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Uniaxial type-II superconductors near the upper critical field

Vladimir G. Kogan and John R. Clem
Ames Laboratory—U. S. Department of Energy and Department of Physics, Iowa State University, Ames, Iowa 50011
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The problem of a uniaxial type-II superconductor near the upper critical field is considered in the framework of the Ginzburg-Landau equations with a phenomenological mass tensor. The currents are shown to flow in planes which are in general no longer orthogonal to the direction of the vortex axes as in the isotropic case; the inclination angle is obtained in terms of anisotropic masses. The magnetic field has a component normal to the vortex axes; equations are derived which relate the transverse and axial fields. The average value of the transverse field (the transverse induction) vanishes. The constitutive relation between the induction and the magnetization is obtained. The components of the magnetization normal and parallel to the vortex direction are simply related in terms of the effective masses.

I. INTRODUCTION

Strongly anisotropic type-II superconductors have been studied already for some time. The dependence of the upper critical field H_{c2} upon the magnetic field orientation in layered (e.g., $NbSe_2$) crystals is described reasonably well by the Ginzburg-Landau (GL) equations with a phenomenological mass tensor M_{ik}[1–4]:

(GL1):

$$\tfrac{1}{2}M_{ik}^{-1}\left(-i\hbar\frac{\partial}{\partial x_i}-\frac{2e}{c}A_i\right)\left(-i\hbar\frac{\partial}{\partial x_k}-\frac{2e}{c}A_k\right)\psi=|\alpha|\psi-\beta|\psi|^2\psi \ .$$

(GL2):

$$(\vec{\nabla}\times\vec{H})_i=\frac{4\pi e}{c}M_{ik}^{-1}\left[\psi^*\left(-i\hbar\frac{\partial}{\partial x_k}-\frac{2e}{c}A_k\right)\psi+\text{c.c.}\right] \ .$$

Here ψ is the order parameter, $\vec{A}$ is the vector potential of the local magnetic field $\vec{H}$, and α and β are the coefficients in the GL free-energy expansion, which can be expressed in terms of the thermodynamic critical field H_c and the order parameter $|\psi_0|$ in the absence of the magnetic field: $|\alpha|/\beta=|\psi_0|^2$, $\alpha^2/\beta=H_c^2/8\pi$. The inverse mass tensor M_{ik}^{-1} has the principal values $1/M_i(i=1,2,3)$. The usual summation convention over the indices repeated twice is adopted hereafter.

Following Ref. 5, we choose $|\psi_0|$ and $H_c\sqrt{2}$ as the units of $|\psi|$ and H, respectively. As the unit of length we take $\lambda=(\overline{M}c^2/16\pi e^2|\psi_0|^2)^{1/2}$ with the mean mass $\overline{M}=(M_1M_2M_3)^{1/3}$. This choice[6] is arbitrary but convenient as is seen below. The GL equations now read

$$\mu_{ik}\Pi_i\Pi_k\psi=\psi(1-|\psi|^2) \ , \tag{1}$$

$$e_{ikl}\frac{\partial H_l}{\partial x_k}=\mu_{ik}\,\mathrm{Re}(\psi^*\Pi_k\psi) \ , \tag{2}$$

where $\vec{\Pi}=(i\kappa)^{-1}\vec{\nabla}-\vec{A}$. Here all quantities are dimensionless, and they are denoted by the same letters as their dimensional counterparts. The GL parameter is defined in the usual way: $\kappa=2\sqrt{2}eH_c\lambda^2/\hbar c$. Note, however, that κ is expressed in terms of $\overline{M}$ via λ^2. The tensor $\mu_{ik}=\overline{M}M_{ik}^{-1}$ has the eigenvalues $\mu_i=\overline{M}/M_i$; therefore , $\mu_1\mu_2\mu_3=1$.

Tilley[2] has shown that H_{c2} can be found from Eq. (1) by neglecting the term $\psi|\psi|^2$ and solving the eigenvalue problem for the resulting linear equation, i.e., essentially in the same fashion as in the isotropic case (see, e.g., Ref. 7). One finds that $H_{c2}=\tilde{\kappa}$, where $\tilde{\kappa}$, which replaces the GL parameter κ_{is} of the isotropic case, depends on the field orientation in the crystal as specified in Refs. 2 and 6 [see Eq. (26)]. Moreover, it is stated in Refs. 2, 6, and 8 that all other κ-dependent results of the isotropic GL theory can be translated to the anisotropic case if one replaces κ_{is} by $\tilde{\kappa}$.

However, as was pointed out first by Takanaka,[9] a new transverse component $\vec{H}_\perp$ of the magnetic field arises in the Abrikosov vortex in an anisotropic material, unless the vortex direction coincides with one of the principal crystal axes. In the vicinity of the lower critical field H_{c1} the transverse field has been discussed recently in Ref. 10. This field cannot be obtained from the isotropic theory by a simple replacement of κ_{is} by its angular dependent analog $\tilde{\kappa}$.[11] Moreover, $H_\perp$ is not necessarily small. For example, for a vortex at 45° to the layers of $NbSe_2$ ($M_1=M_2\simeq0.1M_3$) in the low-field region, Ref. 10 predicts $\langle H_\perp^2\rangle\simeq0.25\langle H_z^2\rangle$, where H_z is the usual axial field in the vortex. Therefore, $\vec{H}_\perp$ should be observable in neutron scattering and nuclear magnetic resonance experiments. Both these techniques are applicable in fields not too close to H_{c2}.

Theoretically, however, the domain near H_{c2} is the simplest one. Besides, already here one can observe all the characteristic features of the field distribution.

Roughly speaking, the field differences are simply amplified as one goes down from H_{c2}. The main purpose of the present paper is to determine how the anisotropy affects the field and current distributions in the vortex lattice near H_{c2}. For the sake of simplicity we consider only layered materials ($M_1 = M_2 < M_3$). However, all results are applicable to fiber materials as well ($M_1 = M_2 > M_3$).

II. TRANSFORMATION OF COORDINATES

We consider now a uniaxial layered crystal. In the system of coordinates (X, Y, Z), where Z is the axis normal to the layers, the tensor μ_{ik} is diagonal ($\mu_{xx} = \mu_{yy} = \mu_1$, $\mu_{zz} = \mu_3$). We are interested in an array of vortices directed arbitrarily with respect to the crystal axes. Let us denote by z the direction of the vortex axes and introduce another system of Cartesian coordinates (x,y,z), which is rotated with respect to the (X, Y, Z) axes through an angle θ about the Y axis (see Fig. 1). All physical quantities related to the lattice of vortices are independent of z. In the frame (x,y,z) the nonzero μ's are

$$\begin{aligned}
&\mu_{xx} = \mu_1 \cos^2\theta + \mu_3 \sin^2\theta \ , \\
&\mu_{yy} = \mu_1, \quad \mu_{xy} = \mu_{yz} = 0 \ , \\
&\mu_{zz} = \mu_1 \sin^2\theta + \mu_3 \cos^2\theta \ , \\
&\mu_{xz} = (\mu_1 - \mu_3) \sin\theta \cos\theta \ .
\end{aligned} \tag{3}$$

The relations

$$\begin{aligned}
&\mu_{xx}\mu_{zz} - \mu_{xz}^2 = \mu_1\mu_3, \quad \mu_{xx} + \mu_{zz} = \mu_1 + \mu_3 \ , \\
&\mu_1^2\mu_3 = 1
\end{aligned} \tag{4}$$

are useful in following calculations.

The solutions of the GL equations near H_{c2} are well known for the isotropic case where $\mu_{ik} = \delta_{ik}$. To make use of this knowledge in our problem, we shall find a new system of coordinates where μ_{ik} has a "unitlike" form. We shall clarify this statement below.

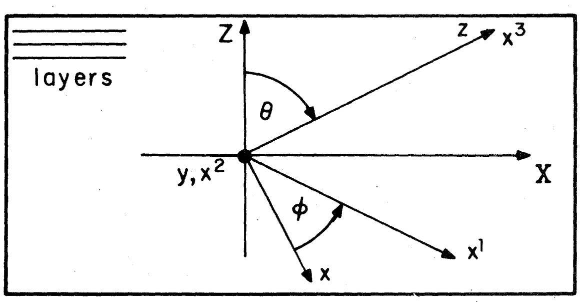

FIG. 1. The axes X, Y, Z are the principal directions of the crystal and the axis z gives the direction of the vortices. The current flows in planes parallel to the plane x^1, x^2.

Let us choose the transformation

$$x^1 = ax, \quad x^2 = by, \quad x^3 = cx + dz \ . \tag{5}$$

The new coordinates x^i have the following important features: (a) the new x^3 axis coincides with the old z axis (this is important because of the exclusive role of the z axis in the problem); and (b) any z-independent quantity η in the old frame is x^3 independent in the new one. Indeed:

$$\frac{\partial \eta}{\partial x^3} = \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial x^3} + \frac{\partial \eta}{\partial y}\frac{\partial y}{\partial x^3} + \frac{\partial \eta}{\partial z}\frac{\partial z}{\partial x^3} = 0 \tag{6}$$

because $\partial x/\partial x^3 = \partial y/\partial x^3 = 0$ and $\partial \eta/\partial z = 0$.

Obviously, the new system is no longer orthogonal: the x^2 and x^3 axes coincide with the y and z axes, while the new x^1 axis is inclined with respect to the old x direction (see Fig. 1). Moreover, the scales along the new axes are no longer the same, so that in the new system one must distinguish between co- and contravariant representations of vectors and tensors.

We want the tensor μ to assume the "unitlike" form in the new coordinates:

$$\mu^{ik} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{7}$$

This gives four independent conditions for the four unknown coefficients a,b,c,d:

$$\begin{aligned}
&\mu^{11} = 1 = a^2\mu_{xx} \ , \quad \mu^{22} = 1 = b^2\mu_{yy} \ , \\
&\mu^{33} = 1 = c^2\mu_{xx} + 2cd\,\mu_{xz} + d^2\mu_{zz} \ , \\
&\mu^{13} = 0 = a(c\,\mu_{xx} + d\,\mu_{xz}) \ .
\end{aligned}$$

Using Eqs. (4), we obtain

$$\begin{aligned}
&a = \mu_{xx}^{-1/2}, \quad b = \mu_{yy}^{-1/2} = \mu_1^{-1/2} \ , \\
&c = -\mu_{xz}\mu_1^{1/2}\mu_{xx}^{-1/2}, \quad d = (\mu_1\mu_{xx})^{1/2} \ ;
\end{aligned} \tag{8}$$

i.e., the linear transformation with the desired features does exist: Eqs. (5) and (8) determine it completely. The inverse transformation is

$$x = x^1/a, \quad y = x^2/b, \quad z = x^3/d - x^1c/ad \ . \tag{9}$$

To avoid misunderstanding, we note here that the form (7) does not imply that μ^{ik} is the unit Kronecker tensor δ^i_k. It simply means that in the particular frame chosen, the diagonal components of μ^{ik} are unity and the others are zero. When transformed back to the system (x,y,z), μ^{ik} again assumes the form (3), unlike the unit tensor δ^i_k, which remains unchanged. Our choice of the contravariant tensor μ^{ik} as being "unitlike" in the new coordinates is a matter of convenience only. We could have chosen the covariant μ_{ik}'s as "unitlike." Then the coefficients a,b,c,d would have been different, whereas all the final physical results would have remained the same.

The geometry in the new frame is completely determined by its metric tensor g_{ik}; the latter is defined by invariance of the line element: $dx^2+dy^2+dz^2=g_{ik}dx^i dx^k$. Making use of Eqs. (9) and (8), we obtain

$$g_{ik}=\begin{pmatrix} \mu_{xx}+\mu_{xz}^2\mu_{xx}^{-1} & 0 & \mu_{xz}\mu_{xx}^{-1}\mu_1^{-1/2} \\ 0 & \mu_1 & 0 \\ \mu_{xz}\mu_{xx}^{-1}\mu_1^{-1/2} & 0 & \mu_{xx}^{-1}\mu_1^{-1} \end{pmatrix} . \tag{10}$$

All the g_{ik} are constants; i.e., the new space is flat, and covariant derivatives reduce to partial ones. It is also easy to verify using (4) that $\det(g_{ik})=1$. This fact is of special convenience, because the Levi-Civita tensor e^{ikl}, transformed from (x,y,z) to the x^i system, preserves its 0, ±1 form (see, e.g., Ref. 12).

In an arbitrary flat system of coordinates with $\det(g_{ik})=1$, Eqs. (1) and (2) read

$$\mu^{ik}\pi_i\pi_k\psi=\psi(1-|\psi|^2) , \tag{11}$$

$$e^{ikl}\frac{\partial h_l}{\partial x^k}=\mu^{ik}\mathrm{Re}(\psi^*\pi_k\psi) . \tag{12}$$

We use here corresponding lower case symbols h, a, π, etc., to denote co- and contravariant components of vectors denoted by capitals $\vec{H}$, $\vec{A}$, $\vec{\Pi}$, etc., in the original frame (x,y,z). Thus, the covariant vector π_i is

$$\pi_i=\frac{1}{i\kappa}\frac{\partial}{\partial x^i}-a_i . \tag{13}$$

The contravariant components of the magnetic field are

$$h^i=e^{ikl}\frac{\partial a_l}{\partial x^k} ; \tag{14}$$

the covariant ones are $h_i=g_{ik}h^k$. Note that there is no connection as simple as (14) between h_i and the vector potential; e.g., $h_i=e_{ikl}(\partial a^l/\partial x^m)g^{mk}$.

The contravariant components of any vector are transformed in the same way as the coordinates; e.g., from Eqs. (5) and (8) we obtain for the magnetic field

$$h^1=\mu_{xx}^{-1/2}H_x, \quad h^2=\mu_1^{-1/2}H_y ,$$
$$h^3=\mu_1^{1/2}\mu_{xx}^{1/2}(H_z-\mu_{xz}\mu_{xx}^{-1}H_x) . \tag{15}$$

For the covariant components we obtain

$$h_1=\mu_{xx}^{1/2}(H_x+\mu_{xz}\mu_{xx}^{-1}H_z), \quad h_2=\mu_1^{1/2}H_y ,$$
$$h_3=(\mu_1\mu_{xx})^{-1/2}H_z . \tag{16}$$

Being covariant, the operator $\vec{\nabla}$ also transforms according to Eq. (16). However, for a quantity independent of z in the original frame, we have simpler relations:

$$\frac{\partial}{\partial x^1}=\mu_{xx}^{1/2}\frac{\partial}{\partial x} , \quad \frac{\partial}{\partial x^2}=\mu_1^{1/2}\frac{\partial}{\partial y} , \quad \frac{\partial}{\partial x^3}=0 . \tag{17}$$

This is not necessarily the case for derivatives of the vector potential and the phase χ of the order parameter: they both can be z dependent, unlike their gauge-invariant combination $(1/\kappa)\vec{\nabla}\chi-\vec{A}$.

Because of the special form of μ^{ik} given by Eq. (7), the GL equations (11) and (12) read in the new frame:

$$\pi_i\pi_i\psi=\psi(1-|\psi|^2) , \tag{18}$$

$$e^{ikl}\frac{\partial h_l}{\partial x^k}=\mathrm{Re}(\psi^*\pi_i\psi) . \tag{19}$$

This form of the GL equations is not covariant; it occurs only in our special coordinates. Indeed, $\pi_i\pi_i=(\pi_1)^2+(\pi_2)^2+(\pi_3)^2$ is not an invariant. Equation (19) relates contravariant components on the left-hand side (the current) to the covariant ones on the right.

Equation (18) looks exactly like the isotropic first GL equation, $\Pi^2\psi=\psi(1-|\psi|^2)$; i.e., we succeeded in removing the anisotropic masses from this equation. This is, however, not the case for Eq. (19), even though the masses do not appear there explicitly. In fact, they are present in this equation; this becomes obvious if one rewrites Eq. (19) as an equation for only the covariant (or contravariant) components of the vector potential:

$$e^{ikl}\frac{\partial}{\partial x^k}g_{lm}h^m=e^{ikl}e^{mpq}g_{lm}\frac{\partial^2 a_q}{\partial x^k\partial x^p}$$
$$=\mathrm{Re}\left[\psi^*\left(\frac{1}{i\kappa}\frac{\partial}{\partial x^i}-a_i\right)\psi\right] , \tag{20}$$

where g_{lm} is mass dependent [see Eq. (10)].

III. UPPER CRITICAL FIELD

To demonstrate how our method works, we first turn to Eq. (18) at H_{c2}, where $|\psi|^2$ can be neglected:

$$\left(\frac{1}{i\kappa}\frac{\partial}{\partial x^i}-a_i\right)^2\psi=\psi . \tag{21}$$

The field is uniform here: $H_x=H_y=0$, $H_z=H_{c2}$ or, as is seen from (15), $h^1=h^2=0$ and

$$h^3=(\mu_1\mu_{xx})^{1/2}H_{c2} . \tag{22}$$

The connection (14) between h^i and a_i is the same as that of $\vec{H}$ and $\vec{A}$ in Cartesian coordinates; therefore we can choose the gauge

$$a_1=a_3=0 , \quad a_2=h^3x^1 . \tag{23}$$

The following treatment is the same as that for the isotropic case[7]: because x^3 does not appear in Eq. (21),

$$\psi=\zeta(x^1,x^2)\exp(ik_3x^3) , \tag{24}$$

where k_3 is a constant. Substituting this in Eq. (21), we obtain the oscillator equation with eigenvalues $\alpha_{n,k_3}=h^3(2n+1)/\kappa+(k_3/\kappa)^2$. In our case $\alpha_{n,k_3}=1$, which gives $h^3=(\kappa-k_3^2/\kappa)/(2n+1)$; i.e., for $n=0$ and

$$k_3=0 \tag{25}$$

we have the maximum possible $h^3=\kappa$. Equation (22) now yields

$$H_{c2}=\tilde{\kappa}=\kappa(\mu_1\mu_{xx})^{-1/2} \; . \tag{26}$$

This is the Tilley result.[2,6] The angular dependence of μ_{xx} is given in (3), so that explicitly $H_{c2}=(\kappa/\mu_1)\times[\cos^2\theta+(\mu_3/\mu_1)\sin^2\theta]^{-1/2}$. This dependence fits quite well the $H_{c2}(\theta)$ observed in $NbSe_2$ (Refs. 3 and 4) if $\mu_1/\mu_3\simeq 11$.

IV. CURRENT ($H_{c2}-H \ll H_{c2}$)

Let us turn now to Eq. (19). First we rewrite it interms of $\psi=\sqrt{\omega}\exp(i\chi)$, the gauge-invariant supermomentum q_i, and the current density j^i:

$$q_i=\frac{1}{\kappa}\frac{\partial\chi}{\partial x^i}-a_i, \quad j^i=e^{ikl}\frac{\partial h_l}{\partial x^k} \; . \tag{27}$$

(Here we use $c\sqrt{2}H_c/4\pi\lambda$ as the unit of current density.) We obtain

$$j^1=\omega q_1, \quad j^2=\omega q_2 \; , \tag{28}$$

$$j^3=\omega q_3 \; . \tag{29}$$

There is a substantial difference between the implications of Eqs. (28) and (29). To see this, we consider the supermomentum in the original frame, $\vec{Q}=(1/\kappa)\vec{\nabla}\chi-\vec{A}$. In the isotropic case $\vec{Q}$ has only x and y components; $|\vec{Q}|$ diverges near the vortex axis as $(\kappa r)^{-1}$, where $r^2=x^2+y^2$, and decreases to a value of order unity near the cell boundary. This is correct even at H_{c2}. The anisotropy does not change this situation drastically; q_1 and q_2 remain large even at H_{c2}. Unlike q_1 and q_2, however,

$$q_3=\frac{1}{\kappa}\frac{\partial\chi}{\partial x^3}-a_3=\frac{1}{\kappa}k_3-a_3=0 \tag{30}$$

at H_{c2} because of (23), (24), and (25). This means that in the region of interest ($H_{c2}-H \ll H_{c2}$), q_3 must be small. Therefore, j^3 is of a higher order in the small parameter $(H_{c2}-H)/H_{c2}$ than j^1 and j^2. In other words, j^3 should be considered as zero in the linear approximation. Going back now to the original frame (x,y,z), we obtain with the help of Eq. (15):

$$J_z=(\mu_{xz}/\mu_{xx})J_x \; . \tag{31}$$

We conclude, therefore, that in the immediate vicinity of H_{c2}, axial currents J_z exist in a system of Abrikosov vortices in an anisotropic material. These currents vanish only if the vortices are oriented along one of the principal crystal directions ($\theta=0,\pi/2$) because of the factor $\mu_{xz}=(\mu_1-\mu_3)\sin\theta\cos\theta$. For strong anisotropy, J_z is of the same order of magnitude as J_x, if θ is not close to 0 or $\pi/2$; e.g., for $NbSe_2$ the maximum value of μ_{xz}/μ_{xx} is about 1.5, which is achieved at $\theta\simeq 73°$.

Consider now a plane passing through the y axis at an angle ϕ with respect to the xy plane. The unit vector normal to this plane is $\hat{n}=(-\sin\phi,0,\cos\phi)$ and $\hat{n}\cdot\vec{J}=-J_x\sin\phi+J_z\cos\phi=J_z\cos\phi[1-(\mu_{xx}/\mu_{xz})\tan\phi]$. We see that $\hat{n}\cdot\vec{J}=0$ for

$$\tan\phi=\mu_{xz}/\mu_{xx} \; ; \tag{32}$$

i.e., the current lines are parallel to this plane. Thus, the remarkably simple relation (31) also can be formulated as follows: in a system of parallel vortices in a uniaxial crystal, vortex currents flow in a plane inclined with respect to the plane normal to the vortex axes. The inclination angle is given by Eq. (32). This angle is in general not small: in $NbSe_2$ ϕ reaches the maximum value of $\simeq 57°$ at the vortex orientation $\theta\simeq 73°$.

The current plane coincides with the plane x^1x^2 in the coordinates x^i used above. This follows, e.g., from $j^3=0$. It is clear now why the nonorthogonal frame (x^1,x^2,x^3) proved to be so convenient in our problem. In Fig. 1, the intersection of the current plane with the figure plane is shown as the axis x^1. The angle $\phi<\theta$ at any θ ($0<\theta<\pi/2$); this is seen from Eq. (32) written explicitly

$$\tan\phi=\tan\theta\frac{\mu_1-\mu_3}{\mu_1+\mu_3\tan^2\theta} \; . \tag{33}$$

Equation (31) can be rewritten in terms of the field $\vec{H}$:

$$\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=\frac{\mu_{xz}}{\mu_{xx}}\frac{\partial H_z}{\partial y} \; . \tag{34}$$

Applying $\partial/\partial y$ to both sides here and using $\vec{\nabla}\cdot\vec{H}=0$, one gets

$$\nabla^2 H_x=-\frac{\mu_{xz}}{\mu_{xx}}\frac{\partial^2 H_z}{\partial y^2} \; , \tag{35}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. Operating by $\partial/\partial x$ we have

$$\nabla^2 H_y=\frac{\mu_{xz}}{\mu_{xx}}\frac{\partial^2 H_z}{\partial x\,\partial y} \; . \tag{36}$$

The field $H_{x,y}$ affects the free energy density and therefore all the thermodynamic results, such as the constitutive relation between the magnetic induction $\vec{B}$ (the average of the microscopic $\vec{H}$ over several intervortex spacings) and the thermodynamic macroscopic field $\vec{H}_M$, and the equilibrium lattice structure. In the following sections we calculate the thermodynamic properties of an anisotropic uniaxial superconductor near H_{c2}, taking into account the transverse field.

V. ABRIKOSOV IDENTITIES IN THE ANISOTROPIC CASE

In our calculations we follow essentially the original Abrikosov approach in which the central role belongs to the couple of well-known Abrikosov identities.[5,7] Perhaps the simplest way to derive the identities has been proposed by one of the authors.[13] We now briefly review this method for the isotropic case and later will apply it to the anisotropic situation.

Let us begin with the first GL equation for an isotropic material

$$\Pi^2\psi=\psi(1-|\psi|^2) \tag{37}$$

and rewrite it in terms of the operators $\Pi^{\pm}=\Pi_x \pm i\Pi_y$:

$$\Pi^-\Pi^+\psi=\psi\left(1-\frac{H_z}{\kappa}-|\psi|^2\right) \ . \tag{38}$$

We know that $|\psi|^2$ is of order $(\kappa-H)/\kappa$ near $H_{c2}=\kappa$. Therefore, neglecting the term $\psi|\psi|^2$ in the region considered, we also have to neglect the term $\psi(1-H_z/\kappa)$. In other words, instead of solving the linearized Eq. (37), $\Pi^2\psi_0=\psi_0$, we have a more convenient form, $\Pi^-\Pi^+\psi_0=0$, for the order parameter ψ_0 in the lowest approximation. The last equation is further simplified to

$$\Pi^+\psi_0=0 \ . \tag{39}$$

Performing the necessary operations $[\vec{\Pi}=(i\kappa)^{-1}\vec{\nabla}-\vec{A}]$, we obtain in terms of $\psi_0=\sqrt{\omega}\exp(i\chi)$ and $\vec{Q}=\vec{\nabla}\chi/\kappa-\vec{A}$:

$$\omega Q_x=-\frac{1}{2\kappa}\frac{\partial\omega}{\partial y},\quad \omega Q_y=\frac{1}{2\kappa}\frac{\partial\omega}{\partial x} \ . \tag{40}$$

Now, the second GL equation $\vec{\nabla}\times\vec{H}=\omega\vec{Q}$ yields

$$\omega Q_x=\frac{\partial H_z}{\partial y}\ ,\ \omega Q_y=-\frac{\partial H_z}{\partial x} \ . \tag{41}$$

Eliminating $\omega\vec{Q}$ from Eqs. (40) and (41), we obtain the first Abrikosov identity

$$H_z=H_0-\omega/2\kappa \ , \tag{42}$$

where H_0 is an arbitrary constant.

The solution ψ_0 of the homogeneous equation (39) has to be normalized. To do that, let us return to the exact nonlinear Eq. (38) and look for its solution in the form $\psi_0+\psi_1$. Making use of (39) and (42), one gets for ψ_1

$$\Pi^-\Pi^+\psi_1=\psi_0\left[\frac{\kappa-H_0}{\kappa}+\left(\frac{1}{2\kappa^2}-1\right)|\psi_0|^2\right] \ . \tag{43}$$

As discussed in the Appendix, this inhomogeneous linear equation for ψ_1 has a solution if its right-hand side is orthogonal to the solution of the corresponding homogeneous equation $\Pi^-\Pi^+\psi=0$, i.e., to ψ_0. Therefore, we have

$$\frac{\kappa-H_0}{\kappa}\langle\omega\rangle+\left(\frac{1}{2\kappa^2}-1\right)\langle\omega^2\rangle=0 \ , \tag{44}$$

the second Abrikosov identity.

Abrikosov further substituted the GL equations into the GL free energy to get $\mathfrak{F}=\int(H^2-|\psi|^4/2)\,dV$, or for the macroscopic free energy density

$$F=\langle H_z^2-\omega^2/2\rangle \ . \tag{45}$$

Then after introducing the structure parameter $\beta_A=\langle\omega^2\rangle/\langle\omega\rangle^2$, he averaged Eq. (42) to obtain the magnetic induction

$$B=H_0-\langle\omega\rangle/2\kappa \ , \tag{46}$$

expressed $\langle\omega\rangle$ and H_0 in terms of B and β_A with the help of Eqs. (44) and (46), and finally obtained

$$F(B)=B^2-(\kappa-B)^2/\delta,\quad \delta=(2\kappa^2-1)\beta_A+1 \ . \tag{47}$$

The coordinate transformation explored earlier reduces the anisotropic GL equations (1) and (2) to the "isotropiclike" form (18), (28), and (29). As in the isotropic case, we introduce the operators $\pi^{\pm}=\pi_1\pm i\pi_2$. Now, $\pi^-\pi^+\psi=(\pi_1^2+\pi_2^2+i[\pi_1,\pi_2])\psi$ where the commutator yields

$$[\pi_1,\pi_2]\psi=\frac{i\psi}{\kappa}\left(\frac{\partial a_2}{\partial x^1}-\frac{\partial a_1}{\partial x^2}\right)=\frac{i\psi}{\kappa}h^3 \ ; \tag{48}$$

Eq. (14) has been used here. Further, it is easy to see that $(\pi_3)^2\psi=(q_3)^2\psi$. [Use the definitions (13) and (27) of π_i and q_i and recall that any quantity independent of z is also independent of x^3, so that $\partial|\psi|/\partial x^3=\partial q_3/\partial x^3=0$.] Thus, $\pi_i\pi_i\psi=[\pi^-\pi^+ +h^3/\kappa+(q_3)^2]\psi$. Now Eq. (18) reads

$$\pi^-\pi^+\psi=\left(1-\frac{h^3}{\kappa}-|\psi|^2-(q_3)^2\right)\psi \ . \tag{49}$$

We have shown before that $q_3=0$ at H_{c2} [see Eq. (30)], so that q_3 is of order of $(H_{c2}-H)/H_{c2}$ near H_{c2} (we shall return to this assumption later). Then all terms on the right-hand side of Eq. (49) $[(\kappa-h^3)/\kappa,\ |\psi|^2,\ (q_3)^2]$ have to be neglected along with $|\psi|^2$, and we have in this approximation $\pi^+\psi_0=0$ or

$$\omega q_1=-\frac{1}{2\kappa}\frac{\partial\omega}{\partial x^2},\quad \omega q_2=\frac{1}{2\kappa}\frac{\partial\omega}{\partial x^1} \ . \tag{50}$$

Equation (28) yields: $\omega q_1=\partial h_3/\partial x^2$, $\omega q_2=-\partial h_3/\partial x^1$, which after combination with Eq. (50) gives $h_3=\text{const}-\omega/2\kappa$, or in the original coordinates

(x,y,z):

$$H_z = H_0 - \frac{\omega}{2\tilde{\kappa}} \; ; \tag{51}$$

here we used Eqs. (16) and (26). The quantity H_0 is a constant still to be determined.

Substituting Eq. (51) into Eqs. (35) and (36), we obtain in the same approximation for the transverse field

$$\nabla^2 H_x = \gamma \frac{\partial^2 \omega}{\partial y^2} \quad , \tag{52}$$

$$\nabla^2 H_y = -\gamma \frac{\partial^2 \omega}{\partial x \, \partial y}, \quad \gamma = \frac{\mu_{xz}}{2\tilde{\kappa}\mu_{xx}} \quad . \tag{53}$$

Thus, in the anisotropic case three equations (51)–(53) appear in place of the first Abrikosov identity (42) of the isotropic case, for which $H_x = H_y = \gamma = 0$.

Equations (52) and (53) show that H_x and H_y are of order ω, i.e., of order $(H_{c2} - H)/H_{c2}$. This means that A_z as well as Q_z are also linear in $(H_{c2} - H)/H_{c2}$, so that $q_3 = (\tilde{\kappa}/\kappa) Q_z$ [see Eqs. (16) and (26)] is of the same order of magnitude. One can neglect now $(q_3)^2$ in Eq. (49) with respect to other terms on the right-hand side.

The same theorem that led to the normalization (44) in the isotopic case now yields $\langle (1 - h^3/\kappa - \omega)\omega \rangle = 0$. Using Eqs. (15) and (51), we obtain the new normalization

$$\frac{\tilde{\kappa} - H_0}{\tilde{\kappa}} \langle \omega \rangle + \left(\frac{1}{2\tilde{\kappa}^2} - 1 \right) \langle \omega^2 \rangle + 2\gamma \langle H_x \omega \rangle = 0 \quad , \tag{54}$$

which coincides with that of the isotropic case if $\tilde{\kappa} = \kappa$ and $\gamma = 0$.

VI. FREE ENERGY DENSITY

It is easy to show that, as in the isotropic case, the free energy

$$\mathfrak{F} = \int dV \left[- |\psi|^2 + \frac{|\psi|^4}{2} + \mu_{ik} \left(\frac{1}{i\kappa} \frac{\partial}{\partial x_i} - A_i \right) \psi \left(\frac{i}{\kappa} \frac{\partial}{\partial x_k} - A_k \right) \psi^* + H^2 \right] \tag{55}$$

can be transformed with the help of the GL equation (1) to the form $\mathfrak{F} = \int dV (H^2 - |\psi|^4/2)$. One needs only to integrate by parts the terms $(\partial\psi/\partial x_i)(\partial\psi^*/\partial x_k)$ and $A_i \psi \partial\psi^*/\partial x_k$; the resulting surface integral vanishes because of the boundary condition

$$n_k \mu_{ik} \left(\frac{1}{i\kappa} \frac{\partial}{\partial x_i} - A_i \right) \psi = 0 \quad , \tag{56}$$

where $\vec{n}$ is the normal to the sample surface. Equation (56) follows from the minimum condition for $\mathfrak{F}$ in the same way as the isotropic boundary condition, which coincides with Eq. (56) if $\mu_{ik} = \delta_{ik}$. We have, therefore,

$$F = \langle H_z^2 + H_\perp^2 - \omega^2/2 \rangle \quad , \tag{57}$$

$(H_\perp^2 = H_x^2 + H_y^2)$ which differs from the isotropic free energy density (45) by the mean square of the transverse field.

To make further progress, we have to calculate the averages $\langle H_x \omega \rangle$ of Eq. (54) and $\langle H_\perp^2 \rangle$ of Eq. (57). For this purpose let us use the Fourier transforms of ω and $\vec{H}$, which are periodic in the flux-line lattice:

$$\omega(\vec{r}) = \sum_{\vec{G}} \omega_{\vec{G}} e^{i\vec{G}\cdot\vec{r}}, \qquad \omega_{\vec{G}} = S^{-1} \int \omega(\vec{r}) e^{-i\vec{G}\cdot\vec{r}} dS \quad , \tag{58}$$

where the integral is taken over the lattice cell, S is the cell area, and the $\vec{G}$ are two-dimensional reciprocal-lattice vectors. Equations (52) and (53) then yield

$$H_{x\vec{G}} = \gamma G_y^2 \omega_{\vec{G}} / G^2, \quad H_{y\vec{G}} = -\gamma G_x G_y \omega_{\vec{G}} / G^2 \quad , \tag{59a}$$

which are valid for $\vec{G} \neq 0$.

We now show that the transverse magnetic induction $\vec{B}_\perp = \langle \vec{H}_\perp \rangle$ vanishes. We stress this, because the erroneous conclusion that $\vec{B} \neq 0$ was made in Ref. 9. Consider the GL current equation (2) and multiply it by $m_{pi} = \mu_{pi}^{-1}$ from the left to get $m_{pi} j_i = \omega(\vec{\nabla}\chi/\kappa - \vec{A})_p$. If one integrates $m_{pi} j_i / \omega$ over a contour C_0 in the plane (xy) normal to the vortex axes (see Fig. 2), one obtains the macroscopic relation $B_z = 2\pi N/\kappa$, where $2\pi/\kappa$ is the flux quantum and N is the number density of z directed vortices.

To evaluate the components of the transverse in-

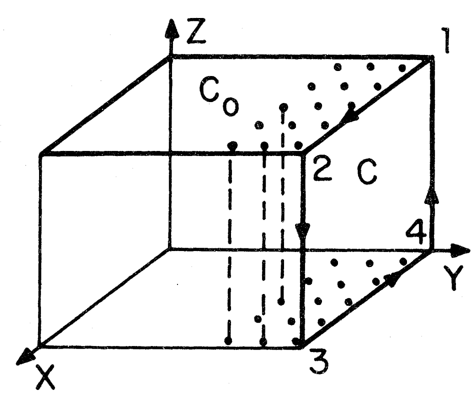

FIG. 2. Some of the vortices are shown by the dashed lines. The plane of the contour C_0 is normal to the z axis and the plane of the contour C is normal to the y axis. The points 1, 2, 3, and 4 are equivalent within the primitive cell.

duction, e.g., B_y, consider a contour C in the plane (xz). The magnetic induction is independent of the shape of the contour. We choose the contour C as shown in Fig. 2 where the points 1, 2, 3, and 4 are equivalent with respect to their positions within the primitive cell. Then, because the paths $1\to 2$ and $4\to 3$ are equivalent, $\int_1^2 dx(m_{xl}j_l/\omega)$ is cancelled by $\int_3^4 dx(m_{xl}j_l/\omega)$. Similarly, the integrals along the paths $2\to 3$ and $4\to 1$ cancel. Therefore, $\oint_C(\vec\nabla\chi/\kappa-\vec A)\cdot d\vec l=0$. Moreover, $\oint_C\vec\nabla\chi\cdot d\vec l=0$, because no singularities of the phase χ are surrounded by C. Thus, $\oint_C\vec A\cdot d\vec l=0$; i.e., $B_y=0$. A similar argument shows that $B_x=0$.

We have now in addition to Eq. (59a):

$$H_{x,\vec G=0}=\langle H_x\rangle_{\rm cell}=B_x=0\ ,\qquad H_{y,\vec G=0}=\langle H_y\rangle_{\rm cell}=B_y=0\ . \tag{59b}$$

The average $\langle H_\perp^2\rangle$ needed for the free energy density (57) is easily found now from Eqs. (59a) and (59b):

$$\langle H_\perp^2\rangle=\langle H_x^2\rangle+\langle H_y^2\rangle=\gamma^2\sum{}'\frac{G_y^2|\omega_{\vec G}|^2}{G^2}\ , \tag{60}$$

where $\sum'$ means $\sum_{\vec G\neq 0}$. Likewise,

$$\langle H_x\omega\rangle=S^{-1}\int H_x\sum_{\vec G}\omega_{\vec G}e^{i\vec G\cdot\vec r}dS=\sum_{\vec G}\omega_{\vec G}H_{x,-\vec G}=\gamma\sum{}'\frac{G_y^2|\omega_{\vec G}|^2}{G^2}\ . \tag{61}$$

Note that both $\langle H_\perp^2\rangle$ and $\langle H_x\omega\rangle$ are expressed in terms of the same sum.

Further, we have

$$\sum{}'|\omega_{\vec G}|^2=\sum_{\vec G}|\omega_{\vec G}|^2-\omega_0^2=\langle\omega^2\rangle-\langle\omega\rangle^2=\langle\omega\rangle^2(\beta_A-1)\ . \tag{62}$$

On the other hand,

$$\sum{}'|\omega_{\vec G}|^2=\sum{}'\frac{G_y^2|\omega_{\vec G}|^2}{G^2}+\sum{}'\frac{G_x^2|\omega_{\vec G}|^2}{G^2}\ , \tag{63}$$

so that the needed sum, $\sum' G_y^2|\omega_{\vec G}|^2/G^2$, is a certain part of $\sum'|\omega_{\vec G}|^2$ dependent on the lattice periodicity and on the orientation of the axes x,y with respect to the lattice. For example, for the square lattice both sums on the right-hand side of Eq. (63) are equal if the axes x and y coincide with symmetry directions.

We introduce now a new constant β_1 of order unity, so that

$$\sum{}'\frac{G_y^2|\omega_{\vec G}|^2}{G^2}=\frac{\beta_1}{2}\sum{}'|\omega_{\vec G}|^2=\frac{\beta_1(\beta_A-1)\langle\omega\rangle^2}{2}\ . \tag{64}$$

Obviously β_1 depends on the lattice structure. Thus,

$$\langle H_\perp^2\rangle=\gamma\langle H_x\omega\rangle=\gamma^2\frac{\beta_1(\beta_A-1)\langle\omega\rangle^2}{2}\ . \tag{65}$$

We have now instead of (51) and (54):

$$B_z=H_0-\langle\omega\rangle/2\tilde\kappa\ , \tag{66}$$

$$\langle\omega\rangle=2\tilde\kappa(\tilde\kappa-H_0)/[\beta_A(2\tilde\kappa^2-1)-2\tilde\kappa^2\gamma^2\beta_1(\beta_A-1)]\ . \tag{67}$$

Equations (66) and (67) allow us to express H_0 and $\langle\omega\rangle$ in terms of B_z. Doing this and using (51) and (65), we obtain the free energy density (57) as a function of induction B_z (recall that $B_x=B_y=0$):

$$F=B_z-(\tilde\kappa-B_z)^2/(\tilde\delta+\delta_1)\ , \tag{68}$$

$$\tilde\delta=(2\tilde\kappa^2-1)\beta_A+1,\quad \delta_1=-2\tilde\kappa^2\gamma^2\beta_1(\beta_A-1)\ . \tag{69}$$

The expressions (68) and (69) replace the isotropic free energy density (47) and reduce to it when $\tilde\kappa=\kappa$ and $\gamma=0$.

VII. MAGNETIZATION

The quantities $\tilde\kappa$, $\tilde\delta$, and δ_1 appearing in the free energy density (68) all depend on the vortex orientation in the crystal; they are all functions of θ. The component of the macroscopic field $\vec H_M$ in the direction of the vortices is

$$H_{Mz}=\frac12\left(\frac{\partial F}{\partial B_z}\right)_\theta=B_z+\frac{\tilde\kappa-B_z}{\tilde\delta+\delta_1}\ . \tag{70}$$

(The factor $\frac12$ arises from the system of units adopted here.)

It is easy to verify with the help of Eqs. (66) and (67) that $H_{Mz}=H_0$. Thus, the constant H_0 has the meaning of the component of $\vec H_M$ along the vortices, i.e., the same as in the isotropic case.

Let now the induction $\vec B$ change its direction by a small angle $\delta\theta$ while $|\vec B|=$const; then $\delta B_x=B_z\delta\theta$ ($\vec B$ is directed along z initially). The corresponding variation of the free energy density is $\delta F=(\partial F/\partial\theta)_B\delta\theta=(\partial F/\partial\theta)_B(\delta B_x/B)$; i.e., $H_{Mx}=(\delta F/\delta\theta)_B/2B$. Performing the differentiation of (68), we have

$$H_{Mx}=-\frac{\tilde\kappa-B}{(\tilde\delta+\delta_1)\tilde\kappa}\frac{\partial\tilde\kappa}{\partial\theta}=-\frac{\mu_{xz}}{\mu_{xx}}\frac{\tilde\kappa-B}{\tilde\delta+\delta_1}\ . \tag{71}$$

We neglect here terms of order $(\tilde\kappa-B)^2/\tilde\kappa^2$ and make use of Eqs. (26) and (3). Formulas (70) and (71) along with $B_x=0$ represent the constitutive relation.

We express now the magnetization $\vec M=(\vec B-\vec H)/4\pi$ from (70) and (71) to obtain

$$-4\pi M_x=-\frac{\mu_{xz}}{\mu_{xx}}\frac{H_{c2}-B}{\tilde\delta+\delta_1}\ ,\quad -4\pi M_z=\frac{H_{c2}-B}{\tilde\delta+\delta_1}\ , \tag{72}$$

in conventional units. Here M_x and M_z are the magnetization components normal and parallel to the vortices, respectively. The ratio

$$M_x/M_z = -\mu_{xz}/\mu_{xx} \tag{73}$$

is especially simple and, what is more important, independent of the still undetermined structure parameters $\tilde{\delta}$ and δ_1 [or β_A and β_1; see Eq. (69)].

In the crystal frame (X,Y,Z), $M_Z = M_z\cos\theta - M_x\sin\theta$, $M_X = M_z\sin\theta + M_x\cos\theta$. After simple algebra we obtain for the component M_X in the layer plane and for M_Z normal to the layers

$$\begin{aligned} -4\pi M_X &= \frac{H_{c2}-B}{\tilde{\delta}+\delta_1}\frac{\mu_3}{\mu_{xx}}\sin\theta \ , \\ -4\pi M_Z &= \frac{H_{c2}-B}{\tilde{\delta}+\delta_1}\frac{\mu_1}{\mu_{xx}}\cos\theta \ ; \end{aligned} \tag{74}$$

their ratio is

$$M_X/M_Z = (\mu_3/\mu_1)\tan\theta \ . \tag{75}$$

Again, this ratio is independent of the lattice structure.

VIII. DISCUSSION

According to the anisotropic effective mass GL theory, the structure of the local magnetic field in a system of vortices in a strongly anisotropic uniaxial material differs substantially from that of the isotropic case. The transverse field, which obeys Eqs. (52) and (53) near H_{c2}, in principle can be found for a known distribution of $|\psi|^2 = \omega(\vec{r})$. The latter depends on the equilibrium lattice structure, which still remains to be determined.

It is clear that the lattice structure depends on its orientation θ within the crystal; in other words, the angular dependence of the parameters $\tilde{\delta}$ and δ_1 (or β_A and β_1) also remains unknown. This will make it difficult to compare Eqs. (72) and (74) with experiment. Moreover, the expressions for M_X and M_Z depend on the sample shape. Indeed, the connection between the magnetic induction $\vec{B}$ inside the sample and the externally applied magnetic field $\vec{H}_{ext}$ is affected by the sample shape via demagnetization effects: $B = H_{ext} - \xi(\theta)(H_{c2} - H_{ext})$, where $\xi(\theta)$ depends upon the demagnetization coefficients. Thus no predictions independent of sample shape and flux-line lattice structure can be made on the basis of Eqs. (72) and (74).

To the contrary, the ratio (75) depends neither on the structure of the vortex lattice nor on $|\vec{B}|$. However, the direction θ_{ext} of the external field differs from the direction θ of $\vec{B}$. At H_{c2}, $\theta_{ext} = \theta$; in the immediate vicinity of H_{c2}, $\theta_{ext} = \theta + O[(H_{c2} - H_{ext})/H_{c2}]$. However, there is no sense in keeping a small correction to θ in Eqs. (73) and (75), because the magnetization components are both proportional to the small parameter $(H_{c2} - H_{ext})/H_{c2}$. This is the maximum accuracy our theory can provide, so that the ratios M_x/M_z and M_X/M_Z can be obtained here only to zero order in the small parameter. For this reason the angle θ can be replaced by θ_{ext} in the results (73) and (75).

Thus, the ratios (73) and (75) are in an especially convenient form for experimental verification. Indeed, the ratio μ_3/μ_1 can be obtained from the angular dependence of H_{c2}; M_X, M_Z, and θ_{ext} all can be measured independently.

Our last remark concerns the limiting case $\tilde{\kappa} >> 1$. We have from Eqs. (69) and (53): $\delta_1 = -(\mu_{xz}/\mu_{xx})^2 \times (\beta_A - 1)\beta_1/2$. This quantity is of order unity and therefore can be neglected with respect to $\tilde{\delta} = 2\tilde{\kappa}^2\beta_A$ when $\tilde{\kappa} >> 1$. The free energy density (68) then can be obtained to good approximation from its isotropic counterpart (47) by a simple replacement of κ_{is} by the angular dependent $\tilde{\kappa}$.

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APPENDIX

Let ψ_0 be a solution of the homogeneous equation $L\psi_0 = 0$, where L is a linear operator. Consider another equation $L\psi = \eta$, which is no longer homogeneous. Multiply it by ψ_0^* from the left and integrate over the domain where L is defined: $\int \psi_0^*\eta dx = \int \psi_0^* L\psi dx = \int \psi\tilde{L}\psi_0^* dx$, where $\tilde{L}$ is the operator transposed to L. We see that if $\tilde{L} = L^*$, i.e., if L is an Hermitian operator, the last integral is zero: $L^*\psi^* = (L\psi_0)^* = 0$. Thus, $\int \psi_0^*\eta dx = 0$; i.e., ψ_0 and η are orthogonal.

In the case considered in the text, the operator $\Pi^-\Pi^+ = \Pi^2 + H_z/\kappa$ is certainly Hermitian: H_z/κ is a real number and Π^2 is the kinetic energy operator [see Eq. (43)].

[1]V. L. Ginzburg, Zh. Eksp. Teor. Fiz. 23, 236 (1952).
[2]D. K. Tilley, Proc. Phys. Soc. London 85, 1977 (1965).
[3]R. C. Morris, R. V. Coleman, and R. Bhandary, Phys. Rev. B 5, 895 (1972).
[4]P. de Trey, S. Gygax, and J.-P. Jan, J. Low Temp. Phys. 11, 421 (1973).
[5]A. A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957).
[6]R. A. Klemm and J. R. Clem, Phys. Rev. B 21, 1868 (1980).

[7]P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).
[8]L. Dobrosavljević and H. Raffy, Phys. Status Solidi B 64, 229 (1974).
[9]K. Takanaka, Phys. Status Solidi B 68, 623 (1975).
[10]V. G. Kogan, Phys. Rev. B 24, 1572 (1981).
[11]The point is that the coordinate transformation explored by Tilley (see Ref. 2) leads to the "isotropiclike" free energy with $\kappa_{\rm is}$ replaced by $\tilde{\kappa}$, only at H_{c2}, where the field is exactly uniform.
[12]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1971), Sec. 83.
[13]V. G. Kogan, J. Low Temp. Phys. 20, 103 (1975).