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London approach to anisotropic type-II superconductors

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The London equations with a phenomenlogical mass tensor are used to analyze the vortex structure near the lower critical field. The anisotropy results in a transverse magnetic field in the Abrikosov vortex. This field attenuates exponentially at large distances from the vortex axis. The strongly anisotropic attenuation length is evaluated. The line energy is found in the London approximation.

Strongly anisotropic type-II superconductors have been studied for some time. Most of the interest has been in the angular dependence of the upper critical field H_{c2}. For anisotropic layered or fiber materials this dependence is described quite well by the Ginzburg-Landau (GL) equations with a phenomenological mass tensor.[1–4] Owing to the linearity of the equations near H_{c2}, a relatively simple solution can be found in this region.[2] However, close to the lower critical field H_{c1} the GL approach leads to nonlinear equations,[5] and it apparently is not possible to follow the Abrikosov approach to obtain H_{c1} in the anisotropic case. In that situation it is helpful to apply the London model, which provides a reasonable approximation at least for large GL parameters κ. Although it has low accuracy, this approach nevertheless makes it possible to predict the existence of a transverse magnetic field in a vortex. This justifies the use of the London model, despite the fact that many important aspects (temperature dependencies, origin of anisotropy, etc.) remain beyond the scope of the theory.

The simplest way to get the London equations is to minimize the energy

$$\epsilon = \int [h^2 + (\lambda_0 \operatorname{curl}\vec{h})^2]\, dV/8\pi \quad ,$$

which is the sum of the magnetic $(h^2/8\pi)$ and the kinetic parts. Here $\lambda_0^2 = \text{const}M_0$ is the squared penetration depth and M_0 is the mass. The generalization to the anisotropic situation[6–8] is obvious: the isotropic mass in λ_0^2 should be replaced by the mass tensor in a way that keeps the kinetic term invariant:

$$8\pi\epsilon = \int (h^2 + \lambda^2 m_{ij} \operatorname{curl}_i \vec{h} \operatorname{curl}_j \vec{h})\, dV \quad . \tag{1}$$

We introduce here $\lambda^2 = \text{const}\overline{M}$ with some mean mass $\overline{M}$; the components m_{ij} represent the effective masses divided by $\overline{M}$. The tensor m_{ij} is diagonal if its principal directions are chosen as coordinate axes ($m_{xx}^0 = M_1/\overline{M}$, $m_{yy}^0 = M_2/\overline{M}$, $m_{zz}^0 = M_3/\overline{M}$). It is convenient[9] to choose $\overline{M}^3 = M_1 M_2 M_3$; then $\det m_{ij} = 1$.

Straightforward minimization of the energy (1) yields the London equations

$$h_i = \lambda^2 m_{kl} e_{lsi} e_{ktj} \frac{\partial^2 h_j}{\partial x_s \partial x_t} \quad , \tag{2}$$

where e_{ikl} is the Levi-Civita tensor. In the isotropic case $m_{ij} = \delta_{ij}$ and Eq. (2) coincides with the usual London equations. Alternatively, Eq. (2) can be derived from the second GL equation

$$\operatorname{curl}_i \vec{h} = (8\pi e/c) f^2 \mu_{ik} (\hbar \vec{\nabla} \eta - 2e\vec{A}/c)_k \quad ,$$

where the order parameter $f^2 = \text{const}$ in the London domain, η in the phase, $\vec{A}$ is the vector potential, and μ_{ik} is the tensor of the inverse masses. Multiplying this by M_{ji} and operating by curl one gets Eq. (2) ($M_{ji}\mu_{ik} = \delta_{jk}$).

To be more specific, we consider here the important case of a layered material: $M_1 = M_2 < M_3$, where M_3 is the principal value of M_{ij} in the direction z_0 normal to the layers. We also use $\overline{M}$ as the unit of mass; then $m_1^2 m_3 = 1$. Bearing in mind that we are going to deal with an arbitrarily directed vortex, we rewrite Eq. (2) in the system of coordinates (x, y_0, z) which is rotated with respect to the principal axes (x_0, y_0, z_0) about the y_0 direction through the angle θ (for a vortex along z, θ is the angle between the vortex axis and the principal direction z_0). The components m_{ij} in the new system are

$$\begin{aligned} m_{xx} &= m_1 \cos^2\theta + m_3 \sin^2\theta \quad , \\ m_{xy} &= m_{yz} = 0, \quad m_{yy} = m_1 \quad , \\ m_{zz} &= m_1 \sin^2\theta + m_3 \cos^2\theta \quad , \\ m_{xz} &= (m_1 - m_3) \sin\theta \cos\theta \quad . \end{aligned} \tag{3}$$

The useful relations

$$m_{xx} + m_{zz} = m_1 + m_3, \quad m_{xx} m_{zz} - m_{xz}^2 = m_1 m_3 \quad , \tag{4}$$

follow from invariancy of m_{ii} and $\det m_{ij}$.

Nothing depends on z for a vortex in the z direc-

tion. The equations (2) now read
($\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$)

$$h_x = \lambda^2 \left[m_{zz}\Delta h_x - m_{xz}\frac{\partial^2 h_z}{\partial y^2} \right] , \tag{5a}$$

$$h_y = \lambda^2 \left[m_{zz}\Delta h_y + m_{xz}\frac{\partial^2 h_z}{\partial x\, \partial y} \right] , \tag{5b}$$

$$h_z = \lambda^2 \left[m_1\frac{\partial^2 h_z}{\partial x^2} + m_{xx}\frac{\partial^2 h_z}{\partial y^2} - m_{xz}\Delta h_x \right] . \tag{5c}$$

The striking feature of these equations is that they cannot be satisfied by the usual Abrikosov vortex solution. For such a vortex in the z direction, $h_x = h_y = 0$; this would mean, as is seen from Eqs. (5a) and (5b), $\partial^2 h_z/\partial y^2 = \partial^2 h_z/\partial x\, \partial y = 0$, which is nonsense. Thus we conclude that in the anisotropic case the vortex must have a more complicated structure than the Abrikosov vortex.

The following simple observation makes a search for a new structure more plausible. Let us imagine a current-carrying cylindrical coil with winding loops inclined from the plane normal to the coil axis z. These inclined loops produce not only an h_z field, but some transverse field $h_{x,y}$ as well. If new current planes are rotated with respect to the xy plane about the y axis, the transverse field has the symmetry of a magnetic moment directed along x.

In the case of a vortex in an isotropic material, inclination of current loops would rise the magnetic energy, and therefore it does not occur.[10] In an anisotropic material, however, the increase of the magnetic energy can be smaller than a corresponding decrease in the kinetic energy, if the current flows in energetically favorable directions.

A brief look at Eqs. (5) makes it quite clear that $\vec{h}(x,y)$ is difficult to determine. However, we are interested not only in the distribution $\vec{h}(x,y)$ itself, but also in the line energy ϵ_L, which determines the lower critical field. The magnetic part of $8\pi\epsilon_L$, namely, $\int h^2 dx\, dy$, is simply expressed in terms of Fourier components

$$\vec{h}_{\vec{k}} = \int \vec{h}(x,y) \exp(-\vec{k}\cdot\vec{r})\, dx\, dy$$

as

$$\int h^2 dS = \sum_j \int |h_{j\vec{k}}|^2 d\vec{k}/4\pi^2 ,$$

where $j = x, y, z$, and $dS = dx\, dy$. The kinetic part is also quadratic in currents, so that the Fourier transform (FT) may prove useful in evaluating ϵ_L.

Let us see first how to get ϵ_L^0 in the isotropic case without use of the actual field distribution. Applying the FT to the isotropic London equation $h_z - \lambda^2 \Delta h_z = \phi_0 \delta(r)$ one gets $h_{z\vec{k}} = \phi_0/(1+\lambda^2 k^2)$, where $\phi_0 = 2\pi\hbar c/2e$. Then,

$$(\mathrm{curl}_x \vec{h})_{\vec{k}} = i k_y h_{z\vec{k}} ,$$

$$(\mathrm{curl}_y \vec{h})_{\vec{k}} = -i k_x h_{z\vec{k}} ,$$

and

$$32\pi^3 \epsilon_L^0 = \phi_0^2 \int \frac{d\vec{k}}{1+\lambda^2 k^2} = 2\pi\phi_0^2 \int_0^\infty \frac{k\,dk}{1+\lambda^2 k^2} . \tag{6}$$

The last integral diverges at ∞ so that one must introduce a cutoff at $k_{\max} = 1/\xi$, where ξ is the vortex core size. Then we have $\epsilon_L^0 = (\phi_0/4\pi\lambda)^2 \ln\kappa$ for $\kappa^2 \gg 1$. Despite the impression that the last inequality is the only one needed for the validity of ϵ_L^0, the comparison with the better GL approach shows that $\ln\kappa$ must also be large.[5]

The following is also important for our treatment below. In the region $\lambda^{-1} \ll k \ll \xi^{-1}$ (or $\xi \ll r \ll \lambda$) which exists in materials with $\kappa \gg 1$ only, $k^2\lambda^2 \gg 1$ and the kinetic part is the main contribution to the energy. To get the same result for ϵ_L^0 one can neglect 1 in the integrand (6) and simultaneously replace 0 in the lower limit by $1/\lambda$ to obtain $\int_{\lambda^{-1}}^{\xi^{-1}} dk/k\lambda^2$. Formally, this replacement is done to get rid of a negative ln for small k's. Actually, the lower limit is in any case of no importance because both field and current are exponentially small at $r \gg \lambda$ ($k \ll \lambda^{-1}$).

We now proceed along these lines to find ϵ_L in the anisotropic case from Eqs. (5). First, one must add the term $\phi_0\delta(\vec{r})/\lambda^2$ on the right-hand side of Eq. (5c) to take into account the singularity on the vortex axis. The FT then yields

$$h_{x\vec{k}} = \phi_0 \lambda_{xz} k_y^2/d , \tag{7a}$$

$$h_{y\vec{k}} = -\phi_0 \lambda_{xz} k_x k_y/d , \tag{7b}$$

$$h_{z\vec{k}} = \phi_0 (1+\lambda_{zz}k^2)/d , \tag{7c}$$

$$d = (1+\lambda_1 k_x^2 + \lambda_{xx} k_y^2)(1+\lambda_{zz}k^2) - \lambda_{xz}^2 k^2 k_y^2 , \tag{7d}$$

where hereafter we adopt the notation $\lambda_{ij} = \lambda^2 m_{ij}$, $\lambda_1 = \lambda^2 m_1$, $\lambda_3 = \lambda^2 m_3$.

Introducing now the FT of the current components $j_{x\vec{k}} = i k_y h_{z\vec{k}}$, $j_{y\vec{k}} = -i k_x h_{z\vec{k}}$, $j_{z\vec{k}} = i(k_x h_{y\vec{k}} - k_y h_{x\vec{k}})$ ($4\pi/c$ is omitted), we get for the kinetic part of ϵ_L:

$$\lambda_{il}\int j_i j_l dS = \lambda_{il}\int j_i(\vec{r}) j_{l\vec{k}} e^{i\vec{k}\cdot\vec{r}} dS\, d\vec{k}/4\pi^2$$

$$= \lambda_{il}\int j_{l\vec{k}} j^*_{i\vec{k}} d\vec{k}/4\pi^2 .$$

We substitute now the $h_{\vec{k}}$'s of Eq. (7) in

$$32\pi^3\epsilon_L = \int (h_{i\vec{k}} h^*_{i\vec{k}} + \lambda_{il} j_{l\vec{k}} j^*_{i\vec{k}})\, d\vec{k}$$

to obtain after elementary manipulations

$$32\pi^3\epsilon_L = \phi_0^2 \int d\vec{k}\,\frac{1+\lambda_{zz}k^2}{d} \quad . \tag{8}$$

The integral here diverges as $k \to \infty$ because d is a polynomial of the 4th order in $\vec{k}$ [see Eq. (7d)]. The integration region, therefore, must be cutoff at k's of the order of $1/\xi$. At first sight a somewhat better result could be achieved by an improved cutoff, if the integration domain in the $\vec{k}$ plane is bounded by the ellipse with semiaxes $\sqrt{m_{xx}}/\xi$, $\sqrt{m_{yy}}/\xi$ in the k_x, k_y directions, respectively. This, however, would mean an attempt to correct the model in the region where the London theory fails to be correct. At least such an "improvement" would be by no means reliable.

We have checked the last thesis by straightforward integration in the elliptical region. The result is $\epsilon_L = \text{const}g(\theta)\ln[\kappa f(\theta)]$, where g and f are angular-dependent functions of order of unity. The integration over the circle of the radius $1/\xi$ gives $\epsilon_L = \text{const}g(\theta)\ln\kappa$ with the same pre-ln factor. We see that the results differ from each other only by a term of order unity added to $\ln\kappa$, but such a term is beyond the accuracy of the London theory.

A further simplification is achieved by neglecting the 1's with respect to λ^2k^2 in the integral (8) and by changing the lower limit of integration over k from zero to $1/\lambda$ (as was discussed above). The integral in Eq. (8), then, is evaluated simply by using polar coordinates in the $\vec{k}$ plane and by integrating first over the polar angle:

$$\epsilon_L = (\phi_0/4\pi\lambda)^2\sqrt{m_{zz}}\ln\kappa \quad . \tag{9}$$

It is seen that ϵ_L depends on the vortex orientation in the material through $m_{zz}(\theta)$ given by Eq. (3).[11] At $\theta=0$ the line energy

$$\epsilon_L(0) = (\phi_0^2\ln\kappa)/16\pi^2\lambda^2 m_1$$

$$[m_{zz}(0) = m_3,\ m_3^{1/2} = m_1^{-1}] \quad .$$

In this case all currents flow in the layers plane and the GL theory can be applied to get[5,12]

$$\epsilon_{\text{GL}}(0) = (\phi_0^2/16\pi^2\lambda^2 m_1)(\ln\kappa m_1 + 0.5)$$

(recall that λ and κ are defined here in terms of $\overline{M}$). As one would expect, our $\epsilon_L(0)$ coincides with $\epsilon_{\text{GL}}(0)$ if a constant addition to $\ln\kappa$ is neglected.

Let us return now to $\vec{h}(\vec{r})$ in real space; in principle, it can be reconstructed by the inverse FT of $\vec{h}_{\vec{k}}$ given by Eqs. (7). Actual integration, however, hardly seems to be possible. The investigation of the asymptotic behavior of $\vec{h}(\vec{r})$ at $r \gg \lambda$ is somewhat simplified by the possibility of factorizing the denominator d in the $\vec{h}_{\vec{k}}$ expressions

$$d = (1+\lambda_{zz}k_x^2+\lambda_3k_y^2)(1+\lambda_1k^2) \quad ;$$

one can check this making use of Eq. (4). Then the integration over k_x in

$$\int \vec{h}_{\vec{k}}\exp(i\vec{k}\cdot\vec{r})\,dk_x\,dk_y/4\pi^2 \quad ,$$

is straightforward, for the poles of the integrand in the complex plane k_x are simply found. Further, the integral over k_y can be evaluated for $r \to \infty$ by the steepest descent method.

Being cumbersome, asymptotic expressions are nevertheless relevant, for just at $r \gg \lambda$ the London approach gives an exact result (i.e., the same as that of the GL theory)

$$h_x \simeq \frac{\phi_0 m_{xz}}{4\pi\lambda}\left(\frac{2\pi}{r\lambda}\right)^{1/2}\sin^2\phi(\alpha_x e^{-r/\Lambda}+\beta_x e^{-r/\Lambda_1}) \quad , \tag{10}$$

$$h_y \simeq \frac{\phi_0 m_{xz}}{4\pi\lambda}\left(\frac{2\pi}{r\lambda}\right)^{1/2}\sin\phi\cos\phi(\alpha_y e^{-r/\Lambda}+\beta_y e^{-r/\Lambda_1}) \quad , \tag{11}$$

$$h_z \simeq \frac{\phi_0}{4\pi\lambda}\left(\frac{2\pi}{r\lambda}\right)^{1/2}(\alpha_z e^{-r/\Lambda}+\beta_z e^{-r/\Lambda_1}) \quad , \tag{12}$$

where (r,ϕ) are polar coordinates in the plane (x,y) in the usual way connected to axes (x,y,z);

$$\alpha_x = -(m_{zz}\Lambda/m_1\lambda)^{1/2}[m_{zz}^2(m_3-m_1)\sin^2\phi + m_3^2 \times (m_{zz}-m_1)\cos^2\phi]^{-1} \quad ,$$

$$\beta_x = \beta_y = m_1^{-3/4}[m_3 - m_1 - (m_3-m_{zz})\cos^2\phi]^{-1} \quad ,$$

$$\alpha_y = (m_1m_3)^{1/2}\alpha_x \quad ,$$

$$\alpha_z = \sin^2\phi(m_{zz}-m_3)m_{zz}(m_1/m_3)^{1/2}\alpha_x \quad ,$$

$$\beta_z = (m_{zz}-m_1)\beta_x \quad .$$

The two different exponential terms in Eqs. (10)–(12) have an origin in two different poles of the denominator $d(k_x,k_y)$ in each of half planes of the complex variable k_x. The attenuation length

$$\Lambda = \Lambda_1(m_{zz}/m_1)^{1/2}[(m_{zz}/m_3)\sin^2\phi + \cos^2\phi]^{-1/2} \tag{13}$$

is nearly always greater than $\Lambda_1 = \lambda\sqrt{m_1}$ because $m_1 \leq m_{zz} \leq m_3$. In other words, the term $\exp(-r/\Lambda)$ is the leading one except in some special situations. The terms $\propto\exp(-r/\Lambda_1)$ should be kept if one is interested in the transition to the isotropic limit ($m_3 \to m_1 \to 1$, $\Lambda \to \lambda \to \lambda_0$); otherwise these terms can be neglected in $h_{x,y}$. Also, the second term in h_z is important if (a) $\theta=0$ (i.e., $m_{zz}=m_3$) and $\alpha_z=0$, and (b) for $\theta\neq0$, α_z is small in a vicinity of $\phi=0$.

In the limiting cases $\theta=0$, $\pi/2$, the components $h_{x,y}=0$. The field $h_z(x,y)$ can be found here by direct integration of Eq. (5c) because at these limits $m_{xz}=0$:

$$h_z(\theta=0)=(\phi_0/2\pi\Lambda_1^2)K_0(r/\Lambda_1) \quad ,$$

$$h_z(\theta=\pi/2)=(\phi_0/2\pi\lambda^2\sqrt{m_1 m_3})K_0(r/\Lambda(\pi/2)) \quad ,$$

where K_0 is the modified Bessel function, and $\Lambda(\pi/2)$ is the Λ of Eq. (13) at $\theta=\pi/2$. One can check that Eq. (12) gives the correct asymptotic behavior in these limits.

It is worth noting that the anisotropy affects very strongly the asymptotics (10)–(12) because it enters the attenuation length Λ itself.

To get an estimate of the transverse field in the intermediate region $\xi << r << \lambda$, we evaluate the magnetic part ϵ_{tr} of the line energy due to the transverse field $h_{x,y}$. This can be done because

$$8\pi\epsilon_{tr}=\int(h_x^2+h_y^2)dS=\phi_0^2\lambda_{xz}^2\int k^2k_y^2 d\vec{k}/4\pi^2 d^2 \quad ,$$

where Eqs. (7a) and (7b) have been used. The main contribution to this energy comes from the domain $\xi << r << \lambda$ ($\lambda^{-1} << k << \xi^{-1}$), and we can treat the last integral as was done in the evaluation of the line energy:

$$\epsilon_{tr}=(\phi_0 m_{xz}/8\pi\lambda)^2(m_3 m_{zz})^{1/2} \quad .$$

Doing the same for the magnetic energy ϵ_{ax} associated with the h_z, one gets

$$\epsilon_{ax}=(\phi_0/8\pi\lambda)^2 m_{zz}^{1/2}(m_{zz}+m_3)m_3^{1/2} \quad .$$

Now, the ratio

$$\epsilon_{tr}/\epsilon_{ax}=m_{xz}^2/m_{zz}(m_{zz}+m_3) \tag{14}$$

gives the relative value of the mean squares of the transverse and axial fields in a vortex. For the layered crystal $NbSe_2$, e.g., $m_1/m_3 \approx 0.09$,[3,4] at 4.2 K and $\epsilon_{tr}/\epsilon_{ax}=0.25$ if a vortex is directed at 45° to the layers. We see that the transverse field is not necessarily small with respect to the usual axial field h_z. If the vortex direction approaches $\theta=0$ or $\pi/2$, $\epsilon_{tr}\rightarrow 0$ due to the factor

$$m_{xz}^2=(m_3-m_1)^2\sin^2\theta\cos^2\theta \quad .$$

Finally, we point out that the transverse field must have a rather strong influence on the vortex-vortex interaction, which at low inductions is determined by the asymptotic behavior of the field. As we saw, the latter is strongly anisotropic even for a moderate value of the ratio m_3/m_1. One can expect, therefore, that the triangular equilateral array is unlikely to occur in an arbitrarily directed vortex lattice in an anisotropic type-II superconductor. The presence of the transverse field should be seen in neutron diffraction experiments. Also, the distribution of $|\vec{h}(x,y)|$ which is more complicated in our case with respect to the isotropic one, is expected to be revealed in NMR experiments.

Note added in proof. K. Takanaka [Phys. Status Solidi (b) 68, 623 (1975)] considered the transverse field in the vortex lattice near H_{c2}; his discussion of the low-field situation is restricted to $\theta=0$, $\pi/2$ cases where $h_{x,y}=0$.

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[11] The pre-ln factor in Eq. (9) coincides with one given in Ref. 6 and predicted in Ref. 2. In both papers it is stated that all results of the GL theory for the isotropic case hold also in an anisotropic material if one replaces the isotropic κ_0 by the angular-dependent quantity $\kappa/m_{zz}^{1/2}$. However, in such a way one cannot get the transverse field in the vortex discussed here, so that the very possibility of "isotropization" of the GL equations is questionable.
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