Transmission of particles through a random one-dimensional potential

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The transmission of a particle through a random chain of delta functions is studied both numerically and analytically. Several statistical ensembles were used. The results show that the logarithm of the transmission coefficient obeys the central-limit theorem and its average scales linearly with the length of the system. The averages of the transmission coefficient and its inverse show ensemble-independent behavior only in the weak-scattering limit. They scale exponentially with different characteristic lengths, which are related to the average of the logarithm of the transmission coefficient. In the strong-scattering limit the averages of the transmission coefficient and its inverse depend strongly on the statistical ensemble used, thus indicating that they are not physically meaningful quantities.

I. INTRODUCTION

Transmission of a particle through a one-dimensional random potential has become a much-studied problem in the theory of disordered systems,[1] because of its relation to the localization of the quantum states and to the zero-temperature transport properties of thin metallic wires.[2-7] The transmission coefficient T has interesting statistical features. It was shown that only its logarithm, which is related to the inverse localization length $\langle \alpha \rangle$, obeys the central-limit theorem,[8] whereas averages of T and T^{-1} become unrepresentative of the ensemble for macroscopically large systems.[9,10] Results for the latter averages, which, for long enough samples, are considered to represent the conductance and the resistance of the system, have been already obtained for a number of different models[9-17] showing their expected exponential dependence on the length of the system. However, relations between $\langle \alpha \rangle$ and the corresponding decrease and growth rates have been derived only for specific limiting cases as, for instance, the white-noise potential.[12] Explicit expressions for $\langle \alpha \rangle$ are mostly lacking, and the question of the influence of the statistical ensemble on the above averages has not yet been addressed systematically.

We want to investigate these questions for the model of randomly spaced delta functions of equal strengths. We use several methods ranging from numerical studies to various analytical approaches to calculate the averages of T, T^{-1}, and $\ln T$ to elucidate the relationships among these quantities. We derive an analytic expression for the inverse localization length $\langle \alpha \rangle$ for low density of delta functions. On the other hand, we study the changes of the averages of T^{-1} upon changing the statistical ensemble while preserving the macroscopic parameters. We confirm earlier statements[4,9,10,12] about the average of T and its inverse in that we observe certain universality in the regime of localization lengths which are large compared with the mean distance between the delta-function potentials (impurities). Here we find that, independent of the ensemble (letting $L \to \infty$),

$$-\langle \ln T \rangle = \langle \alpha \rangle L \,, \tag{1}$$

$$\ln \langle T^{-1} \rangle \equiv \gamma L = 2 \langle \alpha \rangle L \,, \tag{2}$$

$$-\ln \langle T \rangle \equiv \delta L = \tfrac{1}{4} \langle \alpha \rangle L \,, \tag{3}$$

where L is the length of the system. However, for smaller localization lengths, quantities (2) and (3) become ensemble sensitive so that even minor changes in the statistical ensemble cause dramatic variations in their behavior. Thus, as pointed out earlier,[4] it is meaningless to use them for scaling arguments in the strong-scattering regime.

The paper is organized as follows. In Sec. II we present the model and describe the statistical ensembles to be used. Section III gives the numerical data for $\langle \alpha \rangle$, $\langle T \rangle$, and $\langle T^{-1} \rangle$. In Sec. IV we derive the analytical expression for $\langle \alpha \rangle$ and study $\langle T^{-1} \rangle$ for two of the statistical ensembles. Finally, the results are discussed and compared with previous work in Sec. V.

II. THE MODEL

Consider a chain of delta-function potentials of equal positive strengths located at $x_1, x_2, \ldots$: $b\sum_j \delta(x-x_j)$, $b>0$. The Schrödinger equation for a particle of mass m moving in this potential is

$$-\frac{d^2\psi(x)}{dx^2}+2\beta'\sum_j \delta(x-x_j)\psi(x)=k^2\psi(x)\ , \tag{4}$$

where $\beta' = mb/\hbar^2$ and $E=\hbar^2k^2/2m$ is the energy. The positions x_j are random. We study the following statistical ensembles:

Ensemble 1. N coordinates $x_1,\ldots,x_N$ are distributed at random and independently in the interval $(0,L)$. We denote by $n=N/L$ the density of the impurities (δ functions).

Ensemble 2. In this ensemble we specify the length L of the sample and the density n but allow the number of impurities N to fluctuate. Members of this ensemble are realized by cutting randomly segments of length L from a sample of length $\mathscr{L} >> L$ with $\mathscr{N} = n\mathscr{L}$ impurities randomly and independently distributed, as is ensemble 1.

Ensemble 3. Here we fix the number of impurities N and the concentration n. We put the first impurity at the origin, $x_1=0$. $x_2>x_1$ is chosen according to the distribution $P(x_2-x_1) = ne^{-n(x_2-x_1)}$. This procedure is repeated $(N-1)$ times. In this ensemble the length $L=x_N$ is random, its average being $\langle L\rangle = N/n$.

These ensembles are physically equivalent, which is to say that the averages of sensible physical quantities should not depend on the choice. If, however, the average of some quantity does depend on the ensemble, this means that the average is not representative of the "typical" sample and has little physical meaning.

The parameters characterizing the system are the strength of the potential relative to the wave number

$$\beta=\beta'/k\ , \tag{5}$$

and the number of impurities within one wavelength

$$\nu = n/k\ . \tag{6}$$

Ensemble 1 will be studied numerically. For ensemble 2 we calculate the average of the logarithm of the transmission coefficient analytically in the low-density limit ($\nu << 1$). The average of the inverse of the transmission coefficient is calculated for both ensembles 2 and 3 analytically.

III. NUMERICAL CALCULATIONS

For the numerical calculations we used the relation of T to the transfer matrix $\mathscr{M}$:

$$T=2(\tfrac{1}{2}\mathrm{Tr}\mathscr{M}^\dagger\mathscr{M}+1)^{-1}\ . \tag{7}$$

The numerical calculation of $\mathscr{M}$ was done using the same method as in Ref. 10. The statistical properties of $\alpha = \ln T/L$ were investigated using ensemble 1 with as much as 10^4 systems generated in the computer. The maximum number of potentials within one system, located at randomly chosen sites, was $N=800$. Because of the exponential dependence of the matrix elements of $\mathscr{M}$ on the length of the system care had to be taken during the calculation in order to avoid operations involving large and small numbers. Results for the distribution of α are shown in Fig. 1. The full curves are Gaussians with the first and second moment numerically determined. The well-known fact that α obeys the central limit theorem is demonstrated in the insert, where the variance of α, $\langle\Delta\alpha^2\rangle$, is plotted against $\langle\alpha\rangle$ for various chain length. For not too large $\langle\alpha\rangle$ the data are consistent with

$$\langle\Delta\alpha^2\rangle = 2\langle\alpha\rangle/L\ . \tag{8}$$

As noticed earlier[10] it is very hard to calculate $\langle T\rangle$ and $\langle T^{-1}\rangle$ directly, because of the extremely broad distribution of T. Therefore we used the Gaussian distribution of α and calculated these averages from

$$\langle T^{-1}\rangle = \exp(\tfrac{1}{2}L\langle\alpha\rangle + L^2\langle\Delta\alpha^2\rangle)\mathrm{erfc}\left[-\frac{\langle\alpha\rangle+L\langle\Delta\alpha^2\rangle}{(2\langle\Delta\alpha^2\rangle)^{1/2}}\right]\Big/\mathrm{erfc}\left[-\frac{\langle\alpha\rangle}{(2\langle\Delta\alpha^2\rangle)^{1/2}}\right], \tag{9}$$

$$\langle T\rangle = \exp(-\tfrac{1}{2}L\langle\alpha\rangle + L^2\langle\Delta\alpha^2\rangle)\mathrm{erfc}\left[-\frac{\langle\alpha\rangle-L\langle\Delta\alpha^2\rangle}{(2\langle\Delta\alpha^2\rangle)^{1/2}}\right]\Big/\mathrm{erfc}\left[-\frac{\langle\alpha\rangle}{(2\langle\Delta\alpha^2\rangle)^{1/2}}\right]. \tag{10}$$

The second moments corresponding to the distributions of T and T^{-1} are, as expected, exponentially increasing with L. Results for $\langle\alpha\rangle$, $\ln\langle T^{-1}\rangle/L$, and $-\ln\langle T\rangle/L$ are shown in Figs. 2 and 3. Since $\langle\alpha\rangle$ turned out to converge quite rapidly with the length of the system (even for $\nu=1$, $\langle\alpha\rangle$ became independent of the length of system above $L=100$) we have plotted for this quantity only the results for

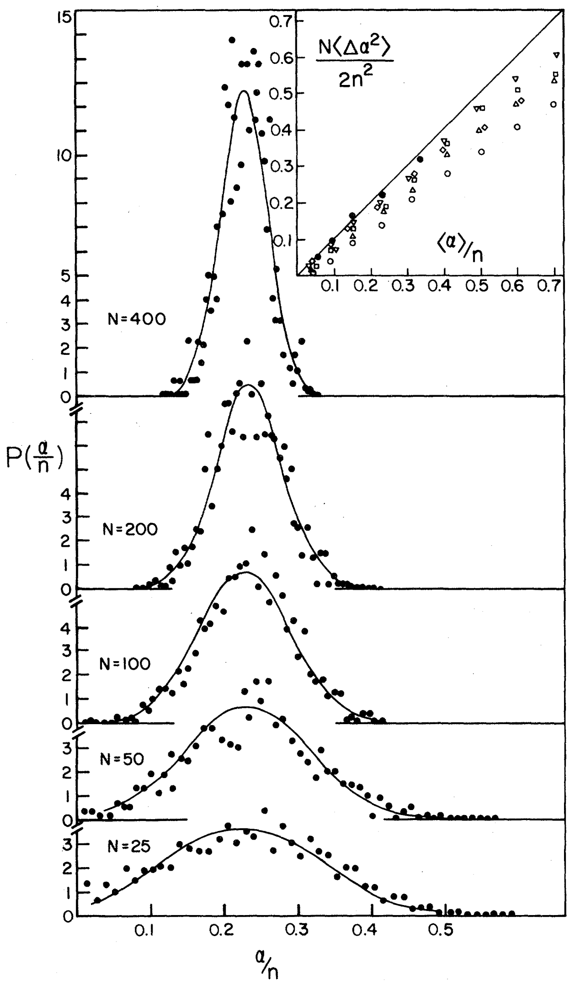

FIG. 1. Distribution of the inverse localization length $\alpha, P(\alpha/n)$, as obtained numerically (dots) from an ensemble of 1000 systems with $N=25$, 50, 100, 200, and 400 impurities. The potential strength was $\beta=2.0$ and the density of the impurities was $v=0.05$. α is measured in units of the average inverse spacing n between the impurities. The full curves are normalized Gaussians with first and second moments as obtained from the numerical data. The inset shows the squared variances of the data, $\langle \Delta\alpha^2 \rangle$, multiplied by N for systems with $N=25$ (○), 50 (△), 100 (□), 200 (▽), 400 (◇), and 800 (●) delta potentials as a function of the mean value of $\alpha, \langle \alpha \rangle$.

$N=400$. We note that for small $\langle \alpha \rangle$, in the limit of large L,

$$\frac{\ln\langle T^{-1}\rangle}{L} = 2\langle \alpha \rangle , \tag{11}$$

$$\frac{-\ln\langle T\rangle}{L} = \frac{\langle \alpha \rangle}{4} . \tag{12}$$

This is readily obtained from Eqs. (8)–(10) in the limit $L \to \infty$. For larger values of $\langle \alpha \rangle$ these relations are no longer valid.

IV. ANALYTICAL THEORY

A. Localization length

In order to solve the Schrödinger equation (4) we decompose the wave function into right- and left-going waves

$$\psi(x) = g(x) + f(x) , \tag{13}$$

where

$$g(x) = e^{ikx} - \sum_{x_j < x} \phi_j e^{ik(x-x_j)} , \tag{14}$$

$$f(x) = \sum_{x_j > x} \phi_j e^{-ik(x-x_j)} . \tag{15}$$

Inserting $\psi(x)$ into Eq. (4), integrating over $x_j - \delta$ to $x_j + \delta$, and taking $\delta \to 0$ we get a system of N inhomogeneous equations for the scattering amplitudes ϕ_j

$$\phi_j = \mu \left[e^{ikx_j} - \sum_{l \neq j} \phi_l e^{ik|x_j - x_l|} \right] , \tag{16}$$

with

$$\mu = \frac{\beta}{\beta - i} . \tag{17}$$

The transmission coefficient is

$$T = |g(L)|^2 . \tag{18}$$

To calculate the average of $\ln T$ we will use a modification of the method developed by Lifshitz and Kirpichenkov[16] for tunneling through barriers with impurities. Using (14) twice, once with x and once with $x + \Delta x$, we get

$$g(x+\Delta x) = e^{ik\Delta x} g(x) - \sum_{x < x_j < x+\Delta x} \phi_j e^{ik(x - x_j + \Delta x)} . \tag{19}$$

For the scattering amplitude ϕ_j corresponding to $x_j \in [x, x+\Delta x]$ we have from (14) and (16)

$$\phi_j = \mu e^{ik(x_j - x)} g(x) - \mu \sum_{x < x_l < L} \phi_l e^{ik|x_j - x_l|} . \tag{20}$$

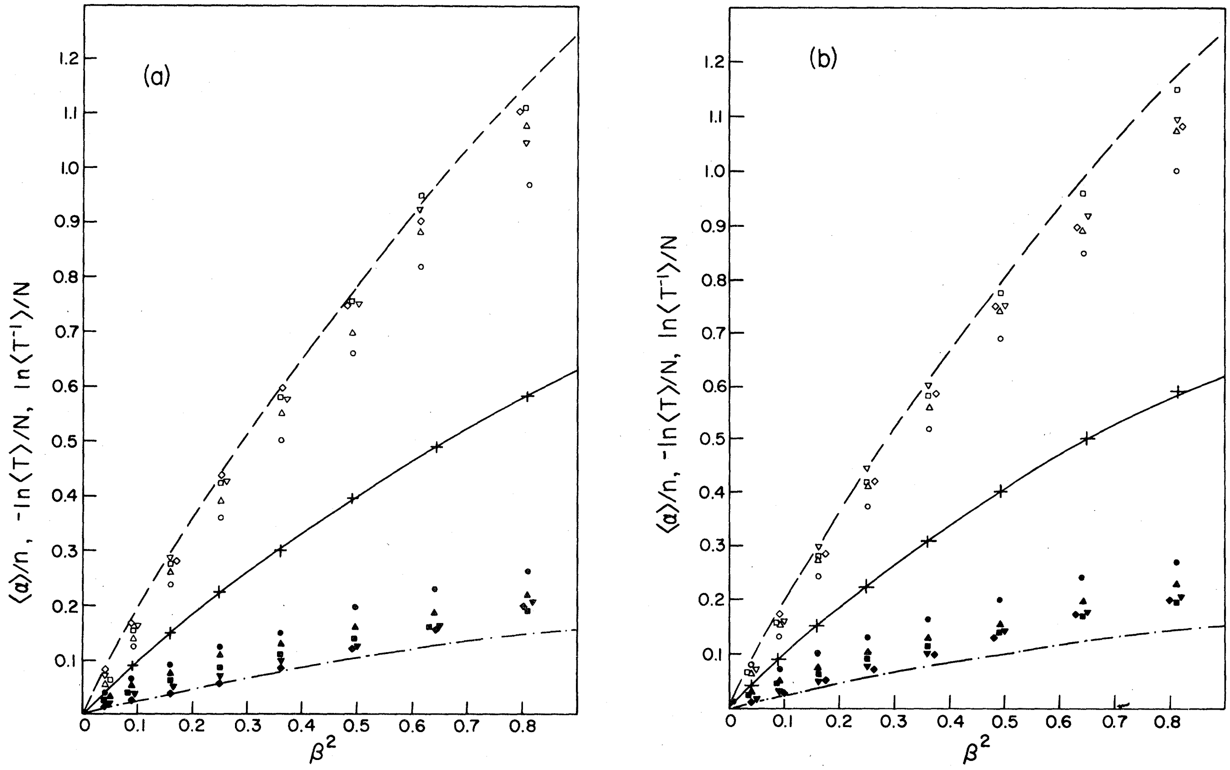

FIG. 2. The numerically determined averages of the transmission coefficient, $\langle T \rangle$ (closed symbols), its inverse, $\langle T^{-1} \rangle$ (open symbols), and the inverse localization length, $\langle \alpha \rangle$ (+) as a function of the strength of the impurity potential, β. The threee averages are appropriately scaled to the same units, namely n. The number of impurities in the systems is $N = 25$(●,○), 50 (Δ,Δ), 100 (□,□), 200 (∇,∇), and 400 (◆,◇). The solid curve is drawn as a guideline for the eye connecting the numerical data for $\langle \alpha \rangle$. The dashed and dashed-dotted curves correspond to $2\langle \alpha \rangle$ and $\langle \alpha \rangle/4$, respectively. The density of the impurities is $\nu = 0.01$ (a), 0.05 (b), 0.5 (c), and 1.0 (d).

For infinitessimally small $\Delta x = dx$ we neglect the possibility of more than one impurity in $[x,x+dx]$ and simplify (19) to

$$g(x+dx) = (1+ikdx)g(x) - \phi_0 \Delta \ , \tag{21}$$

where ϕ_0 is the scattering amplitude for a delta function within $[x,x+dx]$ and the random variable Δ is unity for those configurations which have an impurity just to the right of x (with probability ndx) and $\Delta = 0$ if this is not the case (probability $1-ndx$). The amplitude ϕ_0 is sum of two terms:

$$\phi_0 = \mu g(x) - \mu \sum_{x<x_l<L} \phi_l e^{+ik(x_l - x)} \ , \tag{22}$$

where the first term corresponds to the wave incident from the left and the sum represents the effect of the backscattering from the impurities to the right of x.

So far no approximation has been made but we cannot solve the system of Eqs. (21) and (22) exactly. However, an argument similar to that of Lifshitz and Kirpichenkov can be used when the density of impurities is small, $\nu << 1$. The basis of the approximation is the following fact: To calculate $\langle \ln T \rangle$ we can neglect the backscattering terms in (22) and write

$$\phi_0 \approx \mu g(x) \ . \tag{23}$$

The sum in (22) contributes only terms of order ν^2 to $\langle \ln T \rangle$. We shall verify (23) below. Accepting it for the moment we proceed to derive a differential equation for $\langle \ln |g(x)|^2 \rangle$. Substituting (23) into (21), squaring the resulting equation, and taking the logarithm we get

$$\ln |g(x-dx)|^2 = \ln |g(x)|^2 + \ln |1 - \mu\Delta + ikdx|^2 \ . \tag{24}$$

Averaging over all configurations results into

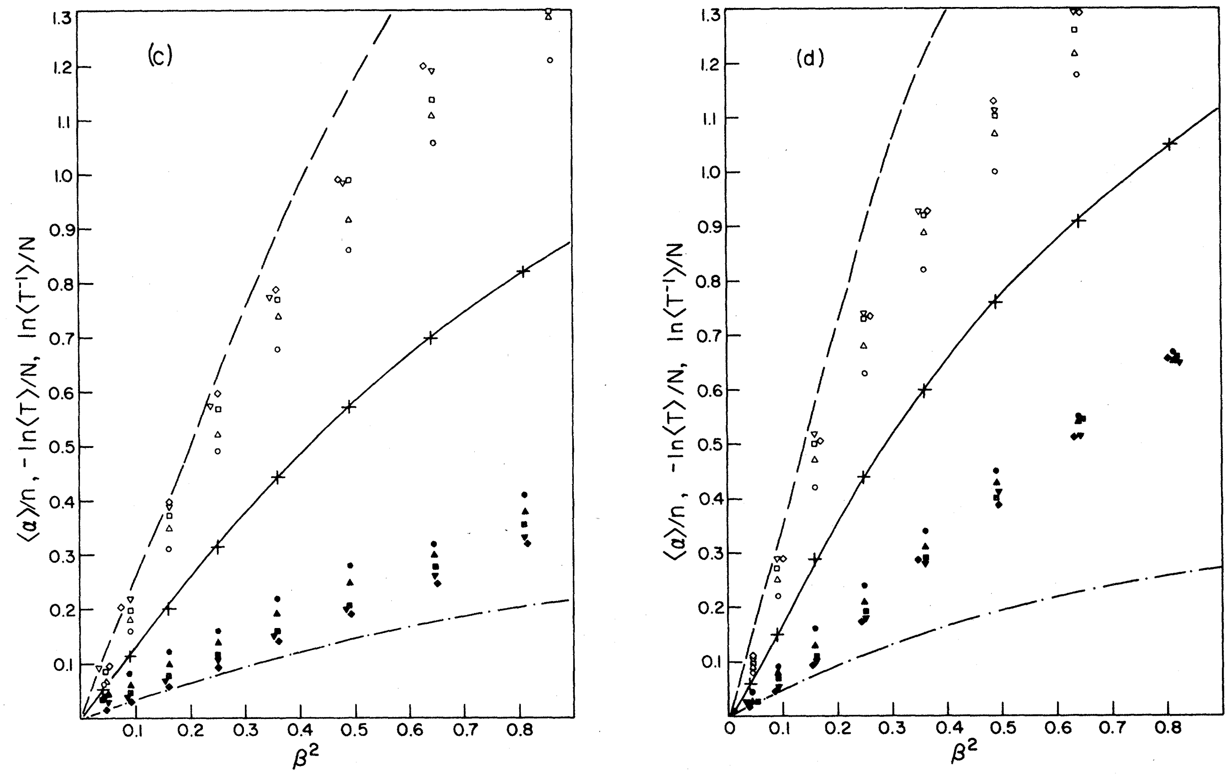

FIG. 2. *(Continued).*

$$\frac{d\langle \ln |g(x)|^2\rangle}{dx} = n \ln |1-\mu|^2 . \tag{25}$$

Setting $x = L$ in the solution of (25) we get for the inverse localization length

$$\langle\alpha\rangle = -\frac{\langle \ln T\rangle}{L} = n\ln(1+\beta^2) + O\left[\frac{n^2}{k}\right] . \tag{26}$$

Now we must return to the justification of (23). It will be done explicitly by including one term from the sum (22) and showing that it leads to higher-order corrections. Let us then assume that there is an additional impurity to the right of the point x at $x+t$, $t>0$. Let us denote by ϕ_1 the scattering amplitude of this impurity. Equation (20) reduces to

$$\phi_0 = \mu g(x) - \mu\phi_1 e^{+ikt} , \tag{27}$$

$$\phi_1 = \mu e^{ikt} g(x) - \mu\phi_0 e^{ikt} . \tag{28}$$

Solving (27) and (28) we have

$$\phi_0 = \frac{\mu-\mu^2 e^{2ikt}}{1-\mu^2 e^{2ikt}} g(x) . \tag{29}$$

Substituting this into (21) we obtain the corrected form of (25)

$$\frac{d}{dx}\langle \ln |g(x)|^2\rangle_t = n\ln|1-\mu^2| - n\ln|1-\mu^2 e^{2ikt}|^2 , \tag{30}$$

where the subscript t at the average denotes averaging over all configurations for which the second impurity to the right of x is located at $x+t$. To get the final result for the localization length we average (30) over t with the probability distribution ne^{-nt}. The correction on the right-hand side of (30) is

$$n^2 \int_0^\infty dt\, e^{-nt} \ln|1-\mu^2 e^{2ikt}|^2 . \tag{31}$$

The crucial point is that the integral multiplying n^2 in (31) does not become of order n^{-1} for small n, because the average of $\ln|1-\mu^2 e^{2ikt}|^2$ is zero:

$$\int_a^{a+\pi/k} \ln|1-\mu^2 e^{2ikt}|^2 = 0 \tag{32}$$

for any a. This shows that our approximation (23) is correct to first order in the density. In Fig. 3 we plotted the low-density expression (26) together with numerical results for low and higher density. The

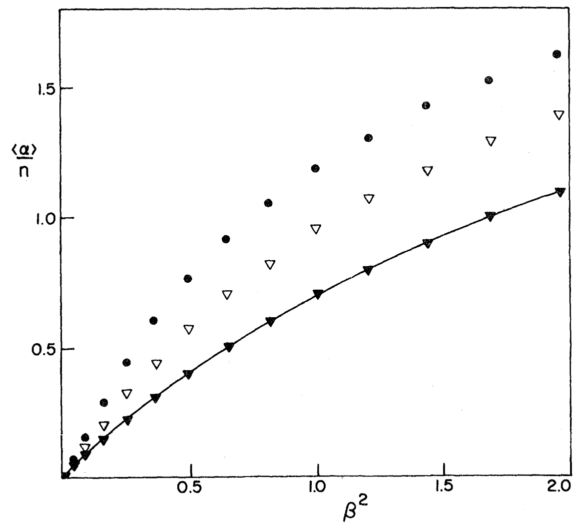

FIG. 3. Comparison between low-density analytical results (full curve) and the numerical data for the averaged inverse localization length $\langle \alpha \rangle$ (in units of the density n) as a function of the potential strength β. The numerical results are extrapolated with $N \to \infty$ for densities $\nu = 0.01$ and 0.05 (▼), 0.5 (▽), 1.0 (●).

agreement is good. The choice of the ensemble is irrelevant for $\langle \alpha \rangle$. It should be noted that this approach does not work for the averages of the transmission coefficient and its inverse, since the correction corresponding to (31) turns out to be of order n in that case.

B. Inverse transmission coefficient

Erdös and Herndon developed a method, based on the transfer-matrix formalism, for the calculation of the average of the inverse transmission coefficient.[17] Their method can be adapted to study the model of this paper. The implementation, as well as the results, depend on the ensemble employed.

We start with ensemble 2. Denoting by $\mathcal{M}(x)$ the transfer matrix of the system of length x we obtain a set of differential equations for the elements of the matrix $\langle \mathcal{M}^\dagger(x)\mathcal{M}(x)\rangle$.[18] Increasing the length of the system by dx the following equation holds to first order in dx:

$$\mathcal{M}^\dagger(x+dx)\mathcal{M}(x+dx) = X^\dagger M^\dagger \mathcal{M}^\dagger(x)\mathcal{M}(x) M X \;, \quad (33)$$

where

$$X = \begin{bmatrix} e^{ikdx} & 0 \\ 0 & e^{-ikdx} \end{bmatrix} \quad (34)$$

and

$$M = \begin{bmatrix} \dfrac{1-i\beta'\Delta}{k} & -\dfrac{i\beta'\Delta}{k} \\ \dfrac{i\beta'\Delta}{k} & 1+\dfrac{i\beta'\Delta}{k} \end{bmatrix} \quad (35)$$

The random quantity Δ is equal to unity if a δ function is in the interval $[x, x+dx]$ and zero otherwise. Introducing the notation

$$\langle \mathcal{M}^\dagger \mathcal{M} \rangle = \begin{bmatrix} A & B+iC \\ B-iC & A \end{bmatrix} \quad (36)$$

and averaging (33) we obtain the differential equations for A, B, and C:

$$\begin{bmatrix} \dfrac{dA}{dx} \\ \dfrac{dB}{dx} \\ \dfrac{dC}{dx} \end{bmatrix} = \begin{bmatrix} 2\beta^2\nu & -2\beta^2\nu & -2\beta\nu \\ 2\beta^2\nu & -2\beta^2\nu & 2(1-\beta\nu) \\ -2\beta\nu & -2(1-\beta\nu) & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} . \quad (37)$$

The average of the inverse transmission coefficient is given by

$$\langle T^{-1} \rangle = \tfrac{1}{2}(1+A) \;. \quad (38)$$

For large x, A is proportional to $e^{\lambda x}$ where λ is the eigenvalue of the matrix (37) with the largest real part.

There are two possibilities: Either all eigenvalues are real or one is real and the remaining two imaginary and complex conjugate. If the latter possibility is realized, the real root is positive and the real part of the imaginary roots are negative. This can be seen as follows: Denoting the matrix (37) by N we define $D(\lambda) = \det|N - \lambda 1|$. The eigenvalues are the roots of D. Since $D(\lambda) \to -\infty$ for $\lambda \to +\infty$ and $D(0) > 0$ there is at least one positive real root. The sum of the roots is seen from (37) to be zero. These two facts imply that the root with the largest real part is real. Thus, in the numerical search for the zeros of $D(\lambda)$ we can restrict ourselves to real values of λ.

For small densities, $\nu << 1$, we get

$$\lim_{L\to\infty} \frac{\ln\langle T^{-1}\rangle}{L} = 2\beta^2 n(1+2\beta\nu) + O(\nu^3) . \qquad (39)$$

The first term in this result is to be compared with the inverse localization length computed above to first order in n. For arbitrary β there is no simple relationship between them but for small β we get from (26)

$$\langle \alpha \rangle = n\beta^2 + O(\nu\beta^3,\nu^2) \qquad (40)$$

which is one-half of the leading term in (39).

To calculate $\langle T^{-1}\rangle$ for ensemble 3 we derive recursion relations for the averages of the elements of the matrix $\mathscr{M}^\dagger\mathscr{M}$

$$\mathscr{M}_{n+1}^\dagger \mathscr{M}_{n+1} = M^\dagger Y^\dagger \mathscr{M}_n^\dagger \mathscr{M}_n Y M , \qquad (41)$$

where $\mathscr{M}_n$ is the transfer matrix for n impurities, M is given by (35), and the propagation matrix Y is

$$Y = \begin{bmatrix} e^{ik\Delta x} & 0 \\ 0 & e^{-ik\Delta x} \end{bmatrix} . \qquad (42)$$

Here Δx is the separation between nth and $(n+1)$st delta functions. We average (41), introduce the notation

$$\langle \mathscr{M}_n^\dagger \mathscr{M}_n \rangle = \begin{bmatrix} A_n & B_n + iC_n \\ B_n - iC_n & A_n \end{bmatrix} \qquad (43)$$

and use

$$\langle \cos 2k\Delta x \rangle = \frac{n^2}{n^2+4k^2} , \qquad (44)$$

$$\langle \sin 2k\Delta x \rangle = \frac{2kn}{n^2+4k^2} . \qquad (45)$$

As a result we get the following recursion relations:

$$\begin{bmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \end{bmatrix} = \begin{bmatrix} 1+2\beta^2 & -2\beta^2 & -2\beta \\ \dfrac{2\beta^2\nu^2-4\beta\nu}{\nu^2+4} & \dfrac{\nu^2-2\beta^2\nu^2+4\beta\nu}{\nu^2+4} & -\dfrac{2\beta\nu^2-2\nu}{\nu^2+4} \\ -\dfrac{4\beta^2\nu+2\beta\nu^2}{\nu^2+4} & -\dfrac{2\nu-4\beta^2\nu-2\beta\nu^2}{\nu^2+4} & \dfrac{4\beta\nu+\nu^2}{\nu^2+4} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} . \qquad (46)$$

Again the exponential growth of $\langle T_n^{-1}\rangle = \frac{1}{2}(1+A_n)$ is determined by the largest eigenvalue of (46). For small density, $\nu << 1$, and arbitrary fixed β we get

$$\lim_{L\to\infty} \frac{\ln\langle T_n^{-1}\rangle}{L} = n\ln(1+2\beta^2) + 4n\nu\frac{\beta^3}{(1+2\beta^2)^2} . \qquad (47)$$

Ensembles 2 and 3 give differing results for $\langle T^{-1}\rangle$. They agree only in the weak-scattering limit and to first order in n (terms of the form $\beta^2 n$).[19]

The largest eigenvalues of the matrices (37) for ensemble 2 and (46) for ensemble 3 were found numerically for a wide range of values of β and ν. The results are shown in Fig. 4 together with the data obtained numerically for ensemble 1.

V. DISCUSSION

A. The inverse localization length

As shown in Fig. 1 the distribution of the logarithm of the transmission coefficient is Gaussian, and obeys the central limit theorem. Its variance is proportional to the square root of the length of the system, as can be seen from the insert of this figure. This behavior is in agreement with earlier results obtained for the isotopically disordered chain,[8] the white-noise potential,[11–13] and the Anderson model of a one-dimensional disordered system.[9] Our numerical result (8) for the variance of the distribution agrees quantitatively with that of the white-noise model in the weak-scattering limit, where the inverse localization length $\langle \alpha \rangle$ is small compared with the density of the impurities. For larger $\langle \alpha \rangle$ the data in Fig. 1 indicate some deviation from the white-noise behavior, the proportionality factor between $L\langle \Delta\alpha^2\rangle/2$ and $\langle \alpha \rangle$ becoming smaller than one and depending on $\langle \alpha \rangle$. However, this does not effect the central-limit behavior. We conclude that the average of $\ln T$ is independent of the statistical ensemble, and can be used to define the inverse localization length through its linear dependence on the size of the system. The ensemble independence of $\langle \alpha \rangle$ is shown in Fig. 3, where we observe close agreement between the numerical results for small density ($\nu < 0.1$), which were obtained from ensem-

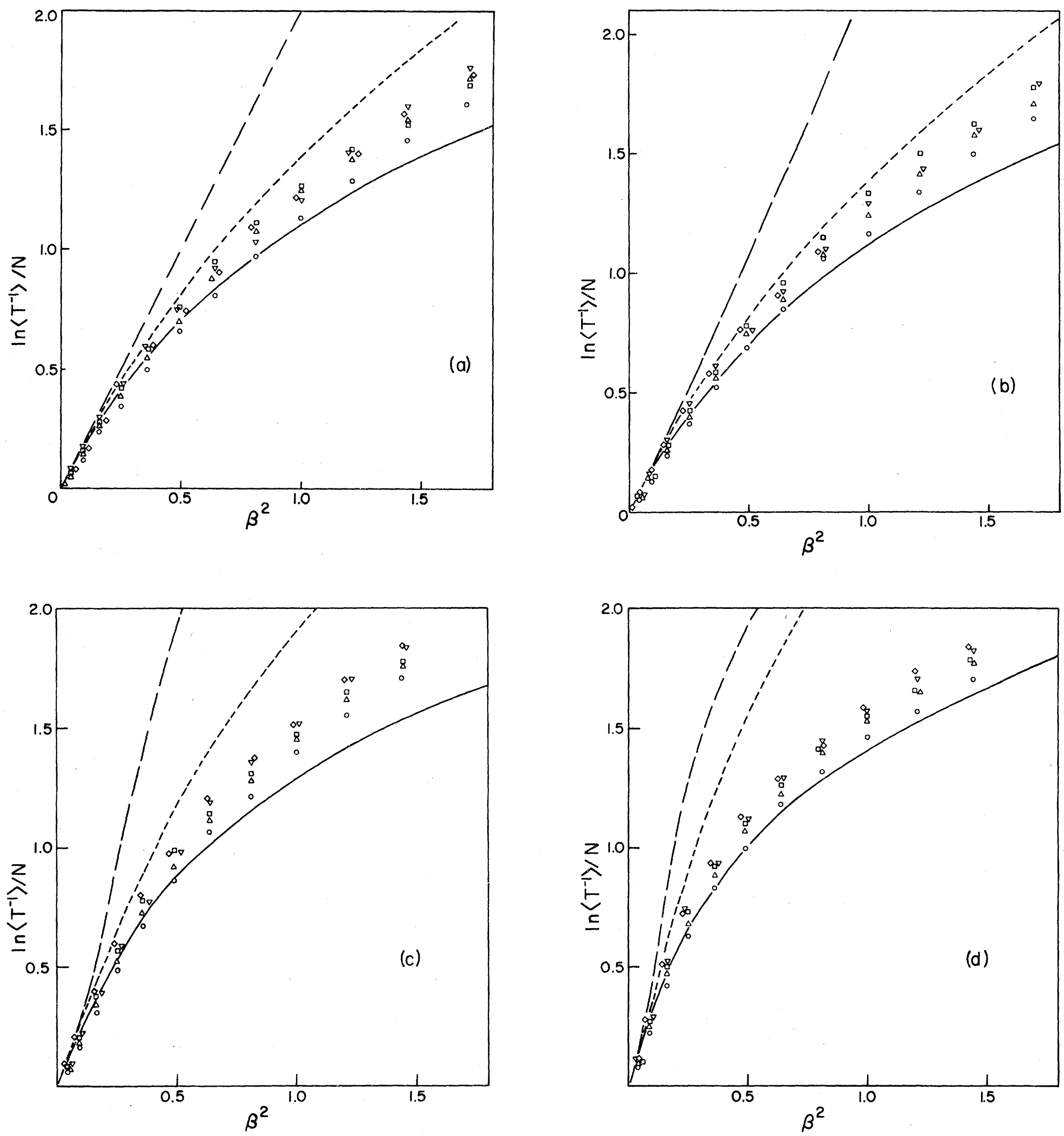

FIG. 4. Comparison between the averages of the inverse of the transmission coefficient $\langle T^{-1}\rangle$ as obtained for the three ensembles described in Sec. II. The density of the impurities is $\nu = 0.01$ (a), 0.05 (b), 0.5 (c), 1.0 (d). Open symbols denote the numerical data (from ensemble 1) for systems containing $N = 25$ (○), 50 (Δ), 100 (□), 200 (∇), and 400 (◇) impurities. The dashed and solid curves denote the results obtained from ensembles 2 and 3 as described in Sec. IV, respectively. For comparison, the large localization length limit $2\langle\alpha\rangle$ is shown as the dotted curve.

ble 1, and the analytical expression (26) derived for small density from ensemble 2 in the preceding section. For weak- ($\beta^2 << 1$) and strong- ($\beta^2 >> 1$) scattering potentials we obtain in this limit $\langle\alpha\rangle = n\beta^2$, and $2n\ln\beta$, respectively.

B. Transmission coefficient

The behavior of the averages of the transmission coefficient and its inverse is more complicated. Since its distribution generally does not obey the central limit theorem we have $\langle T\rangle^{-1} \neq \langle T^{-1}\rangle$. This is shown in Fig. 2. One also expects that these averages depend on the ensemble, which represents the statistical properties of the system. This is shown in Fig. 4. We distinguish two regimes. In the "white-noise limit" ($\langle\alpha\rangle/n << 1$) the localization length is proportional to $(n\beta^2)^{-1}$ and is large compared with the mean spacing between the impurities. The three ensembles yield identical results for $\langle T^{-1}\rangle$. It scales exponentially with the length of the system

$$\ln\langle T^{-1}\rangle = \gamma L ,$$

the growth rate being closely related to the localization length

$$\gamma = 2\langle\alpha\rangle$$

in this regime. This agrees with the scaling law for the resistance found by Anderson *et al.*[4] for a disordered chain consisting of arbitrary scatterers assuming phase randomness between two successive scattering events. It agrees also with the result for the white-noise potential.[11-13]

We were not able to derive analytical expressions for the length dependence of $\langle T\rangle$ for our model. However, the numerical data obtained from ensemble 1 (Fig. 2) indicate that $\langle T\rangle$ decays exponentially with length

$$\ln\langle T\rangle = -\delta L ,$$

and in the limit of large localization lengths the decrease rate δ is related to the inverse localization length by

$$\delta = \langle\alpha\rangle/4 .$$

This agrees with the result for the average conductivity of the white-noise model.[12,14]

For small and intermediate localization length ($\langle\alpha\rangle/n \geq 1$) the averages of T and T^{-1} still depend exponentially on the length of the system but there is no simple relation between the corresponding characteristic lengths and the localization length. There is also a strong dependence on the statistical ensemble as is seen in Fig. 4. For larger density the results obtained numerically from ensemble 1, and analytically from ensemble 3, appear to be close to each other, whereas both disagree with the (much larger) values obtained for ensemble 2. This seems reasonable, since in ensembles 1 and 3 the number of impurities is fixed, whereas in ensemble 2 it is allowed to fluctuate. Thus, in view of the fact that the average value of T^{-1} is dominated by its large values, which correspond to chains with large numbers of impurities, we expect the average of T^{-1} to be larger for ensemble 2 than for ensembles 1 and 3.

VI. CONCLUSION

We summarize our results as follows:

(1) The average logarithm of the transmission coefficient is a physically meaningful quantity, which for large systems scales linearly with the length of the system. Its scaling behavior defines an inverse localization length, which in the low-density limit is given by

$$\langle\alpha\rangle = n\ln(1+\beta^2) + O(n^2) .$$

(2) The transmission coefficient and its inverse do not obey the central limit theorem. The corresponding averages are not representative of the ensemble. In the weak-scattering limit we nevertheless find exponential scaling, which is solely determined by the localization length, in agreement with previous work.

(3) For smaller localization length the averages of the transmission coefficient and its inverse become sensitive to the statistical ensemble which renders these quantities somewhat useless in this regime.

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[19]It can be shown that for a model with an equal number of positive and negative delta functions the terms in (37) and (46), which are linear in β, vanish. The low-density results of (39) and (47) contain then only terms linear in n and are correct to the third power in n.