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Flux bunching in type-II superconductors

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We show that a type-II superconductor at low flux densities ($B \ll H_{c1}$) is globally unstable to formation of a two-phase state with some regions containing flux, and others containing no flux. The basic source of the instability is the dependence of the vortex energy on the matter density, which leads to an effective attractive interaction between vortices. The instability of a uniform array of vortices discovered by P. H. Roberts is shown to signal the tendency of the system to separate into these two phases. The phase separation is expected to occur only at very low flux densities in laboratory superconductors; however, we estimate that the proton superconductor in neutron stars may be in the two-phase state.

I. INTRODUCTION

In a neutron star the protons are likely to be superconducting, and calculations suggest they will be a type-II superconductor.[1] Such a superconductor displays a Meissner effect only for fields less than the lower critical field H_{c1}, and at higher fields flux enters the superconductor in the form of quantized flux lines. At the upper critical field, H_{c2}, the system becomes normal. For a review of properties of type-II superconductors we refer to Ref. 2.

Jones[3] argued that under the conditions expected in neutron stars the magnetic stresses associated with a type-II superconductor would be much larger than those in a normal metal. Subsequently Easson and Pethick[4] showed that this was indeed the case, the magnetic stresses being of order $HB/4\pi$, where B is the magnetic induction. When $B \ll H_{c1}$, which is the case in neutron stars, H is approximately equal to H_{c1}. Hence the magnetic stress is of order $H_{c1}B/4\pi$, which is much larger than the stress in a normal metal, which is of order $B^2/8\pi$.

In a neutron star the protons form a fluid, and so have a vanishing shear modulus. Recently Roberts[5] has investigated the hydrodynamic modes of such a fluid type-II superconductor, and found that the state with a uniform density of vortices is unstable for small values of B. This instability originates in the fact that H depends not only on B, but on the matter density as well.

We shall investigate the properties of these unstable modes and will show that in them a variation in magnetic induction is correlated with a variation in density. If H_{c1} increases with density, variations of the magnetic induction and density have opposite signs, while if H_{c1} decreases with density they have the same sign. Since for $B \ll H_{c1}$ the magnetic energy density is given by $H_{c1}B/4\pi$, it is energetically favorable for flux to be present in regions of lower H_{c1}. This suggests, and we indeed demonstrate, that it may be possible for the energy of the superconductor to be lowered by separating into two phases. In this two-phase configuration, one phase contains magnetic flux and the other is flux free. In addition the two phases differ in density: if H_{c1} increases with density, the lower-density region will contain the flux, while if H_{c1} decreases with density the higher-density region will.

Another way of describing this result is to note that the coupling of vortex lines to density fluctuations leads to an attractive induced interaction between the vortex lines. This induced attraction is analogous to that between ^{3}He atoms in ^{3}He-^{4}He mixtures,[6] where the attraction comes from coupling of ^{3}He atoms to the ^{4}He density fluctuations. At higher values of B the direct interactions between vortex lines, which are primarily repulsive, suppress the instability. The coupling between vortices and the particle density results in the transition at H_{c1}, when flux first enters the superconductor, being a first-order transition. We estimate that conditions in a neutron star are such that the proton type-II superconductor could well be in the two-phase configuration.

The analysis we present indicating the energetic favorability of the two-phase equilibrium state at low B applies equally well to the proton superconductor and to laboratory type-II superconductors. A state in which a superconductor has regions of different magnetic induction is already well known in laboratory type-II superconductors, where it is usually referred to as the intermediate mixed state.[7] However, the mechanism believed to be responsible for this state is different from the one we consider; the intermediate mixed state has been observed only in superconductors with Ginzburg-Landau parameters close to $1/\sqrt{2}$, and is thought to be brought about by particular properties of the direct interaction between vortex lines. The mechanism we consider in this paper leads to the occurrence of the intermediate mixed state even for extreme type-II superconductors.

The intermediate mixed state is similar in some respects to the intermediate state in type-I superconductors. In both cases the superconductor splits into flux-carrying and flux-free regions. However, in the type-I superconductor the flux-carrying regions are normal, while in the type-II case the flux-carrying regions are in the mixed state.

Although the equilibrium analysis of the two-phase configuration applies equally well to the proton superconductor and solid laboratory type-II superconductors, there are a number of differences between the two cases when one considers modes of oscillation. In Roberts's calculations it was assumed that the magnetic flux lines were frozen to the matter. A feature of the fluid superconductor which was crucial to finding unstable modes was the absence of any shear modulus for the fluid (in the absence of magnetic flux). In solid laboratory superconductors the nonzero shear modulus will suppress the Roberts instability, since it is impossible to construct motions of the matter that cause bunching of flux without leading to increases in the elastic energy which will completely dominate any reduction in the magnetic energy. In the fluid superconductor motions in which the flux is tied to the matter can reduce the energy, while in a solid superconductor they cannot, and so in the solid, motion of the vortex lines relative to the matter is necessary to reach lower energy configurations.

The plan of the paper is as follows. In Sec. II we discuss the thermodynamics of the two-phase equilibrium state. This discussion applies both to fluid and solid type-II superconductors. In Sec. III we discuss the instability found by Roberts in fluid type-II superconductors; in particular we point out how the density and field fluctuations in the unstable modes are correlated. Section IV is a brief discussion.

II. TWO-PHASE EQUILIBRIUM

In this section we discuss the equilibrium state, and show that at low flux densities the two-phase state is energetically favorable compared to the uniform state. Consider a volume $V=AL$, having a length L in the $\hat{z}$ direction and cross section A in the xy plane. It is convenient to specify densities in terms of conserved variables. In the case of laboratory superconductors we shall work in terms of the electron number, and will denote the number of electrons in volume V by N. For the case of the proton superconductor in neutron stars, β-decay processes, which change the number of protons, can take place and we shall therefore specify the density in terms of the number of baryons in the volume, which we shall denote by N for that case. We divide the volume V into two regions $V_1=A_1L$ and $V_2=V-V_1=A_2L$. Volume V_1 contains N_1 particles and magnetic induction $B_1\hat{z}$, while volume V_2 contains $N_2=N-N_1$ particles and no induction. Let N_ϕ be the number of flux quanta in V_1, so that $N_\phi\phi_0=B_1A_1$, where $\phi_0=hc/2e$ is the flux quantum. We denote the energy of N_i particles in volume V_i in the absence of magnetic flux by $U(N_i,V_i)$, and we express the energy due to flux lines as $N_\phi LE(n,n_\phi)$, where $E(n,n_\phi)$ is the energy per unit length of a flux line. The densities n_i and n_ϕ are defined by $n_i=N_i/V_i$ and $n_\phi=N_\phi/A_1$. It is convenient to work in terms of the density of flux lines per unit area, n_ϕ, rather than the magnetic induction itself, since this brings out the analogy between this problem and that of phase separation in binary mixtures. The total energy W is then given by

$$W=U(N_1,V_1)+U(N_2,V_2)+N_\phi LE(n_1,n_\phi)\ . \tag{1}$$

(We work with the energy rather than the free energy since we shall confine our discussion to the case of zero temperature.)

The conditions for equilibrium are, first, that W be unaltered when a particle is moved from region 1 to region 2

$$\left(\frac{\partial W}{\partial N_1}\right)_{N,V_1,V_2}=0\ , \tag{2}$$

and, second, that W be unaltered if the volume V_1 is slightly changed

$$\left(\frac{\partial W}{\partial V_1}\right)_{N_1,N_2,V}=0\ . \tag{3}$$

Conditions (2) and (3) express the fact that the particle chemical potentials and the pressures in the two phases must be equal.

We define the chemical potential and the pressure in the absence of the field by

$$\mu_0(n)=\left(\frac{\partial U}{\partial N}\right)_V \tag{4}$$

and

$$p_0(n)=-\left(\frac{\partial U}{\partial V}\right)_N\ . \tag{5}$$

Note that p_0 and μ_0 depend only on n. The equilibrium conditions (2) and (3) then become

$$\mu_0(n_2)=\mu_0(n_1)+n_\phi\frac{\partial E}{\partial n_1} \tag{6}$$

and

$$p_0(n_2)=p_0(n_1)+n_\phi n_1\frac{\partial E}{\partial n_1}+n_\phi^2\frac{\partial E}{\partial n_\phi}\ . \tag{7}$$

The last term in Eq. (6) represents the magnetic contribution to the chemical potential, and the last two terms in Eq. (7) are the magnetic contributions to the pressure. From these equations one can find the phase-separation line $n_\phi(n)$. When the system consists of two phases, the average flux density per unit area must be less than that in the phase containing flux, $n_\phi(n)$. Thus one can see that for a given $n=N/V$, if $N_\phi/A > n_\phi(n)$, the superconductor consists of a single phase, while if $N_\phi/A < n_\phi(n)$ it consists of two phases.

Since we are mainly interested in low values of B ($B << H_{c1}$) it is convenient to write $E(n_1,n_\phi)$ as

$$E(n_1,n_\phi)=\frac{1}{4\pi}H_{c1}(n_1)\phi_0+E_i(n_1,n_\phi) \quad . \tag{8}$$

The first term is the energy of an isolated vortex line, and E_i, the second term, represents the interaction between vortex lines, which vanishes as $n_\phi \rightarrow 0$. Equations (6) and (7) become

$$\mu_0(n_2)=\mu_0(n_1)+\frac{\phi_0}{4\pi}n_\phi\frac{\partial H_{c1}}{\partial n_1}(n_1)+n_\phi\frac{\partial E_i}{\partial n_1} \tag{9}$$

and

$$p_0(n_2)=p_0(n_1)+\frac{\phi_0}{4\pi}n_1 n_\phi\frac{\partial}{\partial n_1}H_{c1}(n_1) + n_1 n_\phi\frac{\partial E_i}{\partial n_1}+n_\phi^2\frac{\partial E_i}{\partial n_\phi} \quad . \tag{10}$$

We can combine Eqs. (9) and (10) to get

$$\mu_0(n_2)-\mu_0(n_1)=\frac{p_0(n_2)-p_0(n_1)}{n_1} - \frac{n_\phi^2}{n_1}\frac{\partial E_i}{\partial n_\phi} \quad . \tag{11}$$

Since $\partial\mu_0/\partial p_0=1/n$ and $\partial n/\partial p_0>0$ if the matter is stable, we see that if $E_i=0$ there is no solution to Eq. (11) except $n_1=n_2$. If we then try to use $n_1=n_2$ as a solution in Eqs. (9) or (10) with $E_i=0$, we see this requires either $n_\phi=0$ or $\partial H_{c1}/\partial n=0$. Thus, the direct interactions among the vortices represented by E_i are necessary to give a sensible equilibrium state; this makes physical sense, since if $E_i=0$ there is nothing to prevent all the vortices from collapsing into an infinitely small area.

To derive the phase-separation line we eliminate n_2 from Eqs. (9) and (10), getting an equation relating n_1 and n_ϕ. To do this we expand quantities in powers of (n_2-n_1), which as we shall see, is a good approximation in all cases of interest. Expanding $\mu_0(n_2)-\mu_0(n_1)$ and $p_0(n_2)-p_0(n_1)$ to second order in (n_2-n_1), and substituting these expressions into Eq. (11) yields

$$(n_2-n_1)^2=2n_\phi^2 n_1\frac{\partial n_1}{\partial p_{01}}\frac{\partial E_i}{\partial n_\phi} \quad . \tag{12}$$

Next we expand $\mu_0(n_2)-\mu_0(n_1)$ in Eq. (9) to first order in (n_2-n_1), square the result, and substitute from Eq. (12) to get

$$2\frac{\partial\mu_0}{\partial n_1}\frac{\partial E_i}{\partial n_\phi}=\left[\frac{\phi_0}{4\pi}\frac{\partial H_{c1}}{\partial n_1}+\frac{\partial E_i}{\partial n_1}\right]^2 \quad . \tag{13}$$

Equation (13) relates the density and flux in region 1, and is the desired phase-separation line.

We can make an estimate of the density of flux lines at phase separation by using results for extreme type-II superconductors at low flux densities. For this case[4]

$$H_{c1}=\frac{\phi_0}{4\pi\lambda^2}\ln\left[\frac{\lambda}{\xi}\right], \quad \lambda >> \xi \quad , \tag{14}$$

where λ is the penetration depth and ξ is the coherence length. To estimate the density dependence of H_{c1} we need the density dependence of λ and ξ. For λ we take the London expression

$$\lambda^2=\frac{mc^2}{4\pi n_c e^2} \quad , \tag{15}$$

where n_c is the number density of the superconducting charged carriers (the protons in a neutron star, the conduction electrons in a laboratory superconductor) and m is the mass of a carrier.

In a neutron star β processes can convert protons into neutrons, and so the composition of the matter will depend on its density. However, inspection of detailed calculations (see, e.g., Ref. 8) shows that it is a reasonable approximation to take

$$\frac{\partial n_c}{\partial n}\approx\frac{n_c}{n} \quad . \tag{16}$$

(Recall that for neutron stars we are taking n to be the baryon density.) Then, if we neglect variations of the logarithmic terms in Eq. (14) we find

$$\frac{\partial H_{c1}}{\partial n}\approx\frac{H_{c1}}{n} \quad . \tag{17}$$

For a triangular lattice in the extreme type-II limit the interaction energy for $B << H_{c1}$ is given by[2]

$$E_i=\frac{3\phi_0^2}{8\pi^2\lambda^2}\left[\frac{\pi}{2}\frac{\lambda}{r_0}\right]^{1/2}e^{-r_0/\lambda} \quad , \tag{18}$$

where $r_0=(\frac{4}{3})^{1/4}n_\phi^{-1/2}$ is the distance between flux lines. For $r_0 >> \lambda$ the most rapid variation of E_i with r_0 comes from the exponential term, and we therefore make the approximation

$$\frac{\partial E_i}{\partial n_\phi}\approx\frac{1}{2}\frac{r_0}{\lambda}\frac{E_i}{n_\phi} \quad . \tag{19}$$

Substituting Eqs. (17), (18), and (19) into Eq. (13), and neglecting $\partial E_i/\partial n$ as compared with $(\phi_0/4\pi) \times (\partial H_{c1}/\partial n)$ on the right-hand side, we find the following equation for the equilibrium spacing of the vortex lattice

$$3(\tfrac{3}{2}\pi)^{1/2}\left(\frac{r_0}{\lambda}\right)^{5/2} e^{-r_0/\lambda} = H_{c1}^2 \Big/ \frac{n\,\partial p_0}{\partial n} \ . \tag{20}$$

Let us now make some estimates. For the superconducting protons in a neutron star we take $H_{c1} \approx 10^{15}$ G (Ref. 4) and $n\,\partial p_0/\partial n \approx 10^{33}$ erg/cm^3 . This gives $r_0/\lambda \approx 15$, or $B \approx 10^{15}/15^2 \approx 4\times 10^{12}$ G. The average B expected in a neutron star is of the order of 10^{12} G, so the star could well be in the two-phase state.

For a laboratory type-II superconductor we take $H_{c1} \approx 100$ G, and $n\,\partial p_0/\partial n \approx 10^{11}$ erg/cm^3. This gives $r_0/\lambda \approx 26$ or a B value at phase separation of $B \approx 100/26^2 \approx 0.1$ G. If a metal with a large value of $\partial H_{c1}/\partial n$ could be found, the effect would be enhanced.

We now examine the two-phase state in a slightly different way. This study will make clear that at low enough flux densities the two-phase state does indeed have a lower energy than the one-phase state. We define ΔW to be the energy of the two-phase configuration minus the energy of the one-phase configuration. The strategy is to minimize ΔW.

The one-phase state has N particles and N_ϕ flux quanta uniformly distributed in a volume V. The two-phase state is the same as that considered at the beginning of this section: N_1 particles and the N_ϕ flux quanta in volume $V_1 = A_1 L$ and N_2 particles and no flux quanta in $V_2 = A_2 L = V - V_1$. We define $n = N/V$, $n_1 = N_1/V_1$, and $n_2 = N_2/V_2$. So far, V_1 is arbitrary, and we will select it to minimize ΔW.

Consider expanding ΔW in powers of $(n_1 - n_2)$, which we expect to be small in the optimum configuration:

$$\Delta W = \alpha_0 + \alpha_1(n_1 - n_2) + \alpha_2(n_1-n_2)^2 \ . \tag{21}$$

The coefficients $\alpha_0, \alpha_1, \alpha_2$ depend on V_1, and have the following significance:

(1) α_0 is positive and takes into account the increase in magnetic energy caused by bunching the flux lines into V_1. It is given by

$$\alpha_0 = N_\phi L\left[E_i\left(n, \frac{N_\phi}{A_1}\right) - E_i\left(n, \frac{N_\phi}{A}\right)\right] \ .$$

(2) α_1 can have either sign. It allows for the density dependence of the magnetic energy. It is given by

$$\alpha_1 = N_\phi \frac{V_2}{V} L\left[\frac{\phi_0}{4\pi}\frac{\partial H_{c1}}{\partial n} + \frac{\partial E_i}{\partial n}\left(n, \frac{N_\phi}{A_1}\right)\right] \ .$$

(3) α_2 is positive, and is mainly due to the non-magnetic compressional energy. We thus ignore the magnetic contribution to this coefficient, and take

$$\alpha_2 = \frac{1}{2n}\frac{V_2 V_1}{V}\frac{\partial p_0}{\partial n} \ .$$

By differentiating Eq. (21) we find the optimum value of $n_1 - n_2$ to be

$$n_1 - n_2 = \frac{-\alpha_1}{2\alpha_2} \ . \tag{22}$$

Substituting this back, we find

$$\Delta W = \alpha_0 - \frac{\alpha_1^2}{4\alpha_2} \ . \tag{23}$$

The key point to notice is that α_0 decreases rapidly as N_ϕ becomes small, because of the exponential fall of the vortex-vortex interaction. Hence, at low enough values of N_ϕ the quantity α_0 will become small enough so that the ΔW given by Eq. (23) is negative.

If we use in Eq. (23) the expressions for α_0, α_1, and α_2 given above, and minimize ΔW with respect to V_1, we find an equation like Eq. (13), except that all the derivatives are evaluated at a particle density of n, rather than n_1. However to the order to which we are working this difference is negligible.

III. SMALL OSCILLATIONS

In this section we review the modes that Roberts[5] found to be unstable, and show how the changes in flux density are associated with changes in particle density. Consider a fluid type-II superconductor. We denote the energy per particle, as a function of the number density n and the magnetic induction $\vec{B}$ by $\epsilon(n, \vec{B})$. As in Sec. II, n represents the baryon density for the case of the proton fluid superconductor in a neutron star. The pressure of the matter is given by

$$p = n^2\left(\frac{\partial \epsilon}{\partial n}\right)_B \ ; \tag{24}$$

note that p has magnetic contributions, and therefore differs from the p_0 used in the last section. The magnetic field is given by

$$\vec{H} = 4\pi n \frac{\partial \epsilon}{\partial \vec{B}} \tag{25}$$

and the total energy is given by

$$W = \int \epsilon n \, d^3 r \ . \tag{26}$$

We want to check the stability of the uniform state $n = n_I$, $\vec{B} = \vec{B}_I = B_I \hat{z}$, where n_I and B_I are the initial equilibrium values. To do this we imagine that the fluid elements are given a slowly varying displacement $\vec{u}(\vec{r})$, where $\vec{r}$ denotes the original position of the element. This displacement can cause a density change, so that the density of the element initially at $\vec{r}$ becomes

$$n=\frac{n_I}{|\det D|} \ , \tag{27}$$

where $D_{ij}\equiv\delta_{ij}+\partial u_i/\partial r_j$. Since the fluid is perfectly conducting, and therefore flux lines are frozen to the matter, we can relate the new magnetic induction to the initial value by[5]

$$B_i=\frac{D_{ij}B_I\hat{z}_j}{|\det D|} \ . \tag{28}$$

This equation implies that the flux through any fluid element is conserved.

We now expand W to second order in $\partial u_i/\partial r_j$. Since we are interested in the intrinsic stability of the fluid, we take the fluid to be in a large box of volume V and impose periodic boundary conditions on $\vec{u}(\vec{r})$. This means that all surface terms vanish; in particular the first-order piece in the expansion of W is zero, and the second-order piece is

$$\delta^2 W=\frac{V}{2}\sum_{\vec{k}}A_{ij}(\vec{k})\,u_i(\vec{k})\,u_j^*(\vec{k}) \ , \tag{29}$$

where we have introduced the Fourier transform of the displacement

$$\vec{u}(\vec{r})=\sum_{\vec{k}}\vec{u}(\vec{k})e^{i\vec{k}\cdot\vec{r}} \ , \tag{30}$$

and the symmetric matrix A_{ij} is given by

$$\begin{aligned}A_{ij}=&(a^2+2f+s^2)k_ik_j\\&-(f+s^2)k_z(k_i\hat{z}_j+\hat{z}_ik_j)\\&+b^2k_z^2(\delta_{ij}-\hat{z}_i\hat{z}_j)+s^2k_z^2\hat{z}_i\hat{z}_j \ ,\end{aligned} \tag{31}$$

where

$$a^2\equiv n_I\left(\frac{\partial p}{\partial n}\right)_I, \quad b^2\equiv\frac{H_IB_I}{4\pi} \ ,$$

$$f\equiv\frac{n_IB_I}{4\pi}\left(\frac{\partial H}{\partial n}\right)_I \quad\text{and}\quad s^2\equiv\frac{B_I^2}{4\pi}\left(\frac{\partial H}{\partial B}\right)_I \ .$$

One can check that A has no negative energy eigenvalue only if

$$\frac{\partial H}{\partial B}\geq\frac{1}{4\pi}\left(\frac{\partial H}{\partial n}\right)^2\Big/\left(\frac{1}{n}\frac{\partial p}{\partial n}\right) \ . \tag{32}$$

The result (32) has a simple physical interpretation, as may be seen if we introduce the chemical potential per unit length of flux line, $\mu_\phi=\phi_0H/4\pi$. Equation (32) then becomes

$$\frac{\partial\mu_\phi}{\partial n_\phi}\frac{\partial\mu}{\partial n}\geq\left(\frac{\partial\mu_\phi}{\partial n}\right)^2 \ . \tag{33}$$

This is just the condition that a binary mixture of two components (in this case magnetic flux and matter) be stable with respect to density fluctuations. It is interesting that this is precisely the same criterion as one would obtain from thermodynamic considerations without invoking the flux-freezing condition, and not allowing for bending of the field lines. We therefore see that relaxation of the flux freezing constraint employed in our calculations above does not make lower-energy configurations accessible to the system. In addition, the fact that the calculations in this section allow for bending of field lines [which is not allowed for in the simple thermodynamic derivation of Eq. (33)], and lead to the same stability condition as when bending is neglected, shows that bending of field lines does not play an essential role in bringing about the instability.

Now, in a type-II superconductor with a uniform array of vortices $\partial H/\partial B=0$ when flux first enters, and then gradually increases as B increases. So unless H is independent of n, for small enough B the uniform state is always unstable. It is energetically favorable for the fluid to separate into two phases of differing densities and fluxes, within the constraints imposed by flux conversation, Eq. (28). To see this explicitly, we will consider in detail the case of low flux densities for the rest of this section. We will take B to be small enough so that it is a good approximation to set $H(n,B)=H_{c1}(n)$, and so $\partial H/\partial B=0$. The rotational symmetry about the $\hat{z}$ axis implies that without loss of generality we can take

$$\vec{k}=k(\cos\theta\hat{z}+\sin\theta\hat{x}) \ , \tag{34}$$

with

$$\sin\theta,\ \cos\theta>0 \ .$$

Before considering the case of a general $H_{c1}(n)$ we consider the more specialized situation of $H_{c1}\propto n$. This is the case considered in detail by Roberts, and is a reasonable approximation for cases of interest. In all cases of interest magnetic energies are small compared with the energy of the matter, and so $b^2\equiv H_IB_I/4\pi$ is much less than $a^2\equiv n_I(\partial p/\partial n)_I$. It is therefore convenient to introduce the ratio $\beta=b^2/a^2$ and expand in powers of β. Then the $\vec{k}$ value with the lowest eigenvalue has

$$\cos^2\theta\approx\tfrac{1}{2}\beta-\tfrac{1}{2}\beta^2 \tag{35}$$

and an eigenvalue λ of

$$\lambda\approx-\tfrac{1}{4}\beta^3a^2k^2 \ . \tag{36}$$

The displacement vector is

$$\vec{u}\approx u[\hat{z}(1-\tfrac{1}{4}\beta)-\hat{x}(\tfrac{1}{2}\beta)^{1/2}(1-\tfrac{3}{2}\beta)]e^{i\vec{k}\cdot\vec{r}} \ . \tag{37}$$

Using Eqs. (27) and (28) we can compute the density and field oscillations this displacement leads to. We get

$$\frac{\delta n}{n_I} \approx -iuk\beta(\tfrac{1}{2}\beta)^{1/2} e^{i\vec{k}\cdot\vec{r}} \quad (38)$$

and

$$\frac{\delta \vec{B}}{B_I} \approx iuk\hat{z}(\tfrac{1}{2}\beta)^{1/2} e^{i\vec{k}\cdot\vec{r}} \ . \quad (39)$$

There is therefore an oscillation between regions of higher B—lower n and lower B—higher n. This is what we would expect, since when $H_{c1} \propto n$, $\partial H_{c1}/\partial n > 0$. Notice that this is an almost purely transverse disturbance; the density change is very small, of order $\beta^{3/2}$. This reflects the fact that magnetic energies are small compared with compressional energies. In order to produce a distortion which lowers the total energy one must lower the magnetic energy without causing large changes in the compressional energy (which is always positive, assuming the matter to be stable). The compressional energy is proportional to $(\vec{\nabla}\cdot\vec{u})^2$, and for this to be small the wave must be essentially transverse.

We now drop the assumption that $H_{c1} \propto n$, and allow for a general $H_{c1}(n)$. We will assume that both $\beta \equiv b^2/a^2$ and $\gamma \equiv f/a^2$ are small, and are of comparable magnitude. Note that γ can have either sign, depending on whether $\partial H_{c1}/\partial n$ is positive or negative. It is straightforward to check that A_{ij} has a negative eigenvalue when the $\vec{k}$ vector has $\cot^2\theta < \gamma(\gamma/\beta)$. Since we are assuming that γ and β are comparable, we will take $\cot^2\theta$ (or, equivalently, $\cos^2\theta$) to be small and of order β or γ. Then, to the order to which we are working, we can take the matrix A to be

$$A_{ij} \approx a^2[(1+2\gamma)k_ik_j - \gamma k_z(k_i\hat{z}_j + \hat{z}_ik_j)] \ . \quad (40)$$

Then, for $\vec{k}$ vectors which have $\cos^2\theta$ of order γ, we find the corresponding displacement eigenvectors to be

$$\vec{u} = ue^{i\vec{k}\cdot\vec{r}}[\hat{z}(1-\tfrac{1}{2}\cot^2\theta) - \hat{x}\cot\theta(1-\gamma)(1-\tfrac{1}{2}\cot^2\theta)] \ . \quad (41)$$

We can then compute the changes in density and field, and we get

$$\delta n/n_I = -iku\gamma\cos\theta e^{i\vec{k}\cdot\vec{r}} \quad (42)$$

and

$$\delta\vec{B}/B_I = iuk\cos\theta e^{i\vec{k}\cdot\vec{r}}\hat{z} \ . \quad (43)$$

So we see that a positive value for γ (or $\partial H_{c1}/\partial n$) means that regions of increased B are associated with regions of decreased n; conversely, a negative value for γ means that regions of increased B are correlated with regions of increased n.

Throughout this section we have assumed that the energy density depended only on $\vec{B}$ and n. Then, the strain D_{ij} affected the energy only in so far as it changed $\vec{B}$ or n. This is equivalent to assuming that the time scales of interest are long compared with the characteristic time τ for a distorted vortex lattice to relax locally back to its triangular form. It is difficult to estimate τ, since it depends on dissipative motion of flux lines relative to the matter and on motion of dislocations in vortex lattices, processes which are difficult to describe quantitatively. For the modes we consider in this section, as long as $|\vec{k}|$ is small enough, the frequency will satisfy $\omega\tau << 1$, and our neglect of the lattice distortion is justified. We note that the energies associated with distortions of a flux line lattice from its equilibrium form have been considered by Fetter, Hohenberg, and Pincus[9] in their studies of the oscillations of a vortex lattice.

We should perhaps point out that there is no inconsistency between (i) allowing for dissipative processes to cause the vortex lattice always to relax to a locally triangular form and (ii) assuming in the evaluation of the energy that the average magnetic induction $\vec{B}$ is frozen to the matter. In the latter case all that is required is that the average density of flux lines (over a region containing many flux lines) should be frozen into the matter, and this is not inconsistent with local rearrangements of vortex lines on a scale of order the spacing between flux lines.

IV. DISCUSSION

We have shown that for $B << H_{c1}$ a type-II superconductor is globally unstable to the formation of a two-phase state due to the attraction between vortex lines induced by coupling to the density of the medium. We have estimated some effects of the phase separation for laboratory superconductors and conclude that it is important at very low flux densities. On the other hand, our estimates suggest that the proton superconductor in neutron stars may well be in the two-phase region.

Our numerical estimates have been based on results for extreme type-II superconductors. For the protons in a neutron star one finds $\kappa = \lambda/\xi \sim 70\Delta/\epsilon_p$ at nuclear matter density, where Δ is the proton energy gap and ϵ_p is the proton Fermi kinetic energy.[1] Δ is estimated to be be $\sim\epsilon_p/10$, and therefore $\kappa \sim 7$. The extreme type-II results are not quantitatively accurate for such low values of κ, but should give a reliable order-of-magnitude estimate of the effects.

Let us now discuss the sizes of domains with flux and without flux. In thermodynamic equilibrium these will be determined by the same sorts of considerations as in the intermediate state of type-I superconductors. The surface energy between the flux-free and flux containing regions tends to increase the thickness of the domains, and the extra magnetic field energy due to splaying of the field at the boundaries of the material tends to decrease the domain size. In practice, however, it is likely that the domains will not be able to reach the equilibrium state, and that the previous history of the material will be decisive in determining the domain size.

We have not yet considered in detail the stability of the two-phase state, or the possibility of more complicated equilibrium structures. However, at least for the extreme type-II limit, no other possibility seems feasible.

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[1] G. Baym, C. J. Pethick, and D. Pines, Nature 224, 673 (1969).
[2] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975).
[3] P. B. Jones, Astrophys. Space Sci. 33, 215 (1975).
[4] I. Easson and C. J. Pethick, Phys. Rev. D 16, 275 (1977).
[5] P. H. Roberts, Q. J. Mech. Appl. Math. (in press).
[6] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. 156, 207 (1967).
[7] See, e.g., R. P. Huebener, *Magnetic Flux Structures in Superconductors* (Springer, Berlin, 1979).
[8] G. Baym, H. A. Bethe, and C. J. Pethick, Nucl. Phys. A175, 225 (1971).
[9] A. L. Fetter, P. C. Hohenberg, and P. Pincus, Phys. Rev. 147, 140 (1966).