$ and $\vec{x}'$. The effect of these bonds is to cause all sites $\vec{x}$ in a cluster to have the same value of $\sigma(\vec{x})$. The trace over all states then reduces to a product of traces over clusters $C(\mathcal{G})$ contained in the graph $\mathcal{G}$. In this way we find that

$$Z' = e^{-K_1 N_B} \sum_{\mathcal{G}} (e^{K_1} - 1)^{N_b(\mathcal{G})} \times \prod_{C(\mathcal{G})} \{1 + (s-1)e^{-HN_s(C)} \times [(e^K - 1)/(e^{K_1} - 1)]^{N_b(C)}\} \ , \tag{2.5}$$

where $n_b(C)$ is the number of occupied bonds of C, $n_s(C)$ the number of sites in C, and $n_b(C)$ the number of occupied bonds in C. In the limit $s \to 1$ we write Eq. (2.5) as

$$-Nf' = \lim_{s\to 1} \frac{1}{s-1} \ln Z' = \sum_{\mathcal{G}} e^{-K_1 N_B} (e^{K_1} - 1)^{N_b(\mathcal{G})} \times \sum_{C(\mathcal{G})} e^{-HN_s(C)} [(e^K - 1)/(e^{K_1} - 1)]^{N_b(C)} \tag{2.6}$$

so that f' is the free energy corresponding to Z'. Now consider a particular cluster, C, at some fixed position on the lattice. For this cluster we define its perimeter bonds as the set of unoccupied bonds at least one end of which intersects the cluster (see Fig. 1). Let C' be the union of C with its perimeter bonds. There will be many graphs $\mathcal{G}(C')$ in which C' appears, and only these graphs contribute to the terms in the sum of Eq. (2.6) involving C. In Eq. (2.6) we write $e^{-K_1 N_B}$ as a product over all bonds of e^{-K_1}. Then we note the following weightings. Unoccupied bonds carry a factor e^{-K_1}. Occupied bonds in C' carry the factor

$$e^{-K_1}(e^{K_1} - 1)[(e^K - 1)/(e^{K_1} - 1)] = e^{-K_1}(e^K - 1) \ .$$

Occupied bonds not in C' carry a factor $e^{-K_1}(e^{K_1} - 1) = 1 - e^{-K_1}$. In other words, to the bonds in C' we may attribute the factor

$$\exp[-K_1 n_p(C')]\{[\exp(-K_1)][\exp(K) - 1]\}^{n_b(C')} \ ,$$

where $n_b(C')$ and $n_p(C')$ are the number of bonds and the number of perimeter bonds, respectively, in C'. The other bonds of $\mathcal{G}$ which are not in C' are weighted by a factor $\exp(-K_1)$ if they are unoccupied and by a factor $1 - \exp(-K_1)$ if they are occupied. The sum over all graphs $\mathcal{G}(C')$ is a sum over these two possibilities and therefore yields a factor

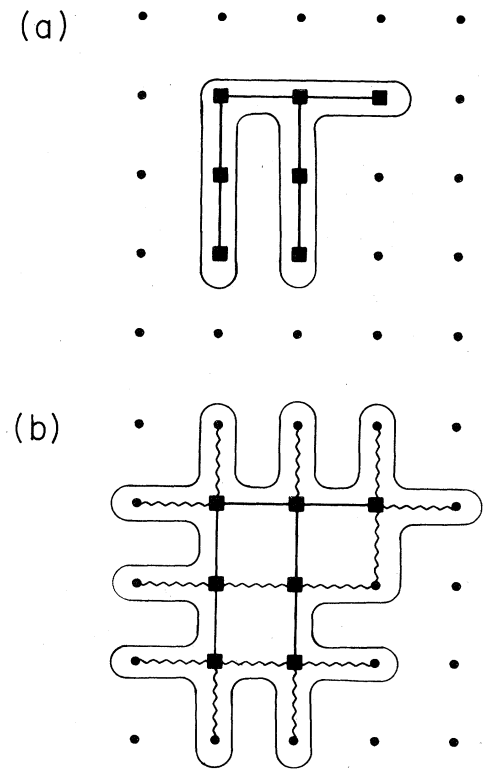

FIG. 1. (a) Cluster, C, of occupied sites, indicated by dark squares, connected by occupied bonds, indicated by lines. The boundary of the cluster is shown. (b) Set C' given by the union of the cluster C and its perimeter bonds, shown here by wave lines.

unity for each bond not in C'. Thus we find that

$$-f' = \sum_{n_p,n_s,n_b} \exp(-K_1 n_p - H n_s) \times [(e^K - 1)e^{-K_1}]^{n_b} A(n_p,n_s,n_b) \quad , \quad (2.7)$$

where $A(n_b,n_p,n_s)$ is the number of clusters per site with n_b bonds, n_p perimeter bonds, and n_s sites. We now set

$$e^{-K_1} = qe^{-J_p} \quad , \quad (2.8a)$$

$$e^{-K_1}(e^K - 1) = pe^{-J_b} \equiv e^{-L} \quad , \quad (2.8b)$$

in which case we may write

$$-f'(K_1,H,L) = \sum_{n_p,n_s,n_b} \exp(-K_1 n_p - H n_s - L n_b) \times A(n_p,n_s,n_b) \quad , \quad (2.9a)$$

or

$$-f'(p,q;J_p,H,J_b) = \sum p^{n_b} q^{n_p} \exp(-J_p n_p - H n_s - J_b n_b) \times A(n_p,n_s,n_b) \quad . \quad (2.9b)$$

Using Eq. (2.2) we relate f' to the free energy, f, corresponding to Z by

$$f'(K_1,H,L) = f(K_1,H,L) + H + \frac{z}{2}K_1 \quad . \quad (2.10)$$

We may define various probability distributions based on $A(n_b,n_p,n_s)$. We may consider the distribution function for the number of clusters per site having a given number of sites, a given number of bonds, or a given number of perimeters. These functions are, respectively,

$$N_s(n,p,q) = \sum_{n_b,n_p} A(n_p,n,n_b) p^{n_b} q^{n_p} \quad , \quad (2.11a)$$

$$N_b(n,p,q) = \sum_{n_p,n_s} A(n_p,n_s,n) p^{n_b} q^{n_p} \quad , \quad (2.11b)$$

$$N_p(n,p,q) = \sum_{n_s,n_b} A(n,n_s,n_b) p^{n_b} q^{n_p} \quad . \quad (2.11c)$$

For the variables the special cases of interest are

$$q = 1 - p, \text{ percolation} \quad , \quad (2.12a)$$

$$q = 1, \text{ lattice animals} \quad , \quad (2.12b)$$

$$p = 1, \text{ perimeter animals} \quad . \quad (2.12c)$$

Specifically we will study the functions

$$B(n,p) = N_s(n,p,1-p) \quad , \quad (2.13a)$$

$$A(n) = N_b(n,1,1) \quad , \quad (2.13b)$$

$$P(n) = N_p(n,1,1) \quad , \quad (2.13c)$$

where $P(n)$ is the number of clusters having a given perimeter, n. The cluster distribution functions of Eq. (2.12) can be obtained from f (or f') by Laplace transformation. We have

$$N_s(n,p,q) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dH\, e^{nH} f'(p,q;0,H,0) \quad , \quad (2.14a)$$

$$N_b(n,p,q) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dJ_b\, e^{nJ_b} f'(p,q;0,0,J_b) \quad , \quad (2.14b)$$

$$N_p(n,p,q) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dJ_p\, e^{nJ_p} f'(p,q;J_p,0,0) \quad . \quad (2.14c)$$

From both a mathematical and physical point of view it is clear that the behavior of the distribution functions for large n reflects the critical-point behavior of the free energy, f'.

Finally, we introduce one two-variable distribution function,

$$N_{b,p}(n_b,n_p;p,q) = \sum_{n_s} A(n_p,n_s,n_b) p^{n_b} q^{n_p} \quad , \quad (2.15)$$

which may be obtained via the relation

$$N_{b,p}(n_b,n_p;p,q) = \frac{1}{4\pi^2}\int_{-i\infty}^{i\infty} dJ_p\, e^{n_p J_p} \int_{-i\infty}^{i\infty} dJ_b\, e^{n_b J_b} f'(p,q;J_p,0,J_b) \quad . \quad (2.16)$$

Note that J_b and J_p are only used as dummy variables for integration. Otherwise, they are set equal to zero.

III. MEAN-FIELD THEORY

We start by recasting the partition function for the Hamiltonian of Eq. (2.1) in terms of a field theory. To this end we first reexpress the Hamiltonian in terms of vectors e_l^σ for $\sigma = 1, 2, \ldots, s$ and $l = 1, 2, \ldots, (s-1)$, in the manner of Zia and Wallace.[17] These vectors satisfy

$$\sum_\sigma e_l^\sigma = 0 \quad , \quad (3.1a)$$

$$\sum_\sigma e_l^\sigma e_{l'}^\sigma = \delta_{l,l'} \quad , \quad (3.1b)$$

$$\sum_l e_l^\sigma e_l^{\sigma'} = \delta_{\sigma,\sigma'} - s^{-1} \quad . \quad (3.1c)$$

We write Eq. (2.1) as

$$\mathcal{H} = -\sum_{\langle \vec{x},\vec{x}' \rangle} \sum_{l,l'} [K\delta_{l,l'} + (K_1 - K) e_l^1 e_{l'}^1] e_l^{\sigma(\vec{x})} e_{l'}^{\sigma(\vec{x}')} - \sum_{\vec{x},l} H_l e_l^{\sigma(\vec{x})} + \tfrac{1}{2}(s-1)NZ(K_1 - K)/s^2 \quad , \quad (3.2)$$

where

$$H_l = \left[H + \frac{Z}{s}(K_1 - K)\right] e_l^1 \equiv H' e_l^1 \ . \tag{3.3}$$

Now we use the Hubbard-Stratonovich transformation to write

$$Z = e^{-N(s-1)f_0} Z_\Psi \ , \tag{3.4}$$

where

$$Nf_0 = \tfrac{1}{2} Nz(K_1 - K)/s^2 + \tfrac{1}{2}(s-1)^{-1} \sum_{\vec{x},\vec{x}'} \sum_{l,l'} H_l(\vec{x}) K_{l,l'}(\vec{x},\vec{x}') H_{l'}(\vec{x}') \ , \tag{3.5a}$$

and

$$Z_\Psi = C_0 \int \prod_{\vec{x},l} d\psi_l(\vec{x})\, e^{-F(\{\psi\})} \ , \tag{3.5b}$$

where C_0 is a normalization constant and

$$F(\{\Psi\}) = \tfrac{1}{2} \sum_{\vec{x},\vec{x}'} \sum_{l,l'} \psi_l(\vec{x}) K_{l,l'}(\vec{x},\vec{x}') \psi_{l'}(\vec{x}') - \sum_{\vec{x},l} H_l(\vec{x}) \psi_l(\vec{x}) + \sum_{\vec{x}} \mathcal{S}(\vec{x}) \ , \tag{3.6}$$

with

$$\mathcal{S}(\vec{x}) = -\ln\left[\sum_\sigma \exp\left(\sum_{l,l'} \sum_{\vec{x}'} K_{l,l'}(\vec{x},\vec{x}') \psi_{l'}(\vec{x}') e_l^\sigma\right)\right] \ . \tag{3.7}$$

Here we set

$$K_{l,l'}(\vec{x},\vec{x}') = \gamma(\vec{x},\vec{x}')[K\delta_{l,l'} + (K_1 - K) e_l^1 e_{l'}^1] \ , \tag{3.8}$$

where

$$\gamma(\vec{x},\vec{x}') = \begin{cases} 1 \text{ if } x \text{ and } x' \text{ are nearest neighbors} & \text{(3.9a)} \\ 0 \text{ otherwise.} & \text{(3.9b)} \end{cases}$$

Since all the critical effects are contained in Z_Ψ, we concentrate our attention on it.

Mean-field theory is obtained by ignoring configurations in $F(\{\Psi\})$ in which $\psi_l(x)$ is spatially varying. That is we set

$$\psi_l(x) = \Psi_l = \Psi e_l^1 \ . \tag{3.10}$$

Then we find

$$\Gamma(\Psi) \equiv \lim_{s\to 1}\left[\frac{1}{N(s-1)} F(\{\Psi\})\right] + H'\Psi \tag{3.11a}$$

$$= \tfrac{1}{2} zK\Psi^2 - zK\Psi - e^{-zK\Psi} \ . \tag{3.11b}$$

The value of Ψ is determined by the equation of state:

$$\Gamma^{(1)}(\Psi) \equiv \frac{\partial\Gamma(\Psi)}{\partial\Psi} = zK\Psi - zK(1 - e^{-zK\Psi}) = H' \tag{3.12}$$

and the inverse susceptibility is given by

$$\Gamma^{(2)}(\Psi) \equiv \frac{\partial^2\Gamma(\Psi)}{\partial\Psi^2} = zK(1 - zKe^{-zK\Psi}) \equiv r \ . \tag{3.13}$$

For the purposes of mean-field theory it is not necessary to consider the general tensor susceptibility, $\partial^2\Gamma/\partial\psi_l\partial\psi_{l'}$, because its anisotropy vanishes as $s \to 1$. This anisotropy will be treated in the next section, inasmuch as it becomes relevant for $d < 8$.

Criticality occurs when $r = 0$, i.e., when $\exp(zK\Psi) = zK$. However, Ψ must satisfy the equation of state, Eq. (3.12). Thus, there is a critical line in the (H', zK) plane, $H_c' = H_c'(zK)$, determined by the relation

$$H_c' = 1 - zK + \ln zK \ . \tag{3.14}$$

One can verify that $H_c'(zK)$ has a single maximum at $zK = 1$, with $H_c' = 0$. Thus, there is no critical point for $H' > 0$. For $H' < 0$ there are two solutions to Eq. (3.14), one for $zK < 1$, which we shall call the weak-coupling transition, and one for $zK > 1$, which we shall call the strong-coupling transition. In the latter regime there is a singularity at $H' = 0, K_1 - K = 0$ corresponding to the instability of the ordered phase with instantons giving rise to the essential singularities discussed by Kunz and Souillard.[12] We will refer to the special point $H' = 0$, $zK = 1$ as the percolation point, since percolation is described by $K_1 - K = 0$, and $H = 0$, which gives $H' = 0$. The critical surface and the essential singularity surface are shown in Fig. 2.

It is often convenient to consider the free energy in terms of particular sets of variables. To relate the phase diagram to values of K_1, L, and H we write

$$H' = H - z\ln(e^{-K_1} + e^{-L}) \ , \tag{3.15a}$$

$$K = \ln(1 + e^{K_1 - L}) \ . \tag{3.15b}$$

Then the relation for criticality may be written as

$$e^{H + zK_1} = ze\ln(1 + e^{K_1 - L}) \ . \tag{3.16}$$

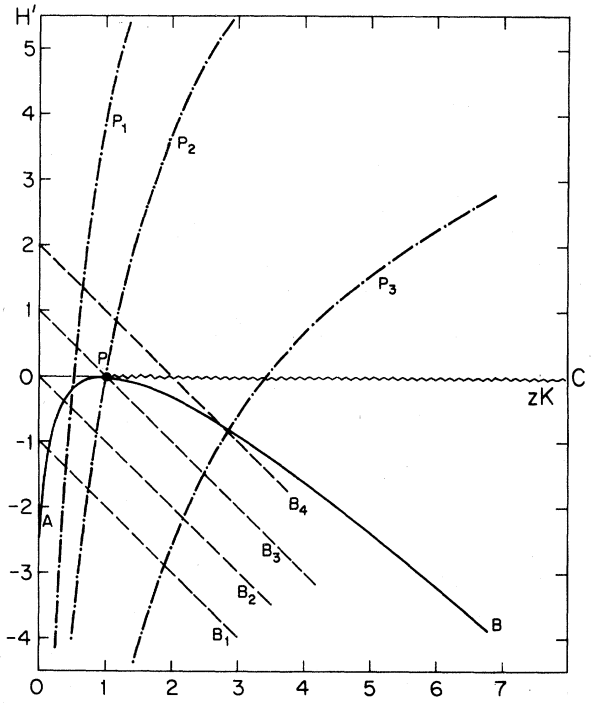

FIG. 2. Phase diagram within mean-field theory. The critical surface (APB), the essential singularity surface (PC), and the trajectories for evaluating $N_b(n,p,q)$ (curves B) and $N_p(n,p,q)$ (curves P) projected onto the plane $K_1 = K$ for $z = 6$. Trajectories for evaluating $N_s(n,p,q)$ are vertical lines at positions along the horizontal axis determined by p and q (the trajectory for $p = 1 - q = p_c$ passes through P). Curves B_1, B_2, B_3, and B_4 correspond, respectively, to $q > 1$, $q = 1$, $q = q_c = \exp(-1/z)$, and $q < q_c$. Curves P_1, P_2, and P_3 correspond, respectively, to $p < p_c = 1 - \exp(-1/z)$, $p = p_c$, and $p > p_c$. Bond curves for $q < q_c$ (e.g., B_4), perimeter curves for $p > p_c$, and site curves for $p > p_c$ intersect both the essential singularity and critical surfaces indicating that there will be an essential singularity and a critical singularity in N_s and N_p and N_b for these cases. The critical surface (APB) is independent of the value of $\Delta K = K_1 - K$. The essential singularity surface is not independent of $K_1 - K$, but we have not calculated its form in the general $(H', K, \Delta K)$ plane. Fluctuations will quantitatively affect the shapes of the critical surface and of the essential singularity surface. The bond, site, and perimeter trajectories being determined by constitutive equations will not be affected by fluctuations.

For percolation we set $H = 0$, $\exp(-L) = p$, and $\exp(-K_1) = 1 - p$ in which case Eq. (3.16) yields the critical percolation concentration, p_c, as

$$p_c = 1 - e^{-1/z} \ . \tag{3.17}$$

For bond animals one sets $H = 0$, $K_1 = 0$, and $\exp(-L) = p$, whence criticality occurs at $p = p_c^{(a)}$ with

$$p_c^{(a)} = e^{1/ze} - 1 < p_c \ . \tag{3.18}$$

For perimeter animals one sets $H = 0$, $L = 0$, and $\exp(-K_1) = q$, whence criticality occurs at $q = q_c^{(a)}$ with

$$1 - p_c > q_c^{(a)} \sim 1 - \frac{1}{z} \ln z \ , \tag{3.19}$$

for large z. Except in Sec. VI we confine ourselves to the disordered regimes: $p < p_c$ for percolation, $p < p_c^{(a)}$ for bond animals, and $q < q_c^{(a)}$ for perimeter animals.

In evaluating the inverse Laplace transforms, e.g., Eq. (2.14), we will set two of the variables equal to the fixed values for the model and integrate over the third one. In all cases p and q are taken to be fixed. For the site distribution function $J_b = J_p = 0$, for the bond, $H = J_p = 0$, and for the perimeter bond $H = J_b = 0$. The integration trajectories in the H', K, $\Delta K = z(K_1 - K)$ plane for the three cases are

$$H' = H - z \ln(p+q) \ , \quad K = \ln\left[\frac{p+q}{q}\right] = \text{const} \ ,$$
$$\Delta K = -z \ln(p+q) = \text{const} \ , \tag{3.20a}$$

for sites;

$$H' = -z(\ln q + K) \ , \quad \Delta K = -z(\ln q + K) = H' \ ,$$
$$K = \ln\left[1 + \frac{p}{q} e^{-J_b}\right] \ , \tag{3.20b}$$

for bonds; and

$$H' = -z \ln p + z \ln(1 - e^{-K}) \ , \quad \Delta K = H' \ ,$$
$$K = \ln(1 + \frac{p}{q} e^{J_p}) \ , \tag{3.20c}$$

for perimeter bonds. These trajectories (for real values of the variables H, J_b, or J_p) are shown in Fig. 2. All the site trajectories are parallel to the H' axis intersecting the critical curve in the weak-coupling regime for $(p+q)/q < \exp(1/z)$ and in the strong-coupling regime for $(p+q)/q > \exp(1/z)$. Note that both low-density percolation ($p + q = 1$, $p < p_c$) and animals ($q = 1$, $p < p_c^{(a)}$) lead to a weak-coupling transition, whereas the percolation problem with $p > p_c$ corresponds to strong coupling.

The bond trajectories begin at $H' = \Delta K = -z \ln(q)$, $K = 0$ and run to $H' = \Delta K = -zK \to -\infty$, intersecting the critical curve in the weak-coupling regime for $z \ln q^{-1} < 1$ and in the strong-coupling regime otherwise. Note that pure bond animals with $p = q = 1$ as well as low-density percolation correspond to the weak-coupling regime whereas percolation for $p > p_c$ again leads to the strong-coupling case. Finally, the

perimeter trajectories begin at $K=+\infty$, $H'=\Delta K = z\ln p^{-1}$ and run to $K=0$, $H'=\Delta K=-\infty$, intersecting the critical curve in the weak-coupling regime for $p<p_c$ and the strong-coupling regime otherwise independently of q, for $q>0$. Thus pure perimeter animals with $p=q=1$ correspond to the strong-coupling regime.

All trajectories in the strong-coupling regime are expected to intersect an essential singularity surface giving rise to large-n behavior of the form of Eq. (1.10). In Sec. VI we show that this surface intersects the line $H'=0$, $K>K_c$ when $K_1-K=0$. We have not calculated the shape of the general surface when $K_1-K\neq 0$ and cannot say whether it always lies above the extension of the mean-field critical surface shown in Fig. 2.

We now present a few detailed calculations. We begin our analysis by considering the case of percolation near p_c. In this case Ψ is small and we can expand $\Gamma^{(1)}$ and $\Gamma^{(2)}$ in powers of Ψ. In this way we obtain

$$\Psi = w^{-1}[-r+(r^2+2wH')^{1/2}] \ , \tag{3.21}$$

$$\Gamma^{(2)} = (r^2+2wH')^{1/2} \ , \tag{3.22}$$

where

$$r = zK(1-zK) \sim z(K_c-K) \sim \frac{z(p_c-p)}{(1-p_c)} - zp(J_b-J_p) \ , \tag{3.23}$$

$$w=(zK)^3 \sim (zK_c)^3 \sim 1 \ , \tag{3.24}$$

$$H' \cong H + z(pJ_b+qJ_p) - \tfrac{1}{2}zpq(J_b-J_p)^2 \ . \tag{3.25}$$

For $H'=0$ we have $\Gamma^{(2)}\sim (p_c-p)^\gamma$, with $\gamma=1$, as expected for pure percolation. For $H'<0$, we have $\Gamma^{(2)}\sim [r_0-r_c(H')]^\gamma$ and $\Psi\sim[r_0-r_c(H')]^\beta$ with $r_c(H')=(2wH')^{1/2}$ and $\gamma=\beta=\frac{1}{2}$. Thus within mean-field theory the one-state Potts model in a negative field has the same critical exponents as for animals.[4] Substituting Eq. (3.21) into Eq. (3.11b) we find, apart from some unimportant terms from f_0

$$-f = -\frac{1}{3}\frac{r^3}{w^2} - \frac{r}{w}H' + \frac{1}{3w^2}(r^2+2wH')^{3/2} \ . \tag{3.26}$$

Henceforth we set $w=1$ near the critical point. From this result we obtain $B(n,p)$ via Eq. (2.14a) as

$$B(n,p) = (2\pi)^{-1/2}n^{-5/2}\exp(-\tfrac{1}{2}nr_0^2) \ , \tag{3.27}$$

with $r_0=z(p_c-p)/(1-p_c)$, with p_c given by Eq. (3.17). This result agrees with that obtained previously by Stephen.[13] Similarly we obtain

$$N_b(n,p,1-p) = (2\pi)^{-1/2}n^{-5/2}(zp_c)^{3/2}\exp\left[-\frac{1}{2}\frac{nr_0^2}{zp_c}\right] \ , \tag{3.28}$$

$$N_p(n,p,1-p) = (2\pi)^{-1/2}n^{-5/2}[z(1-p_c)]^{3/2} \times \exp\left[-\frac{1}{2}\frac{nr_0^2}{z(1-p_c)}\right] \ . \tag{3.29}$$

The above results are of the form

$$N_\alpha(n,p,1-p) = A_\alpha n^{-5/2}e^{-\lambda_\alpha n} \ , \tag{3.30}$$

where α is s, b, or p. For the Cayley tree the exact analysis[6] gives $\lambda_s=\lambda_b=(z-2)\lambda_p$. Our mean-field results do not quite reproduce this relation. As Stephen[13] has pointed out in a similar context, a more delicate form of mean-field theory is needed to reproduce these consistency relations near p_c.

Results away from the percolation threshold can also be easily obtained. Let (K_c,H_c') be the coordinates of some point on the critical line given by Eq. (3.14). For this point Ψ_c is found via

$$\Gamma^{(2)}(\Psi_c)=0 \ . \tag{3.31}$$

Using the equation of state and Eq. (3.13) we can also obtain $\Delta\Psi=\Psi-\Psi_c$ away from criticality. The nonanalytic part of the free energy in the vicinity of the critical point is then quite generally

$$f = \tfrac{1}{6}(zK_c)^2(\Delta\Psi)^3 - (\Delta H')(\Delta\Psi) \ , \tag{3.32}$$

where $\Delta H'=H'-H_c'$ and

$$\Delta\Psi = \left[2\left[z-\frac{1}{K_c}\right](K-K_c)+2\Delta H'\right]^{1/2} \ . \tag{3.33}$$

Using Eqs. (3.32) and (3.33) we can determine the dominant contribution to any of the $N_\alpha(n,p,q)$ for cases when the trajectories in general do not intersect the percolation point, $zK=1$, $H'=0$. We find that

$$N_s(n,p,q) = (2\pi)^{-1/2}(3/2)n^{-5/2}[1-\tfrac{1}{3}(zK_c)^2]e^{nH_c} \ , \tag{3.34}$$

where

$$K_c = \ln\left[\frac{p+q}{q}\right] \ , \tag{3.35a}$$

$$e^{H_c} = zeK_c\left[\frac{q}{p+q}\right]^z \ , \tag{3.35b}$$

and

$$N_b(n,p,q) = \frac{3z}{2}\left(\frac{K_c}{2\pi}\right)^{1/2}(1-\tfrac{1}{3}zK_c)$$

$$\times\left(\frac{p}{qeK_c}\right)^{3/2} n^{-5/2} e^{(n-3/2)J_b^c} \quad , \tag{3.36}$$

where

$$K_c = (zeq^z)^{-1} \quad , \tag{3.37a}$$

$$e^{-J_b^c} = \frac{q}{p}(e^{K_c}-1) \quad . \tag{3.37b}$$

Similar results for $N_p(n,p,q)$ can be obtained but since the result is more complicated, we omit it. Special cases of the above results are

$$A(n) \sim n^{-5/2}\left[\exp\left(\frac{1}{ze}\right)-1\right]^{-n} \tag{3.38}$$

and

$$P(n) \sim n^{-5/2}(q_c^{(a)})^{-n} \quad . \tag{3.39}$$

Finally we consider the two variable distribution functions near percolation obtained via Eq. (2.16). We find that

$$N_{b,p}(n_b,n_p;p,1-p) = (2\pi)^{-1} n_+^{-3} a^{-1/2} z^{3/2}$$

$$\times \exp\left(-\frac{r_0^2 n_+}{2z} - \frac{n_-^2}{2an_+}\right) \quad , \tag{3.40}$$

where

$$n_+ = n_b + n_p \quad , \tag{3.41}$$

$$n_- = (1-p)n_b - pn_p \quad , \tag{3.42}$$

$$a = p(1-p-pz) \quad . \tag{3.43}$$

This result has the same form as the site-bond distribution function calculated by Stephen[13] and suffers from the same limitations: a is negative for some values of p. This presumably results from the incompleteness of mean-field theory.

IV. ϵ EXPANSION

To obtain an ϵ expansion leading to expressions for generating functions in the critical region, we express Z_Ψ in terms of deviations of $\psi_l(x)$ from its equilibrium value:

$$\psi_l(\vec{x}) = \Psi e_l^1 + \phi_l(\vec{x}) \quad , \tag{4.1}$$

where we require

$$\langle \phi_l(\vec{x}) \rangle = 0 \quad . \tag{4.2}$$

We may now express $F(\{\Psi\})$ in terms of $\phi_l(x)$ and write

$$F(\{\Psi\}) = \sum_n F_n(\{\phi\}) \quad ,$$

where F_n contains all contributions to F of order ϕ^n. We can obtain F_0 from Eq. (3.11). Also

$$F_1 = -h\sum_{l,\vec{x}} e_l^1 \phi_l(\vec{x}) \quad , \tag{4.3}$$

with

$$h = zK(1-e^{-zK\Psi}) + H' - zK\Psi \quad , \tag{4.4}$$

and

$$F_2 = \tfrac{1}{2}\sum_{\vec{x},\vec{x}',l}[K\gamma(\vec{x},\vec{x}') - K^2 e^{-zK\Psi}\gamma^2(\vec{x},\vec{x}')]\phi_l(\vec{x})\phi_l(\vec{x}')$$
$$+\tfrac{1}{2}\sum_{\vec{x},\vec{x}'}\sum_{l,l'}[(K_1-K)\gamma(\vec{x},\vec{x}') - K^2 e^{-zK\Psi}\gamma^2(\vec{x},\vec{x}')(2K_1-K-Ke^{-zK\Psi})]\,\phi_l(\vec{x})\phi_{l'}(\vec{x}')e_l^1 e_{l'}^1 \quad , \tag{4.5}$$

where $\gamma^2_{\vec{x},\vec{x}'} = \sum_{\vec{x}''}\gamma_{\vec{x},\vec{x}''}\gamma_{\vec{x}'',\vec{x}'}$. The cubic terms have a symmetry similar to that of the animals problem:

$$F_3 = \sum_{\vec{x}_1,\vec{x}_2,\vec{x}_3}\sum_{l_1,l_2,l_3}[-\tfrac{1}{6}\lambda_{l_1,l_2,l_3}w(\vec{x}_1,\vec{x}_2,\vec{x}_3) + \tfrac{1}{6}e_{l_1}^1 e_{l_2}^1 e_{l_3}^1 v(\vec{x}_1,\vec{x}_2,\vec{x}_3) + e_{l_1}^1\delta_{l_2,l_3}u_3(\vec{x}_1,\vec{x}_2,\vec{x}_3)]\phi_{l_1}(\vec{x}_1)\phi_{l_2}(\vec{x}_2)\phi_{l_3}(\vec{x}_3) \quad , \tag{4.6}$$

where u_3, v, and w are short-ranged potentials. Near and above eight dimensions u_3, v, and also all the potentials in F_n with $n > 4$ are irrelevant and need not be considered further. It is necessary to retain w even though it is irrelevant because it occurs in the recursion relations in combination with a relevant variable. Keeping only

relevant terms in the gradient expansion of F we then obtain a model Hamiltonian for d near 8 of the form

$$\mathcal{H} = E_0 - \sum_l \int d^d\vec{x}\, h_l \phi_l(\vec{x}) + \frac{1}{2}\sum_{l,l'} \int d^d\vec{x}\, \{[r\phi_l(\vec{x})^2 + |\nabla\phi_l(\vec{x})|^2]\delta_{l,l'} + Te_l^1 e_{l'}^1 \phi_l(\vec{x})\phi_{l'}(\vec{x})\}$$
$$- \frac{1}{6} w \sum_{l_1,l_2,l_3} \lambda_{l_1,l_2,l_3} \int d^d\vec{x}\, \phi_{l_1}(\vec{x})\phi_{l_2}(\vec{x})\phi_{l_3}(\vec{x}) \ , \tag{4.7}$$

where

$$r = Kz(1 - Kze^{-Kz\Psi}) \ , \tag{4.8a}$$

$$T = z[K_1 - K - Kze^{-Kz\Psi}(2K_1 - K - Ke^{-Kz\Psi})] \ . \tag{4.8b}$$

Note that r, T, and h depend not only on the external fields K, H, and $K_1 - K$, but also on Ψ. Recursion relations for the potentials of Eq. (4.7) can be developed in the usual way leading to[4]

$$\frac{dr}{dl} = (2-\eta)r + \frac{1}{2}w^2K_d(1+r)^{-2} + K_d w^2 T(1+r)^{-3} \ , \tag{4.9}$$

$$\frac{dT}{dl} = (2-\eta)T - \frac{1}{2}w^2K_d T^2(1+r)^{-4} - 2w^2K_d T(1+r)^{-3} \ , \tag{4.10}$$

$$\frac{dw}{dl} = \frac{1}{2}(6-d-3\eta)w - 2w^3K_d(1+r)^{-3} - 3w^3K_d T(1+r)^{-4} \ , \tag{4.11}$$

$$\frac{dh}{dl} = \frac{1}{2}(d+2-\eta)h + \frac{1}{2}wK_d T(1+r)^{-2} \ , \tag{4.12}$$

$$\eta = -\frac{1}{2}w^2K_d T - \frac{1}{6}w^2K_d \ , \tag{4.13}$$

where $K_d^{-1} = 2^{d-1}\pi^{1/2}\Gamma(d/2)$.

For the percolation problem $K_1 = K$ and $H = 0$, so that $\Psi = 0$ and $T = 0$. In this case one can show that the symmetry of the Potts model is preserved and hence that $u_3 = v = 0$. Thus percolation corresponds to a multicritical point in which the most relevant nonzero potentials are r and w. Then, as expected, one finds a fixed point in $6-\epsilon$ dimensions with $K_d(w^*)^2 = 2\epsilon/7$, $\eta_p = -\epsilon/21$, $\nu_p = \frac{1}{2} + 5\epsilon/84$, $\beta_p = 1 - \epsilon/7$, and $\gamma_p = 1 + \epsilon/7$. In addition the crossover exponent for turning on T is

$$\phi_p = 1 - \frac{\epsilon}{7} = \beta_p \ . \tag{4.14}$$

Zia and Wallace[17] have shown that $\phi_p = \beta_p$ to all orders in ϵ. Since $\phi_p > 0$, the percolation fixed point is unstable with respect to turning on T. In the usual percolation problem, T is zero at the fixed point because Ψ and $K_1 - K$ are zero. From Eq. (4.8b) one sees that it is possible, however, to reach the percolation fixed point even if H and Ψ are nonzero, provided $K_1 - K$ is adjusted to make $T = 0$. This is not the usual situation at all. Here a critical point can be reached even for finite order parameter induced by an external ordering field.

When T is nonzero, the system goes to a different fixed point that first appears in $8-\epsilon$ dimensions.[4] It is controlled by the variable $g = K_d w^2 T$ which satisfies

$$\frac{dg}{dl} = (8 - d - 4\eta)g - \frac{13}{2}g^2(1+r)^{-4} \tag{4.15}$$

near 8 dimensions. This equation and Eqs. (4.9)–(4.13) are identical to those obtained in Ref. 4 for the statistics of animals or dilute branched polymers using a slightly different model to generate cluster statistics. Thus, we can use those results without modification. We obtain

$$\nu = \frac{1}{2} + \frac{5}{36}\epsilon \ , \tag{4.16a}$$

$$\eta = -\frac{1}{9}\epsilon \ , \tag{4.16b}$$

$$\gamma = 1 + \frac{1}{3}\epsilon \ , \tag{4.16c}$$

$$g^* = \frac{2}{9}\epsilon \ . \tag{4.16d}$$

In addition, the exponent μ_3 describing the behavior of the three-point vertex function[18] near criticality is of some importance. We find

$$\mu_3 = \frac{1}{3}\epsilon \ . \tag{4.17}$$

These exponents refer to behavior when r is varied keeping the order parameter fixed. If the external fields $(H, K_1 - K)$ are varied, then, except on a special surface, Ψ will change in response to changes in K leading to a renormalization of critical exponents. To treat this case, the equation of state for Ψ must be solved. This rather complicated procedure is described in detail in Ref. 4. Here, we will outline

only the essential details. The equation for g can be integrated to yield

$$g(l) = e^{\epsilon l} g(0)/G(l) \ , \tag{4.18a}$$

$$G(l) = 1 + \tfrac{9}{2} g(0)(e^{\epsilon l} - 1)/\epsilon \ . \tag{4.18b}$$

Similarly, we find

$$\exp\left[\int_0^{l^*} \eta(l)\, dl\right] = G(l^*)^{-1/9} \ , \tag{4.19a}$$

$$w(l^*) = w(0) \exp[(3 - \tfrac{1}{2}d) l^*] G(l^*)^{-1/2} \ , \tag{4.19b}$$

$$r(l^*) = r(0) e^{2l^*} G(l^*)^{-5/9} \ , \tag{4.19c}$$

where, to avoid introducing extra notation, we assume that r measures distance from the actual rather than the mean-field critical point. As usual l^* is chosen so that $r(l^*) \sim 1$. The functions Γ and $\Gamma^{(1)}$ consist of regular and singular parts, with

$$\Gamma^{(1)}_{\text{sing}} = \tfrac{1}{2} r^2(l^*) \exp\left[-\int_0^{l^*} \tfrac{1}{2}(d + 2 - \eta)\, dl\right] \Big/ w(l^*) \tag{4.20a}$$

$$= \tfrac{1}{2} e^{-4l^*} G(l^*)^{4/9} / w(0) \ , \tag{4.20b}$$

and

$$\Gamma_{\text{sing}} = -\tfrac{1}{6} r^3(l^*) e^{-dl^*} / w^2(l^*) \tag{4.21a}$$

$$= -\tfrac{1}{6} e^{-bl^*} G(l^*)/w^2(0) \ . \tag{4.21b}$$

Equations (4.18) to (4.21) suffice to describe crossover and critical behavior near eight dimensions providing Ψ is held constant as the critical point is reached. Usually, however, the critical point is approached along a path in the $K_1 - K,H$ plane for which Ψ is not fixed. In this case, Ψ and r are determined by the equation of state, $\Gamma^{(1)} = H$. To solve this equation we note that to a given critical point there corresponds a critical value, Ψ_c, of Ψ. Away from the critical point, one has $\Psi = \Psi_c + \Delta\Psi$. Since $\Delta\Psi$ should scale as an order parameter, we define

$$\Delta\Psi(l^*) = \Delta\Psi(0) \exp\left[\tfrac{1}{2} \int_0^{l^*} (d - 2 + \eta)\, dl\right] \ . \tag{4.22}$$

One can easily verify using Eqs. (4.22) and (4.9)–(4.13) that

$$r(l) = t(l) + w(l) \Delta\Psi(l) \ , \tag{4.23}$$

where $t(l)$ is $r(l)$ evaluated at $\Psi = \Psi_c$. We are now in a position to investigate the equation of state. At the critical point, $\Gamma^{(1)}_{\text{sing}}$ is zero, so that $\Gamma^{(1)}_{\text{reg}}(K,g,\Psi_c)$ is in fact $H_c(K,g)$, and

$$\Gamma^{(1)}_{\text{sing}}(t, \Delta\Psi, g) = H - H_c = \Delta H \tag{4.24a}$$

$$= \tfrac{1}{2}(r_0^2 + 2wH)/w \ , \tag{4.24b}$$

where Eq. (4.24b) applies to a critical point in the vicinity of $H = 0$ and $r = 0$ (percolation). To order ϵ,

$$t(l^*) + w(l^*) \Delta\Psi(l^*) = e^{2l^*} G(l^*)^{-5/9} [t(0) + w(0) \Delta\Psi(0)] \ . \tag{4.25}$$

From Eqs. (4.24) and (4.21) we find that

$$e^{-4l^*} \cong 2w(0) \Delta H \times \left[1 + \frac{9g(0)}{2\epsilon} \{[2w(0)\Delta H]^{-\epsilon/4} - 1\}\right]^{-4/9} \ , \tag{4.26}$$

and from Eqs. (4.21) and (4.24)

$$f_{\text{sing}} = \Gamma - (\Delta H) \Delta\psi(0) \ . \tag{4.27}$$

Thus we have

$$f_{\text{sing}} \cong -[3w(0)^2]^{-1} [2w(0)\Delta H]^{3/2} \times \left[1 + \frac{9g(0)}{2\epsilon} \{[2w(0)\Delta H]^{-\epsilon/4} - 1\}\right]^{1/3} \tag{4.28}$$

$$\sim (\Delta H)^{3/2}, \quad g(0)[2w(0)\Delta H]^{-\epsilon/4} << \epsilon \tag{4.29a}$$

$$\sim (\Delta H)^{\theta_a - 1}, \quad g(0)[2w(0)\Delta H]^{-\epsilon/4} >> \epsilon \ , \tag{4.29b}$$

where $\theta_a = \frac{5}{2} - \epsilon/12$ is the animals exponent.[4] From Eq. (4.8b) we see that $g(0)$ depends on Ψ, and thus on r and H. If $g(0, \Psi_c)$ is of order unity, the Ψ dependence of g can be neglected and animals exponents are obtained for all $\Delta H < 1$. One can easily see from Eqs. (3.21)–(3.23) that this is the case for bond animals when considering the average number of percolating clusters with n_s sites or n_b bonds, if $p < p_c$, so that

$$B(n,p) \sim n^{-\theta_a} e^{-n\lambda_s} \ , \tag{4.30a}$$

$$A(n) \sim n^{-\theta_a} K^n \ , \tag{4.30b}$$

where λ_s and K are constants.

On the other hand, for cluster statistics near percolation, $g(0)$ is zero at the critical point and the Ψ dependence of $g(0)$ is important:

$$g(0) = w^2 T = w^3 \Psi \tag{4.31a}$$

$$\sim w^2[-r_0 + (r_0 + 2wH)^{1/2}] \ , \tag{4.31b}$$

and

$$f = -(3w^2)^{-1}(2w\Delta H)^{3/2} \left\{1 - \frac{9w^2 r_0}{2\epsilon}\left[1 - \left(\frac{2w\Delta H}{r_0^2}\right)^{1/2}\right][(2w\Delta H)^{-\epsilon/4} - 1]\right\}^{1/3} \ . \tag{4.32}$$

Using Eq. (2.14a) we then find that

$$B(n,p) \sim (3w^2\pi)^{-1} n^{-5/2} \exp(-r_0^2 n/2w)$$

$$\times \int_0^\infty dy\, e^{-ny} \mathrm{Im}\left\{ (-2wy)^{3/2} \left\{ 1 - \frac{9w^2 r_0}{2\epsilon} \left[1 - \left[\frac{-2wy}{r_0^2 n} \right]^{1/2} \right] \left[\left[\frac{-2wy}{n} \right]^{-\epsilon/4} - 1 \right]^{1/3} \right\} \right\} \tag{4.33}$$

$$\sim (3w^2\pi n^{5/2})^{-1} (2w^2)^{3/2} \exp\left[-\frac{nr_0^2}{2w} \right] \Gamma(5/2), \quad r_0 n^{\epsilon/4} << 1 \tag{4.34a}$$

$$\sim (3w^2\pi n^{5/2})^{-1} (2w^2)^{3/2} \exp\left[-\frac{nr_0^2}{2w} \right] \left[\frac{9w^2}{2\epsilon} (2w^2)^{-\epsilon/4} \right]^{1/3} r_0^{1/3} n^{\epsilon/12}, \quad r_0 n^{\epsilon/4} >> 1 \ . \tag{4.34b}$$

As expected, the result, Eq. (4.34a), for the percolation regime, $r_0 n^{\epsilon/4} << 1$, agrees with the mean-field result, Eq. (3.27), since the percolation problem is mean-field-like for $d = 8 - \epsilon > 6$.

V. CLUSTER STATISTICS IN GENERAL DIMENSION

From the mean-field theory and ϵ expansion of the previous two sections, we can surmise the scaling behavior of the cluster-generating function in general dimension. First we recall that there is a critical surface $H' = H_c'(K, \Delta K)$ on which $\Gamma^{(2)}$ vanishes. In almost all instances, we are interested in critical behavior as a particular point, which we denote $S_c = (H_c', K_c, \Delta K_c)$, on the critical surface is approached along a particular trajectory. At S_c, there is a critical value, Ψ_c, of Ψ. Motion away from S_c can be described in terms of the variables $\Delta H' = H' - H_c'$, $t \sim K - K_c$, and $\delta K_1 = \Delta K - \Delta K_c$, or in terms of $\Delta\Psi = \Psi - \Psi_c$, t, and δK_1. To simplify our discussion, we will consider only the calculation of the site distribution function $B(n,p)$. In this case $\Delta K_c = 0$, $\delta K_1 = 0$, $H' = H$, and the trajectory of interest is that described in Eq. (3.20) with $(p+q) = 1$. Note that t measures distance along the K axis from the point S_c, so that it is zero for the site trajectory under consideration.

Before proceeding, we make the following observations. First, if g is zero at S_c, there is no crossover to animals behavior and the critical behavior is that of percolation (even if S_c does not correspond to the percolation point). Therefore there is a line of critical points in the $(H', K, \Delta K)$ space passing through the percolation point with percolation exponents. Second, $\Gamma^{(1)}$ consists of a regular and a singular part both of which participate in determining $\Delta\Psi$. Finally, above six dimensions w^2 is zero at the percolation fixed point, so that the crossover exponent for g in the vicinity of percolation is $\phi_p = (8-d)\nu = \epsilon/2$, whereas below six dimensions w^2 is nonzero at percolation and the crossover exponent for g is simply that for T: $\phi_p = \beta_p$.

In the vicinity of S_c the correlation functions scale as

$$\Gamma_{\mathrm{sing}}^{(1)} = t^{\Delta_p} Y^{(1)}\left(\frac{\Delta\Psi}{t^{\beta_p}}, \frac{g}{t^{\phi_p}} \right) = \Delta H' \ , \tag{5.1a}$$

$$\Gamma_{\mathrm{sing}} = t^{2-\alpha_p} Y\left(\frac{\Delta\Psi}{t^{\beta_p}}, \frac{g}{t^{\phi_p}} \right) \ , \tag{5.1b}$$

and

$$f_{\mathrm{sing}} = \Gamma_{\mathrm{sing}} - (\Delta H')\Delta\Psi \ . \tag{5.2}$$

Since we are interested in a trajectory with $t = 0$ and $\Delta K = 0$, we write

$$\Gamma_{\mathrm{sing}}^{(1)} = (\Delta\Psi)^{\Delta_p/\beta_p} L^{(1)}\left(\frac{g}{(\Delta\Psi)^{\phi_p/\beta_p}} \right) = \Delta H \ , \tag{5.3a}$$

$$\Gamma_{\mathrm{sing}} = (\Delta\Psi)^{(2-\alpha_p)/\beta_p} L\left(\frac{g}{(\Delta\Psi)^{\phi_p/\beta_p}} \right) \ , \tag{5.3b}$$

which implies that

$$\Delta\Psi = (\Delta H)^{\beta_p/\Delta_p} X^{(1)}\left(\frac{g}{(\Delta H)^{\phi_p/\Delta_p}} \right) \ . \tag{5.4}$$

Of course, in general g is a function of $\Delta\Psi$, δK_1, and t. From Eq. (4.8b), we can write

$$g = g_c + c\Delta\Psi \tag{5.5}$$

for $t = \Delta K = 0$, where c is a constant and g_c is the value of g at the critical point. Using Eqs. (5.4) and (5.5) we find that

$$g = g_c Q\left(\frac{g_c}{(\Delta H)^{\phi_p/\Delta_p}}, (\Delta H)^{(\beta_p - \phi_p)/\Delta_p} \right) \ . \tag{5.6}$$

For $6 < d < 8$, $Q(x,y) \to 1$ as $x \to \infty$, $y \to 0$; for $d < 6$, $y = 1$, Q is a function only of x and $Q(x,1) \to 1$ as $x \to 0$. We can now combine Eqs. (5.6), (5.4), and (5.2) to arrive at

$$f_{\mathrm{sing}} = (\Delta H)^{1+(\beta_p/\Delta_p)} X\left(\frac{g_c}{(\Delta H)^{\phi_p/\Delta_p}}, \Delta H^{(\beta_p - \phi_p)/\Delta_p} \right) \ . \tag{5.7}$$

Thus, by Laplace transforming, we find that

$$B(n,p) = n^{-\tau_p} e^{-n|H_c(p)|} \times \mathrm{Im} \int_0^\infty dy\, e^{-y}(-y)^{1+(\beta_p/\Delta_p)} \times X\left(g_c\left(\frac{n}{-y}\right)^{\phi_p/\Delta_p}, \left(\frac{-y}{n}\right)^{\beta_p-\phi_p/\Delta_p}\right) . \tag{5.8}$$

We see immediately that when $n \gg 1$, we can replace $X(x,y)$ by $X(x)$, a function of x only. Thus we have

$$B(n,p) \sim n^{-\tau_p} e^{-n|H_c(p)|} R(n g_c^{\Delta_p/\phi_p}) . \tag{5.9}$$

When $g_c n^{\phi_p/\Delta_p} \gg 1$, we know that there is a crossover to animals behavior, so that

$$R(x) \to \begin{cases} x^{\tau_p-\theta_p}, & x \to \infty \quad (5.10a) \\ \text{const}, & x \to 0. \quad (5.10b) \end{cases}$$

Equation (5.9) gives the general scaling form for $B(n,p)$. When p is not near p_c, g_c is of order unity, so that animals exponents are obtained for all $n \gg 1$. Near p_c, however, both g_c and H_c go to zero as

$$H_c = -a r_0^{\Delta_p} , \tag{5.11a}$$

$$g_c \sim w^3 \Psi_c \sim r_0^{\beta_p} , \tag{5.11b}$$

where a is some constant and $r_0 \sim p - p_c$. Using Eq. (5.11) in Eq. (5.9) we find

$$B(n,p) \sim n^{-\tau_p} e^{-anr_0^{\Delta_p}} R(n r_0^{\beta_p \Delta_p/\phi_p}) . \tag{5.12}$$

This function agrees with Eqs. (1.6) and (1.9). Note that $r_0^{\beta_p \Delta_p/\phi_p} n$ controls crossover to animals behavior. Thus, for $6 < d < 8$, the naive scaling form of Eq. (1.2) does not apply. For $d < 6$, however, $\phi_p = \beta_p$ and Eq. (1.2) does apply.

VI. ESSENTIAL SINGULARITIES

Langer in his classic paper[15] on the droplet model showed that there is an essential singularity associated with passing from external field $H = 0^+$ to $H = 0^-$ for all temperatures $T < T_c$ and that for $H = 0^-$ there is an imaginary part to the partition function. The detailed mathematical analysis of this problem is quite complicated. One can, however, obtain the correct behavior of the leading singularity by reasonably simple arguments. When H is positive, the system is in stable equilibrium with spins pointing up. When H is allowed to become negative, the state with all spins down is stable. The state with the average magnetization up is, however, metastable. The metastable up-spin state can be studied by constraining the spins to be up at spatial infinity. Under this constraint, one expects there to be droplets of down spins in the sea of up spins. If H is near 0^-, the density of the down-spin droplets should be small and one can calculate properties of the metastable partition function by considering a noninteracting gas of the down-spin droplets. To do this, we need to calculate the free energy, f_D, of an average droplet. There is a bulk as well as a surface contribution to f_D:

$$f_D = f_B + f_s \tag{6.1}$$

$$= -R^d E + \sigma R^{d-1} , \tag{6.2}$$

where R is the radius of the droplet and E is the difference in energy between the uniform down phase and the uniform up phase, and σ is the surface tension. The metastable droplet is obtained by maximizing f_D with respect to R:

$$-dR^{d-1}E + \sigma(d-1)R^{d-2} = 0 , \tag{6.3}$$

so that $R \sim (\sigma/E)$. Thus we have

$$f_D \sim \sigma^d / E^{d-1} . \tag{6.4}$$

Here σ is a condensation energy times a correlation length and E is proportional to H. The droplet, of course, only exists for $H < 0$. Thus we have

$$\mathrm{Im} Z \sim e^{-\sigma^d/E^{d-1}} \sim e^{-1/|H|^{d-1}} . \tag{6.5}$$

There is an imaginary part to Z because the constraint that the spin be up at infinity requires deformation of contours to ensure convergence.

We now wish to use this general droplet picture to calculate the p dependence of the essential singularity predicted by Kunz and Souillard.[12] For simplicity we restrict our attention to the case $K_1 - K = 0$. We proceed exactly as outlined above. First, we demonstrate within mean-field theory the existence of local equilibrium states with two different symmetries in the presence of a field. The first, which we will call the "up" state, is the one discussed extensively in Sec. III. It is singly degenerate. The second, which we will call the "down" state, has a degeneracy of $(s-1)$. It is this degeneracy which ensures that the partition function for the Potts model tends to unity as $s \to 1$. For $H > 0$ the up phase is stable and down-spin regions are unstable. For $H < 0$, small finite regions of down spins are unstable because of their positive surface energy, whereas large regions of down spins grow uncontrollably due to their negative volume energy. It is clear that for $H < 0$ regions of a critical radius $R_c(H)$ are metastable and in our picture it is this metastability that leads to the essential singularity discussed by Kunz and Souillard.[12]

When H' and $K_1 - K$ are zero, spontaneous ordering can occur into any of s different equivalent states

when $p > p_c$. These states have order parameters

$$\langle \psi_l \rangle = e_l^k \Psi_k, \quad k = 1, 2, \ldots, s \ . \tag{6.6}$$

When H' is positive, the state with $k = 1$ is preferred and stable. When H' becomes negative, states along other directions become stable. In view of the symmetry of the Potts model, we do not expect $\langle \psi_l \rangle$ to be along a single direction e_l^k. Instead, we seek solutions of the form

$$\langle \psi_l \rangle = \Psi_1 e_l^1 + \Psi_2 e_l^k, \quad k = 1, 2, \ldots, s \ , \tag{6.7}$$

so that there are $(s-1)$ solutions corresponding to the $(s-1)$ possible choices for k. The mean-field free energy is easily calculated from Eq. (3.6). We find that

$$f = -zK\Psi_1\Psi_2 + \Psi_2 H - \ln(e^{-zK\Psi_1} + e^{-zK\Psi_2} - e^{-zK(\Psi_1+\Psi_2)}) + (s-1)\left[\tfrac{1}{2}zK(\Psi_1^2+\Psi_2^2) - H\Psi_1 - \frac{zK(\Psi_1 e^{-zK\Psi_2} + \Psi_2 e^{-zK\Psi_1} + e^{-zK(\Psi_1+\Psi_2)})}{e^{-zK\Psi_1} + e^{-zK\Psi_2} - e^{-zK(\Psi_1+\Psi_2)}}\right] . \tag{6.8}$$

To minimize f we see that Ψ_1 and Ψ_2 satisfy the equations of state

$$\Psi_1 = H + \frac{e^{-zK\Psi_2}(1 - e^{-zK\Psi_1})}{e^{-zK\Psi_1} + e^{-zK\Psi_2} - e^{-zK(\Psi_1+\Psi_2)}} + O(s-1) \ , \tag{6.9}$$

$$\Psi_2 = \frac{e^{-zK\Psi_1}(1 - e^{-zK\Psi_2})}{e^{-zK\Psi_1} + e^{-zK\Psi_2} - e^{-zK(\Psi_1+\Psi_2)}} + O(s-1) \ . \tag{6.10}$$

A general solution to these equations is complicated. However, we can investigate their solution for small H. For $H = 0$, of course, there are the solutions $\Psi_1 = \Psi_0$, $\Psi_2 = 0$ and $\Psi_1 = 0$, $\Psi_2 = \Psi_0$, where Ψ_0 is given by Eq. (3.12) with $H = 0$. When $H \neq 0$, there is the up solution with $\Psi_2 = 0$ and a down solution with $\Psi_2 \sim \Psi_0$ and $\Psi_1 \sim H$. We therefore write

$$\Psi_1 = \alpha H \ , \tag{6.11a}$$

$$\Psi_2 = \Psi_0 + \beta H \ , \tag{6.11b}$$

and find

$$\alpha = zK(1 - zKe^{-zK\Psi_0})^{-1} \ , \tag{6.12a}$$

$$\beta = \Psi_0 e^{-zK\Psi_0}(1 - zKe^{-zK\Psi_0})^{-2} \ . \tag{6.12b}$$

Finally, we calculate the free energies $f_{\rm up}$ of the up state and $f_{\rm down}$ of the down state described by Eq. (6.11) as

$$f_{\rm up} = O(s-1) \ , \tag{6.13a}$$

$$f_{\rm down} = \Psi_0 H \ . \tag{6.13b}$$

Thus we see that $f_{\rm down} < f_{\rm up}$ for $H < 0$ and $\Psi_0 > 0$. This region is indicated in Fig. 2. When $K_1 - K \neq 0$, there will be a surface in the $(K_1 - K, K, H')$ plane where the uniform solution along H becomes unstable.

The free-energy difference between the up and down phases is thus given by

$$E = \Psi_0 H \tag{6.14}$$

and the surface tension by

$$\sigma = f_0 \xi \ , \tag{6.15}$$

where f_0 is the free energy of Eq. (6.8) evaluated at $K_1 - K = H = 0$ and ξ is a correlation length. We are particularly interested in properties near percolation where

$$f_0 \sim r_0^3 \ , \tag{6.16}$$

$$\xi \sim r_0^{-1/2} \ , \tag{6.17}$$

$$\Psi_0 \sim |r_0| \ . \tag{6.18}$$

Substituting the evaluations of Eqs. (6.14) through (6.18) into Eq. (6.4) and remembering that there are $(s-1)$ equivalent down solutions, we obtain

$$\mathrm{Im}Z \sim N(s-1)\exp(-a|r_0|^{3d/2+1}/|H|^{d-1}) \ , \tag{6.19}$$

where a is a constant. We included a factor N in Eq. (6.19) because there are N sites on which to center the metastable solution. The above form is valid for $d > 6$, where mean-field theory gives a correct description of percolation. The results below six dimensions can be obtained by integrating the renormalization-group recursion relations until $r(1^*) \sim 1$ and matching to Eq. (6.19). The result[19] is that predicted by scaling

$$\mathrm{Im}Z \sim N(s-1)\exp\left[-a'\left(\frac{(p-p_c)^{\Delta_p}}{|H|}\right)^{d-1}\right] \tag{6.20}$$

and leads to

$$B(n,p) \sim n^{-\tau'} \exp[-c(n(p-p_c)^{\Delta_p})^{1-1/d}] \ , \qquad (6.21)$$

where c is a constant and τ' is another critical exponent. The calculation of τ' requires a more careful treatment of the metastability problem than presented here. It will be given in a separate publication.

We close this section by noting that the partition function in the metastable region takes the form

$$Z = e^{-N(s-1)f_1} + N(s-1)c'e^{-f_D} \ , \qquad (6.22)$$

where c' is a complex constant. This form satisfies the requirement that Z tend to unity as $s \to 1$, and gives a free energy of the form

$$f = f_1 - c'e^{-f_D} \ . \qquad (6.23)$$

This requirement would not be met if the system were allowed to condense into a single one of the down solutions for negative H.

VII. SUMMARY

In this paper, we have presented a detailed analysis of the statistics of clusters on a lattice using mean-field theory, the ϵ expansion, scaling arguments, and a generalization of the droplet model. Clusters are defined as groups of adjacent sites connected by occupied bonds. Perimeter bonds are unoccupied bonds connected to sites in a cluster (cf., Fig. 1). Lattices are characterized by the number of clusters per site, $A(n_b,n_p,n_s)$ containing n_b bonds, n_p perimeter bonds, and n_s sites. Generating functions for $A(n_b,n_p,n_s)$ can be obtained by assigning weight factors p, q, and e^{-H} to bonds, perimeter bonds, and sites. Familiar statistical problems correspond to special cases of these weighting factors. For example the probability $P(n_s,p)$ of having a percolating clusters with n sites is obtained by setting $p < 1$, $q = 1-p$, $H = 0$, and summing over n_b and n_p. Similarly, the number of animals, $A(n_b)$, with n_b bonds is obtained by setting $p = q = 1$, $H = 0$ and summing over n_s and n_p.

We have shown how one can use the free energy, f, of the one-state Potts model in a field and in the presence of a uniaxial anisotropy field to obtain the generalized generating function for $A(n_b,n_p,n_s)$. We find that two types of critical points appear in f. One describes the statistics of animals and has critical corrections to mean-field theory below eight dimensions; the other describes percolation and has corrections to mean-field theory below six dimensions. The percolation critical point is a multicritical point that is unstable to the animals critical point. The animals critical behavior occurs only at negative H. Thus the critical properties of any function such as moments of $P(n,p)$ evaluated at $H = 0$ will be controlled by the percolation critical point. Most functions, however, involving negative H will exhibit crossover from percolation to animals critical behavior. Thus $P(n,p)$ which is obtained from f by a Laplace transform with respect to H is controlled by the percolation critical point for n less than a crossover value, n^*, and by the animals critical point for n larger than n^*. The value of n^* depends on dimensionality, d. For $6 < d < 8$, $n^* \sim (p-p_c)^{-4/(8-d)}$, whereas for $d < 6$, $n^* \sim (p-p_c)^{-\Delta_p}$, where Δ_p is the gap exponent, $\beta_p + \gamma_p$, for percolation.

Cluster functions such as $P(n,p)$ are known to exhibit essential singularities at large n of the form $\exp(-\text{const}\, n^{1-1/d})$ when an infinite cluster exists. We have shown that these singularities emerge from metastable states of the Potts model in a negative field analagous to those studied by Langer for the Ising model. Using scaling arguments, we determine the dependence of the coefficient of $n^{1-1/d}$ on $p - p_c$.

Note added in proof. Since this paper was submitted, several other papers relevant to this paper have appeared. The lattice animals problem has been related to the Yang-Lee edge singularity in a random imaginary magnetic field [G. Parisi and N. Sourlas, Phys. Rev. Lett. 46, 871 (1981)]. A Flory-like approximation for ν_{animals} is remarkably good for all dimensions between 2 and 8 [Joel Isaacson and T. C. Lubensky, J. Phys. (Paris) 41, L469 (1980)]. Using the Flory approximation and the results of Parisi and Sourlas, one can obtain an excellent approximation for θ_a for $2 < d < 8$ [T. C. Lubensky and A. J. McKane, J. Phys. (Paris) Lett. (in press)]. The exponent θ' has been calculated [T. C. Lubensky and A. J. McKane, J. Phys. A 14, L157 (1981)]. We should also mention a very complete computer study of $B(n,p)$ by H. Nakanishi and H. E. Stanley [Phys. Rev. B 22, 2466 (1980); J. Phys. A 14, No. 3 (1980)]. This work does not, however, address the question of crossover from percolation to animals behavior.

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[1]C. Domb, J. Phys. A 9, L141 (1976).
[2]D. S. Gaunt, M. F. Sykes, and H. Ruskin, J. Phys. A 9, 1899 (1976).
[3]D. S. Gaunt and H. Ruskin, J. Phys. A 11, 1369 (1978); D. S. Gaunt, *ibid.* 13, L97 (1980).
[4]T. C. Lubensky and J. Isaacson, Phys. Rev. Lett. 41, 829 (1978); 42, 410(E) (1978); Phys. Rev. A 20, 2130 (1979).
[5]D. A. Klarner, Can. J. Math. 19, 851 (1967).
[6]M. E. Fisher and J. W. Essam, J. Math. Phys. 2, 609 (1961).
[7]J. C. Walker and A. B. Harris (unpublished).
[8]D. Stauffer, J. Stat. Phys. 18, 125 (1978); Phys. Rep. 54, 1 (1979).
[9]P. L. Leath and G. R. Reich, J. Phys. C 11, 4017 (1978).
[10]A. B. Harris, T. C. Lubensky, W. K. Holcomb, and C. Dasgupta, Phys. Rev. Lett. 35, 327 (1975); 35, 1397(E) (1975).
[11]R. G. Priest and T. C. Lubensky, Phys. Rev. B 13, 4159 (1976).
[12]H. Kunz and B. Souillard, J. Stat. Phys. 19, 77 (1978).
[13]M. J. Stephen, Phys. Rev. B 15, 5674 (1977).
[14]M. Giri, M. J. Stephen, and G. S. Grest, Phys. Rev. B 16, 4971 (1977).
[15]J. S. Langer, Ann. Phys. (N.Y.) 41, 108 (1967).
[16]P. W. Kasteleyn and C. M. Fortuin, J. Phys. Soc. Jpn. Suppl. 26, 11 (1969); C. M. Fortuin and P. W. Kasteleyn, Physica (Utrecht) 57, 536 (1972).
[17]R. K. P. Zia and D. J. Wallace, J. Phys. A 8, 1495 (1975).
[18]The three-point vertex $\Gamma^{(3)}$ is defined by $\Gamma^{(3)} = \partial^3\Gamma/\partial\Psi^3$. It is defined precisely in Ref. 6.
[19]A. Houghton and T. C. Lubensky Phys. Lett. 77A, 479 (1980).