PHYSICAL REVIEW B VOLUME 24, NUMBER 3 1 AUGUST 1981

Dynamic critical exponent z in some two-dimensional models

Gene F. Mazenko
The James Franck Institute and The Department of Physics, The University of Chicago, Chicago, Illinois 60637

Oriol T. Valls
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455

(Received 6 February 1981)

We discuss the current methods for determining the dynamic critical index z for the dynamic universality class $n=1$, $d=2$ where the nonconserved order parameter is the only slow mode (model A). We conclude that essentially all known methods (ϵ expansions, high-temperature expansions, Monte Carlo calculations, Monte Carlo renormalization-group calculations, and the real-space dynamic renormalization method) are, at their present level of development, inconclusive. We show, in particular, that if we analyze the available high-temperature expansion data using methods similar to those used in carrying out the ϵ expansions, the resulting series is too short to extract any nonconventional value of z. At this level of expansion, the series is compatible with a conventional value of z. We show that these difficulties appear to be associated with the existence of an asymptotic dynamic critical region much narrower than the asymptotic static critical region.

I. INTRODUCTION

A problem of current interest in the field of dynamic critical phenomena is the determination of the dynamic critical index z for the dynamic universality class where the nonconserved order parameter is the only slow mode. This is "model A" in the enumeration of dynamic universality classes due to Halperin, Hohenberg, and Ma.[1] This problem has been studied using two rather different models which are believed to belong to the same dynamic universality class. In both cases the value of z is very uncertain relative to the accuracy that we have for the static critical indices for the same problem. It is worth noting that methods which work extremely well for determining static critical properties give widely differing or inconclusive values for z. These difficulties are particularly acute in two dimensions where most of the work has been carried out. In this paper we will discuss these discrepancies and indicate reasons for these difficulties.

The two models mentioned above which have been studied extensively are the time-dependent Ginzburg-Landau (TDGL) model[2] and the single-spin-flip[3] kinetic Ising (KI) or Glauber[4,5] model. The TDGL model is developed in terms of n-vector fields. It is believed that the $n=1$ TDGL model should be in the same universality class as the KI model.

These models are apparently ideally suited for the application of techniques which have been extremely successful in investigating static critical phenomena. The TDGL model is a prime candidate for the application of momentum-space renormalization-group methods, ϵ expansions, and $1/n$ expansions. The KI model seems well suited to analysis using high-temperature expansions, Monte Carlo methods, and real-space renormalization-group analysis.

Let us briefly recall some of the results in the literature associated with these various approaches. The ϵ expansion ($\epsilon=4-d$, $d=$spatial dimensionality) gives the result for z[1,6,7]:

$$z=2-\eta+z' ,$$

$$z'=\epsilon^2\frac{n+2}{(n+8)^2}\times 3\ln\frac{4}{3}\left[1+\epsilon\left(\frac{6(3n+14)}{(n+8)^2}-1.475\right)+\cdots\right] , \tag{1.1}$$

$$z'=\epsilon^2 0.0320(1-0.216\epsilon)+O(\epsilon^4) \quad (n=1) ,$$

which leads to $z'=0$ for $d=4$, $z'=0.025$ for $d=3$, and $z'=0.073$ for $d=2$. This series seems reasonably well behaved, but we have many fewer terms than exist for the static exponents. Using resummation methods[8] we now have very accurate determinations of static exponents using the ϵ and related expansions.[9] We should have much less confidence in these dynamic results. One can also perform large-n calculations[1] and $\epsilon'=d-2$, $n>2$ expansions[10] for these TDGL models. These calculations are probably not very relevant to the $n=1$ case which is qualitatively different than $n\geq 2$ because of the lack of Nambu-Goldstone modes for $n=1$.

Let us turn next to the KI model. Some of the earliest work in this field was the use of high-

temperature expansions to analyze z for the two-dimensional square lattice case, Yahata and Suzuki[11] analyzed a 9th-order series using the Padé approximant method and obtained $z=2\pm0.05$ while Yahata,[12] and Rácz and Collins,[13] analyzed a 12th-order series, and obtained $z=2.0197$ and $z=2.125\pm0.01$, respectively. Typically high-temperature results and Monte Carlo studies of the same systems have been in good agreement in determining static critical properties. Direct Monte Carlo studies[14] of this problem have given values of $z=1.85\pm0.10$. Here there is a substantial disagreement.

There have been a large number of other types of calculations to determine z using renormalization-group methods. We list a number of these calculations in Table I.[15–28]

An objective survey of these results supports our earlier statement that there are serious difficulties in obtaining reliable estimates of z in two dimensions and in this paper we investigate some sources for these difficulties. We will not resolve the basic question of the precise values of z, but we hope to indicate the degree of quality required of a calculation to do so.

In the next section we define more carefully the models of interest. We carry out in Sec. III some formal development, discuss some exact results, and indicate how one can set up perturbation-theory calculations with the purpose of calculating z for both the TDGL and KI models. In Sec. IV we use the results of the previous sections to analyze the results of high-temperature expansions. In Sec. V we conclude with some discussion of the basic difficulties in this problem.

II. MODELS OF INTEREST

A. TDGL model

The TDGL model is defined in terms of a field $\phi(\vec{x})$ which can vary continuously from $-\infty$ to $+\infty$. The equilibrium statics of this field are governed by the probability distribution

$$P[\phi]=e^{-F[\phi]}/Z \ , \tag{2.1}$$

where $F[\phi]$ is the Ginzburg-Landau-Wilson free-

TABLE I. Results for the dynamic exponent z for the models discussed in the present work, obtained by different methods. (NA indicates not available.)

Ref.	z	ν	Comments
1	2.005	0.989	$0(\epsilon^2)$ calculation for TDGL model
1	2.0	NA	$0(1/n)$ for TDGL
6	1.82	0.837	$0(\epsilon^3)$ calculation for TDGL model
11	2 ± 0.05	1.00	9-term HTE (high-temperature expansion)
12	2.0197	1.00	12-term HTE (square lattice)
12	2.0249	1.005	10-term HTE (triangular lattice)
13	2.125 ± 0.01	1.00	12-term HTE (square lattice)
14	1.85 ± 0.10	(assume $\nu=1$)	Monte Carlo (square lattice)
15	1.4 ± 0.4	$0.96\pm.04$	MCRNG (Monte Carlo renormalization group)
16	1.99	1	MCRNG
17	2.22 ± 0.13	1.07 ± 0.15	MCRNG
18	2.19	0.950	2nd-order cumulant, triangular lattice RSRNG (real space renormalization group)
19	2.23	0.950	same as Ref. 18.
20	2.24	NA	Bond moving, square lattice. RSRNG
21	1.67	2.24	1st-order cumulant, triangular lattice (RSRNG)
22	1.70	1.264	Triangular lattice. RSRNG
23	1.96	0.662	Migdal approx. RSRNG
24	1.879	1	Migdal approx. RSRNG
25	Arbitrary	NA	Triangular lattice. RSRNG
26	1.85 ± 0.15	0.68	Square lattice. numerical RSRNG
27	1.70	1.76	Triangular lattice, (cumulant) RSRNG
27	2.22	0.950	Triangular lattice (different model) RSRNG
28	1.796	0.989	Square lattice RSRNG

energy functional,

$$F[\phi]=\int d^dx\left[\frac{r_0}{2}\phi^2(\vec{x})+\frac{c}{2}[\nabla\phi(\vec{x})]^2+\frac{\bar{u}}{4}\phi^4(\vec{x})\right]$$
$$r_0=r'(T-T_c^0)\ . \tag{2.2}$$

T is the temperature and r', c, $\bar{u}$ are positive temperature-independent constants. Z in Eq. (2.1) is the partition function

$$Z=\int\mathfrak{D}(\phi)e^{-F[\phi]}\ , \tag{2.3}$$

where we carry out a functional integral over all ϕ. The equilibrium correlation functions are given by

$$\langle\phi(\vec{x})\phi(\vec{x}')\rangle=\int\mathfrak{D}(\phi)P[\phi]\phi(\vec{x})\phi(\vec{x}')\ . \tag{2.4}$$

The dynamics of this model are assumed to be generated by a generalized Fokker-Planck operator $\tilde{D}_\phi$ such that time-correlation functions are given by

$$\langle\phi(\vec{x},t)\phi(\vec{x}')\rangle=\langle\phi(\vec{x}')e^{\tilde{D}_\phi t}\phi(\vec{x})\rangle\ , \tag{2.5}$$

where

$$\tilde{D}_\phi\equiv\int d^dx\,\Gamma_0\left[\frac{\delta}{\delta\phi(\vec{x})}-\frac{\delta F[\phi]}{\delta\phi(\vec{x})}\right]\frac{\delta}{\delta\phi(\vec{x})} \tag{2.6}$$

and Γ_0 is a temperature-independent relaxation rate. In this development it is assumed that $\phi(\vec{x})$ contains only wave-vector components less than a cutoff Λ. Many properties of $\tilde{D}_\phi$ and its adjoint D_ϕ are discussed in Ma and Mazenko.[29]

B. KI model

Kinetic Ising models involve dynamics driven by a stochastic spin-flip operator (SFO) acting on a system of Ising spins. We will restrict ourselves here to a two-dimensional square lattice. We shall denote a given Ising spin ($\sigma_i=\pm1$) configuration by $\sigma\equiv\{\sigma_1,\sigma_2,\ldots,\sigma_N\}$ where N is the number of spins and i an index which numbers the lattice points. The Hamiltonian, multiplied by $(-\beta)$, is

$$H[\sigma]=\tfrac{1}{2}K\sum_{\langle ij\rangle}\sigma_i\sigma_j \tag{2.7}$$

with the sum extending to all nearest-neighbor pairs. The equilibrium probability distribution is given by

$$P[\sigma]=e^{H[\sigma]}/Z\ , \tag{2.8}$$
$$Z=\sum_\sigma e^{H[\sigma]}\ . \tag{2.9}$$

Equilibrium correlation functions are defined by

$$\langle\sigma_i\sigma_j\rangle=\sum_\sigma P[\sigma]\sigma_i\sigma_j\ . \tag{2.10}$$

The dynamics of this model are such that the spin-spin time-correlation function is given by

$$C_{ij}(t)=\langle\sigma_j e^{\tilde{D}_\sigma t}\sigma_i\rangle \tag{2.11}$$

and the SFO corresponding to model A can be written in the form

$$\tilde{D}[\sigma|\sigma']=-\frac{\alpha}{2}\sum_i\Lambda^{[i]}_{\sigma,\sigma'}W_i[\sigma]\sigma_i\sigma_i'\ ,$$
$$\Lambda^{[i]}_{\sigma,\sigma'}\equiv\prod_{j\neq i}\delta_{\sigma_j,\sigma_j'}\ , \tag{2.12}$$

where we use a convenient matrix notation: thus $\tilde{D}_\sigma F[\sigma]$ means

$$\tilde{D}_\sigma F[\sigma]\equiv\sum_{\sigma'}\tilde{D}[\sigma|\sigma']F[\sigma']\ . \tag{2.13}$$

The function $W_i[\sigma]$ is restricted[30] by the condition that the equilibrium probability be stationary under time translations

$$D_\sigma P[\sigma]=0;\quad D[\sigma|\sigma']\equiv\tilde{D}[\sigma'|\sigma] \tag{2.14}$$

High-temperature series expansions for this model are invariably performed using the Glauber[4] choice of

$$W_i[\sigma]=(1-\sigma_i\tanh E_i[\sigma])\ , \tag{2.15a}$$
$$E_i[\sigma]=K\sum_j\sigma_j\equiv K\sigma_i^S\ , \tag{2.15b}$$

where, in Eq. (2.15b), the sum extends to the four neighbors of σ_i. We shall, in this paper, use this form of $W_i[\sigma]$. Other forms have also been used[31] in the renormalization-group (RG) methods, but similar considerations would apply. One easily shows[13] that

$$\sigma_i\tanh E_i[\sigma]=A\sigma_i\sigma_i^S+B\sigma_i\sigma_i^T \tag{2.16a}$$
$$\sigma_i^T=\sum_{\langle jkl\rangle}\sigma_j\sigma_k\sigma_l\ , \tag{2.16b}$$

where the last sum extends over the four triplets made up of nearest neighbors of i and

$$A=\tfrac{1}{8}(2\tanh2K+\tanh4K)\ , \tag{2.17}$$
$$B=-\tfrac{1}{8}(2\tanh2K-\tanh4K)\ . \tag{2.18}$$

C. Common development

We can develop much of the formal structure for the TDGL and KI models in terms of a slightly more general model. Let us introduce a variable

$$\psi_q=\begin{cases}N^{-1/2}\sum_i e^{i\vec{q}\cdot\vec{R}_i}\sigma_i & \text{(KI)} \qquad (2.19a)\\ V^{-1/2}\int d^dx e^{i\vec{q}\cdot\vec{x}}\phi(\vec{x}) & \text{(TDGL)}\ , \qquad (2.19b)\end{cases}$$

which is the Fourier transform of the order-parameter of the two problems. $\vec{R}_i$ is the position of the ith Ising spin and V the volume containing the TDGL fields. If we also introduce

$$\tilde{D}_\psi = \begin{cases} \tilde{D}_\sigma \quad \text{(KI)} & \text{(2.20a)} \\ \tilde{D}_\phi \quad \text{(TDGL)}, & \text{(2.20b)} \end{cases} \tag{2.20b}$$

then the order parameter-order parameter time-correlation function for both cases can be written as

$$C(q,t) = \langle \psi_{-q} e^{\tilde{D}_\psi t} \psi_q \rangle \ . \tag{2.21}$$

We will also work with the Laplace transform

$$C(q,z) = -i \int_0^\infty dt e^{izt} C(q,t) \tag{2.22}$$

$$= \langle \psi_{-q} R(z) \psi_q \rangle \ , \tag{2.23}$$

where the resolvent operator is defined by

$$R(z) = (z - i\tilde{D}_\psi)^{-1} \ . \tag{2.24}$$

A useful identity is that

$$\tilde{D}_\psi \psi_q = \phi_q^0 \psi_q + I_q \ , \tag{2.25}$$

where

$$\phi_q^0 = \begin{cases} -\Gamma_0(r_0 + cq^2) \quad \text{(TDGL)} & \text{(2.26a)} \\ -\alpha[1 - 2A(\cos q_x a + \cos q_y a)] \quad \text{(KI)} \ , & \text{(2.26b)} \end{cases}$$

where a is the lattice spacing, and

$$I_q = \begin{cases} -\int \frac{d^d x}{V^{1/2}} e^{i\vec{q}\cdot\vec{x}} \Gamma_0 \bar{u} \phi^3(\vec{x}) \quad \text{(TDGL)} & \text{(2.27a)} \\ +\alpha N^{-1/2} B \sum_i e^{i\vec{q}\cdot\vec{R}_i} \sigma_i^T \quad \text{(KI)} \ . & \text{(2.27b)} \end{cases}$$

III. FORMAL DEVELOPMENT

A. Memory functions

It is now well understood[32] that it is a profitable procedure to shift one's attention from the correlation function $C(q,z)$ to the associated memory function $\phi(q,z)$ defined by the equation

$$[z + i\phi(q,z)]C(q,z) = \tilde{C}(q) \ , \tag{3.1}$$

where $\tilde{C}$ is the static correlation function $C(q,t=0)$. There are two procedures one can develop for calculating $\phi(q,z)$. The first, which was developed in Ref. 33, will be discussed in this subsection. The other method, which involves the introduction of response functions, will be discussed in Sec. III B.

It seems rather natural to divide ϕ into a static part $\phi^{(s)}$ and a dynamic part $\phi^{(d)}$:

$$\phi(q,z) = \phi^{(s)}(q) + \phi^{(d)}(q,z) \tag{3.2}$$

and some straightforward algebra shows that

$$\phi^{(s)}(q)\tilde{C}(q) = -\langle \psi_{-q} \tilde{D}_\psi \psi_q \rangle \ , \tag{3.3}$$

$$\phi^{(d)}(q,z)\tilde{C}(q) = -i[\langle (\tilde{D}_\psi \psi_{-q}) R(z) (\tilde{D}_\psi \psi_q) \rangle - \langle (\tilde{D}_\psi \psi_{-q}) R(z) \psi_q \rangle C^{-1}(q,z) \langle \psi_{-q} R(z) (\tilde{D}_\psi \psi_q) \rangle] \ . \tag{3.4}$$

We can go a bit further with this development if we notice first that we can evaluate

$$\Gamma^{(s)} \equiv -\langle \psi_{-q} \tilde{D}_\psi \psi_q \rangle \tag{3.5}$$

exactly for both of our models. Using the explicit form of the equilibrium probability distribution we easily show for the TDGL model that $\Gamma^{(s)} = \Gamma_0$ (TDGL). We evaluate $\Gamma^{(s)}$ for the KI case in Appendix A with the help of the known exact calculations of short-range static correlations[34] and an identity derived from the minimal coupling[31] form of $W_i[\sigma]$. It follows from Eq. (A5) that $\Gamma^{(s)} = \alpha f(K)$ where $f(K)$ is a positive definite function of K which we plot as a function of $u \equiv \tanh K$ in Fig. 1. Notice that it is a rapidly varying function of u at all $u < u_c$. It has the finite value at the critical coupling $u_c = \sqrt{2} - 1$, $f(u_c) = 0.2823$.

We can also say something about the dynamic part of ϕ: $\phi^{(d)}$ is one-body irreducible. That is, any part of $\tilde{D}_\psi \psi_q$ that is linear in ψ_q will not contribute to $\phi^{(d)}$. Consequently we can write

$$\phi^{(d)}(q,z)\tilde{C}(q) \equiv \Gamma^{(d)}(q,z) = -i[\langle I_{-q} R(z) I_q \rangle - \langle I_{-q} R(z) \psi_q \rangle C^{-1}(q,z) \langle \psi_{-q} R(z) I_q \rangle] \ . \tag{3.6}$$

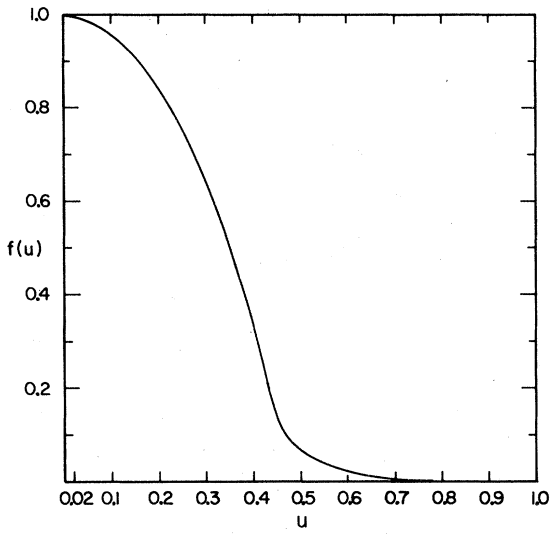

FIG. 1. The quantity $f(K) \equiv \Gamma^{(s)}/\alpha$ evaluated exactly for the KI model, as a function of $u \equiv \tanh K$.

We immediately see that $\Gamma^{(d)}(q,z)$ is of second order in the coupling between components of ψ_q. In the TDGL case $\Gamma^{(d)} \sim O(\bar{u}^2)$ while in the KI case $\Gamma^{(d)} \sim O(B^2) \sim O(K^6)$. As a first approximation in an ϵ expansion [where $\bar{u} \sim O(\epsilon)$] or in a high-temperature expansion in K we would neglect the dynamic part of the memory function relative to the static part.

This approximation leads to the conventional theory for the dynamic index z. We have in this case that the characteristic frequency for the system is simply

$$\omega_c^0 = \phi^{(s)}(q=0) = \Gamma^{(s)}\chi^{-1} \ , \tag{3.7}$$

where χ is the exact static susceptibility. The dynamic critical index z is defined by $\omega_c \sim \xi^{-z}$ where ξ is the correlation length. Since $\chi \sim \xi^{2-\eta}$, we obtain in this case $z \equiv z_0 = 2 - \eta$.

We are more concerned with the role of the nonlinear corrections to the conventional theory given by $\phi^{(d)}$. One can show, following the analysis of Kawasaki,[5] that $\phi^{(d)}(0,0) \leq 0$. On the other hand the characteristic frequency for the system can be defined by

$$\omega_c \equiv \phi(0,0) \tag{3.8}$$

and if the system is stable then

$$\phi(0,0) \geq 0 \ . \tag{3.9}$$

This tells us immediately that

$$\phi^{(s)}(0) \geq \phi^{(d)}(0,0) \tag{3.10}$$

and

$$\phi^{(s)}(0) \geq \phi(0,0) \ . \tag{3.11}$$

Then for large enough ξ

$$\phi^{(s)}(0) = c\xi^{-z_0} \ , \tag{3.12}$$

$$\phi(0,0) = c'\xi^{-z} \ , \tag{3.13}$$

and

$$c\xi^{-z_0} \geq c'\xi^{-z} \tag{3.14}$$

so that

$$z \geq z_0 = 2 - \eta \ , \tag{3.15}$$

which is the Kawasaki[5] bound in the problem.

We then ask what must occur if z is to take on "nonclassical" values $x > z_0$? For this to occur $\phi^{(s)}(0)$ and $\phi^{(d)}(0,0)$ must cancel identically at the critical point. We can state this result in the form

$$\Gamma^{(s)} + \Gamma^{(d)}(0,0) = D\xi^{-(z-z_0)} \ . \tag{3.16}$$

As $T \to T_c$, $\Gamma^{(s)}$ goes to a constant and therefore, for $z \neq z_0$ $\Gamma^{(d)} \doteq -\Gamma^{(s)} + D\xi^{-(z-z_0)}$. This seems like a theoretically discouraging result. Any approximate calculation, including a high-temperature series expansion, is very likely, particularly considering (Fig. 1) the rapid variation of $f(K)$ with the temperature, to treat inadequately this delicate cancellation, and hence to lead to inaccurate results for z.

B. Response functions

We show in this section that we can rearrange our calculation of the characteristic frequency $\omega_c \equiv \phi(0,0)$ in a way that (i) the Kawasaki bound is manifestly satisfied and (ii) a result $z - z_0 > 0$ arises from the divergence of a quantity, not from a delicate cancellation. The approach we use is essentially that used in the development of the ϵ-expansion method for the TDGL model.[1] Ma and Mazenko[29] introduced a response function

$$G(q,z) = \langle \psi_{-q} \bar{R}(z) \psi_q \rangle \ , \tag{3.17}$$

where the resolvent $\bar{R}(z)$ is related to that introduced earlier by

$$\bar{R}(z) = R(z)(-i\tilde{D}_\psi) \ . \tag{3.18}$$

This quantity has the property that

$$G(q,z=0) = \tilde{C}(q) \ . \tag{3.19}$$

Since

$$zR(z) = 1 - \bar{R}(z) \tag{3.20}$$

the response and correlation functions are related by

$$zC(q,z) = \tilde{C}(q) - G(q,z) \ . \tag{3.21}$$

Formally we can define a memory function $\bar{\phi}(q,z)$ to be associated with the correlation function

$$G(q,z) = \langle \psi_{-q} R(z) (-i\tilde{D}_{\psi}\psi_q) \rangle \tag{3.22}$$

via

$$G(q,z) \equiv \frac{\langle \psi_{-q}(-i\tilde{D}_{\psi}\psi_q)\rangle}{z + i\bar{\phi}(q,z)} \ . \tag{3.23}$$

It is then straightforward to show using Eqs. (3.1), (3.8), (3.21), and (3.23) that in the small-q and -z limit

$$\omega_c = \Gamma(0,0)\chi^{-1} \tag{3.24}$$

and

$$\frac{\Gamma^{(s)}}{\Gamma(0,0)} \equiv Q = \left[1 + i\left[\frac{\partial \bar{\phi}^{(d)}(0,z)}{\partial z}\right]_{z=0}\right] \tag{3.25}$$

and the dynamic part of $\bar{\phi}^{(d)}(q,z)$ is given by

$$\bar{\phi}^{(d)}(q,z)\Gamma^{(s)} = -[\langle I_{-q}\bar{R}(z) I_q\rangle - \langle I_{-q}\bar{R}(z)\psi_q\rangle G^{-1}(q,z)\langle \psi_{-q}\bar{R}(z) I_q\rangle] \ . \tag{3.26}$$

With this formulation we see that we retain a value for $z > z_0$ only if the quantity Q diverges. It seems completely reasonable to develop any perturbation-theory analysis in terms of Q. We discuss the results of such calculations in the next section.

IV. HIGH-TEMPERATURE EVALUATION OF Q

Once we have developed the formal structure of the last section we must come up with some way of evaluating Q. It was a direct expansion of Q in powers of u for the TDGL model which led to the result (1.1) when combined with renormalization-group ideas. Therefore it is not necessary for us to reanalyze these results other than to say that one would be comfortable with their numerical reliability only if one had a much longer series and resummation methods could be applied.[35]

An apparently much more reliable method for determining Q is, for the KI model, the use of high-temperature expansion methods. A great deal of work in computing series-expansion coefficients for this problem has been carried out and we shall set up our calculation so as to take advantage of this previous work. In particular, the author of Ref. 12 has worked out directly the quantity

$$a_1 = \int_0^{\infty} dt C(q=0,t) \tag{4.1}$$

to order $u^{12}(u = \tanh K)$. We can use these results to evaluate Q as a power series in u since

$$\frac{1}{\Gamma(0,0)} = \frac{a_1}{\chi^2} \tag{4.2}$$

we have that

$$Q = \frac{\Gamma^{(s)}}{\Gamma(0,0)} = \frac{\Gamma^{(s)} a_1}{\chi^2} \ . \tag{4.3}$$

We need then the expansions for $\Gamma^{(s)}$ and χ as power series in u. The required expansion for $\Gamma^{(s)}$ [see Eq. (A5)] requires the expansion first, of A and B:

$$A = u - 3u^3 + 15u^5 - 85u^7 + 493u^9 - 2871u^{11} + O(u^{13}) \tag{4.4a}$$

$$B = -2u^3 + 14u^5 - 84u^7 + 492u^9 - 2870u^{11} + O(u^{13}) \ . \tag{4.4b}$$

The expansions for $\epsilon(0,1)$, $\epsilon(1,1)$, and $\epsilon(2,0)$ (see Appendix A for definitions), which are needed to get ϵ_T and $\Gamma^{(s)}$ [Eq. (A5)], are obtained from Ref. 36. To the order needed:

$$\epsilon(0,1) = u + 2u^3 + 4u^5 + 12u^7 + 42u^9 + 164u^{11} + O(u^{13}) \ , \tag{4.5a}$$

$$\epsilon(1,1) = 2u^2 + 4u^4 + 10u^6 + 32u^8 + 118u^{10} + O(u^{12}) \ , \tag{4.5b}$$

$$\epsilon(2,0) = u^2 + 6u^4 + 16u^6 + 46u^8 + 158u^{10} + O(u^{12}) \ . \tag{4.5c}$$

From these, the result for ϵ_T using Eq. (A11) is found:

$$\epsilon_T = 3u^3 + 10u^5 + 26u^7 + 80u^9 + O(u^{11}) \ . \tag{4.6}$$

These are sufficient to calculate $\Gamma^{(s)}$ to order u^{12}. The results are listed as the first column of Table II. The expansion coefficients for χ are available[37] to order u^{21}. We list the expansion coefficients of a_1, χ^{-2}, and $\Gamma(0,0)^{-1}$ in Table II. Putting all of these results we obtain for Q the expansion

$$Q = 1 + \tfrac{16}{3}u^6 + 368/27u^8 + 4576/27u^9 + 4912/27u^{10} - 7912/27u^{11} + 887.433u^{12} + O(u^{13}) \ . \tag{4.7}$$

Note first, as pointed out before, that these "dynamical" correlations do not begin until order u^6 [see Eq. (3.26)]. This serves as a check of our algebra in multiplying out the independent calculations of $\Gamma^{(s)}$, a_1, and χ^2. Next we note that it is obvious from inspection of Eq. (4.7) that the series is very difficult to analyze; first, one simply does not have many coeffi-

TABLE II. Coefficients of the expansion in powers of $u \equiv \tanh K$ of the quantities indicated.

Order	$\Gamma^{(s)}$	a_1	χ^{-2}	$\Gamma^{-1}_{(0,0)}$	Q
0	1	1	1	1	1
1	0	8	−8	0	0
2	−4	44	24	4	0
3	0	200	−40	0	0
4	4	804	72	12	0
5	0	2984	−168	0	0
6	−28	31 372/3	360	196/3	16/3
7	0	105 272/3	−776	0	0
8	132	3 069 692/27	1704	5588/27	368/27
9	0	9 674 456/27	−3752	4576/27	4576/27
10	−900	29 807 204/27	8184	35 324/27	4912/27
11	0	3 336 900.146	−17768	10 392/27	−7912/27
12	4668	9 921 382.374	38808	5062.692	887.433

cients to work with, because of the initial string of zeros, and, next, the sign of the u^{11} coefficient shows that the sign is neither constant nor alternated. The ratio method cannot be used. We have worked out the Padé approximant results for the logarithmic derivative of Q. Again, because of the initial string of zeros, not many approximants can be constructed. The results are in Table III and are clearly hopeless. Since u_c is known exactly, one can also attempt to estimate $(z - z_0)$ by evaluating the Padé approximants for $(u - u_c)(Q'(u)/Q)$ at $u = u_c$. The results (Table III) are clearly inconclusive. It is well known that the Padé approximant method works best when the exponent is large. Hence, one can take these results as an indication that $z - z_0$ is probably small. It is interesting to simply plot Q vs u as given by Eq. (4.7) as shown in Fig. 2. We have also plotted the high-temperature expansion for $1/\Gamma^{(s)}$ including terms up to u^{12}. We know that $1/\Gamma^{(s)}$ is finite for all u. We see, however, that it is always greater than Q. An objective observer might well conclude that for correlation lengths associated with a 12 term high-temperature series the effective value of z is given by its conventional value.

These results are to be contrasted with those of Ref. 11: $z = 2.0 \pm 0.05$ from analysis of the series for a_1 directly, while Rácz and Collins[13] analyzed $a_1 \chi^{-1}$ and obtained $z = 2.125 \pm 0.01$.

TABLE III. Top: results for u_c and $z - z_0$ obtained from a Padé analysis of the series for Q'/Q. An asterisk means no solution for u_c was found. Bottom: results for $z - z_0$ obtained from a Padé analysis of the series for $(u - u_c)\ Q'/Q$.

$N \backslash D$	2	3	4
2	*	$u_c = 0.3$ $z - z_0 = 0.013$	*
3	*	$u_c = 0.43$ $z - z_0 = 0.295$	
4	$u_c = 2.85$ $z - z_0 >> 1$		
2	0.143	0.490	0.145
3	0.223	0.217	
4	0.216		

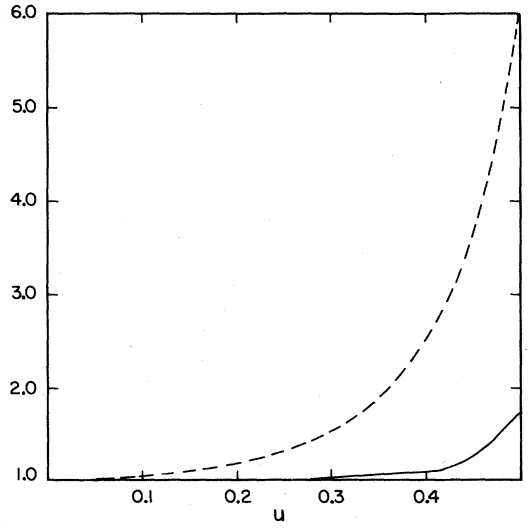

FIG. 2. Comparison of the high-temperature series results for Q and $\Gamma^{(s)-1}$ in the KI model. The solid line is Q from Eq. 4.7 and the dashed line is the series expansion of $1/\Gamma^{(s)}$.

V. DISCUSSION

In the light of the preceding results and considerations, let us review the status of the various methods for computing z in two dimensions.

A. Field theory

The ϵ-expansion method works near four dimensions and leads to a plausible estimate for z in three dimensions. Extrapolation of these results to two dimensions gives a value of $z=1.823$ but one expects the ϵ expansion to be rather poorly behaved in two dimensions.

B. High-temperature expansions

A main point of this paper is that the high-temperature expansion method, which has been seen as the most reliable for treating this problem, is not yet useful because the series is too short. We are becoming increasingly aware[38] that we need rather long series if we are to be able to make quantitative statements. In this case we need many more terms.

C. Monte Carlo calculations

These calculations have been inconclusive for much the same reason the high-temperature expansions have been inconclusive. As Rácz and Collins[13] have pointed out the asymptotic dynamic critical region appears to be much narrower than the asymptotic static critical region. Thus, while one may obtain very good results for static quantities one may not be within the dynamic critical region and may therefore obtain poor values for the dynamic critical properties (these will probably be close to the conventional values). In the case of the high-temperature expansion, going to small reduced temperature and larger correlation length requires sampling larger distances on the lattice which in turn requires going to higher order in the expansion. We should point out that the one-dimensional KI model does not provide a good test in this regard. In this case the interesting quantity Q defined by Eq. (3.26) is exactly one and there is no crossover from conventional dynamics to a $z > z_0$ regime.

D. Monte Carlo RG methods

These methods do not appear to be sensitive enough to obtain z. From what we said above one can see that it is not sufficient to have good static results and reproduce the one-dimensional result. We also add that the existence of a dynamical scaling regime does not guarantee that it is the asymptotic scaling regime. We expect that there will be a region of temperatures (effective correlation length) where one has dynamic scaling with a conventional value of z and this region may be rather broad. We point out that a possibly confusing feature in these calculations is the rather rapidly varying (although noncritical) temperature dependence associated with the initial decay rate $\Gamma^{(s)}$. One must be careful not to confuse the rapid decrease of $\Gamma^{(s)}$ near T_c with a source of additional critical slowing down.

E. Real-space dynamic RG (RSDRG)

This method in its various current manifestations is not sufficiently accurate to give reliable values for z. In the most recent version of our own RSDRG method[39] we find qualitative agreement with the picture of a very narrow region of asymptotic dynamic critical phenomena. At low order in a systematic expansion in the coupling between cells we find that z is given by the conventional value. At the same order of development, however, we find that the static critical indices have values very close to their known asymptotic values.

We must conclude that we do not yet have in two dimensions an accurate determination of z. This is due in the KI model to the narrowness of the asymptotic dynamic critical region relative to the asymptotic static critical region. This narrowness follows from the nonuniversal feature that $Q=1+O(u^6)$ in a high-temperature expansion and we must go very near T_c (have a very large correlation length) for Q to deviate substantially from 1. To resolve the basic question of the value of z apparently requires either a much higher-order high-temperature expansion, a more clever expansion method, or analysis of another model in the same universality class but with a broader asymptotic dynamic critical region.

ACKNOWLEDGMENTS

G.F.M. was supported by NSF Grant No. DMR 77-12637 and an Alfred P. Sloan Foundation Fellowship. O.V. was supported by the University of Minnesota Graduate School Research Fund.

APPENDIX A: EXACT EVALUATION OF $\phi^{(s)}$ FOR THE KI MODEL

From Eq. (3.3) we have

$$\phi^{(s)}(q)\tilde{C}(q)=-\frac{1}{N}\sum_{ij}e^{i\vec{q}\cdot(\vec{R}_i-\vec{R}_j)}\Gamma_{ij} \ , \tag{A1}$$

$$\Gamma_{ij}=\langle\sigma_j\tilde{D}_\sigma\sigma_i\rangle \ . \tag{A2}$$

From Eq. (2.12) it follows that

$$\Gamma_{ij} = -\alpha \langle \sigma_j \sigma_i W_i[\sigma] \rangle \ . \tag{A3}$$

But since the product $W_i[\sigma]P[\sigma]$ is independent of σ_i it is clear that Γ_{ij} will vanish unless $i=j$:

$$\Gamma_{ij} = -\alpha \delta_{ij} \langle W_i[\sigma] \rangle \tag{A4}$$

from which it immediately follows that

$$\phi^{(s)}(q)\tilde{C}(q) = \alpha[1 - 4A\,\epsilon(0,1) - 4B\,\epsilon_T] = \Gamma^{(s)} \ . \tag{A5}$$

Here, $\epsilon(0,1)$ is the nearest-neighbor correlation function: if we denote by $\vec{\delta}_\alpha$ ($\alpha = 1, 2, 3, 4$) the four shortest lattice vectors

$$\epsilon(01) = \langle \sigma_i \sigma_{i+\vec{\delta}_\alpha} \rangle \ , \tag{A6}$$

while ϵ_T is the four-spin–static-correlation function

$$\epsilon_T = \langle \sigma_i \sigma_{i+\vec{\delta}_1} \sigma_{i+\vec{\delta}_2} \sigma_{i+\vec{\delta}_3} \rangle \ . \tag{A7}$$

An exact expression for $\epsilon(01)$ is given in Ref. 34.

To evaluate ϵ_T consider the function

$$\tilde{W}_i[\sigma] = 1 + A_1 \sigma_i \sigma_i^s + A_1^2 \sigma_i'' \ , \tag{A8}$$

where

$$\sigma_i'' = \sum_\alpha \sigma_{i+\vec{\delta}_\alpha} \sigma_{i+\vec{\delta}_{\alpha+1}} \ , \tag{A9a}$$

$$A_1 = -\tfrac{1}{2} \tanh 2K \ . \tag{A9b}$$

It is known[31] that $\tilde{W}_i[\sigma]P[\sigma]$ is independent of σ_i, and, hence, for any $i \neq j$ one has

$$\langle \sigma_i \sigma_j \tilde{W}_i[\sigma] \rangle = 0 \ . \tag{A10}$$

Choosing for j a nearest neighbor to i leads to the identity

$$0 = \epsilon(0,1) + A_1[1 + 2\epsilon(1,1) + \epsilon(2,0)] + 2A_1^2[\epsilon(0,1) + \epsilon_T] \ , \tag{A11}$$

where $\epsilon(m,n)$ is the static-correlation function between two spins separated by m lattice points horizontally and n vertically. An exact expression for $\epsilon(1,1)$ is given in Ref. 34. For $\epsilon(2,0)$ one can use the general formulas for $\epsilon(m,0)$ which are also given in Ref. 34. The calculation is a tedious but straightforward exercise in Jacobi elliptic functions. We shall not write down explicitly here the lengthy final result for $\epsilon(0,2)$. In the notation of Ref. 34 [see their Eq. (4.1)] five matrix elements Σ_0, $\Sigma_{\pm 1}$, $\Sigma_{\pm 2}$ are needed, of which the first three are given by Eqs. (68) and (70) of Ref. 34. We obtain for $\Sigma_{\pm 2}$ the result (for $T \geq T_c$):

$$\begin{aligned}\pi \Sigma_{\pm 2} = &-(1-\kappa^2)K(\kappa)\frac{1}{\kappa(1+\kappa)^{1/2}}[\kappa^2 - \kappa - 1 \mp \kappa^{1/2}(1+\kappa)^{1/2}(\kappa-1)] \\ &+ \frac{(1+\kappa)^{3/2}}{\kappa}\left\{2 - 3(1+\kappa)\left[1 \mp \left(\frac{\kappa}{1+\kappa}\right)^{1/2}\right]E(\kappa_1)\right\} \\ &+ \pi\left\{\frac{(1+\kappa)^{1/2}}{\kappa}\left[1 \mp \left(\frac{\kappa}{1+\kappa}\right)^{1/2}\right][(1+\kappa^2)+\kappa] - \tfrac{1}{2}(1+\kappa)^{1/2}\left[1 \mp 2\left(\frac{\kappa}{1+\kappa}\right)^{1/2}\right] - (1+\kappa)^{3/2}\frac{1}{\kappa}\right\}\end{aligned} \tag{A12}$$

with:

$$\kappa = \sinh^2 2K \ , \tag{A13a}$$

$$\kappa_1 = \frac{2\kappa^{1/2}}{1+\kappa} \ , \tag{A13b}$$

and $K(\kappa)$, $E(\kappa_1)$ are complete elliptic integrals. Once $\epsilon(0,2)$ is known, it is trivial to obtain ϵ_T exactly from Eq. (A11).

[1]B. Halperin, P. Hohenberg, and S. K. Ma, Phys. Rev. B 10, 139 (1974).
[2]See S. K. Ma, *The Modern Theory of Critical Phenomena* (Benjamin, Reading, 1976).
[3]We restrict ourselves to the single-spin flip KI model.
[4]R. Glauber, J. Math. Phys. (N.Y.) 2, 263 (1966).
[5]K. Kawasaki, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 2.
[6]C. De Dominicis, E. Brézin, and J. Zinn-Justin, Phys. Rev. B 12, 4945 (1975).
[7]In Ref. 1, z is written as $z = z + c\eta$ where $c = 0.7621\ (1 - 1.687\epsilon) + O(\epsilon^4)$. The poor behavior for c as a function of ϵ is due to the behavior of η. This does not appear to be the best representation for z.
[8]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev.

D 15, 1554, 1558 (1977).
[9]J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. 39, 95 (1977).
[10]C. De Dominicis, S. K. Ma, and F. Peliti, Phys. Rev. B 15, 4313 (1977).
[11]H. Yahata and M. Suzuki, J. Phys. Soc. Jpn. 27, 1421 (1969).
[12]H. Yahata, J. Phys. Soc. Jpn. 30, 657 (1971).
[13]Z. Rácz and M. F. Collins, Phys. Rev. B 13, 3074 (1976).
[14]E. Stoll, K. Binder, and T. Schneider, Phys. Rev. B 8, 3266 (1973).
[15]S. K. Ma, Phys. Rev. Lett. 37, 461 (1976).
[16]M. P. Nightingale and H. W. J. Blöte (unpublished).
[17]J. Tobochnik, S. Sarker, and R. Cordery (unpublished).
[18]Y. Achiam and J. Kosterlitz, Phys. Rev. Lett. 41, 128 (1978).
[19]Y. Achiam, J. Phys. A 11, L129 (1978).
[20]Y. Achiam, J. Phys. A 13, 1355 (1980).
[21]G. F. Mazenko, M. Nolan, and O. T. Valls, Phys. Rev. Lett. 41, 500 (1978).
[22]W. Kinzel, Z. Phys. B 29, 361 (1978).
[23]M. Suzuki, K. Sogo, I. Matsuba, H. Ikada, T. Chikama, and H. Takano, Prog. Theor. Phys. 61, 864 (1979).
[24]S. T. Chui, G. Forgács, and H. L. Frisch, Phys. Rev. B 20, 243 (1979).
[25]U. Dekker and F. Haake, Z. Phys. B 36, 379 (1980).
[26]S. K. Ma, Phys. Rev. B 19, 4824 (1979).
[27]G. F. Mazenko, M. Nolan, and O. T. Valls, Phys. Rev. B 22, 1275 (1980).
[28]G. F. Mazenko, J. Hirsch, M. Nolan, and O. T. Valls, Phys. Rev. Lett. 44, 1083 (1980).
[29]S. K. Ma and G. F. Mazenko, Phys. Rev. B 11, 4077 (1975).
[30]For a full discussion see: G. F. Mazenko, M. Nolan, and O. T. Valls, Phys. Rev. B 22, 1263 (1980).
[31]See Ref. 28, and G. F. Mazenko, J. Hirsch, M. Nolan, and O. T. Valls [Phys. Rev. B 23, 1431 (1981)] for a discussion of a more "local" single spin-flip operator.
[32]For a discussion in this context see G. F. Mazenko, in *Correlation Functions and Quasiparticle Interactions,* edited by J. W. Halley (Plenum, New York, 1978).
[33]G. Mazenko, M. Nolan, and R. Freedman, Phys. Rev. B 18, 2281 (1978). This method is very useful in the case where $z < z_0$. That is where ϕ blows up as $T \to T_c$.
[34]B. Kaufmann and L. Onsager, Phys. Rev. 76, 1244 (1949).
[35]See Refs. 8 and 9.
[36]M. E. Fisher and R. J. Burford, Phys. Rev. 156, 583 (1967).
[37]C. Domb, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1974), Vol. 3.
[38]B. Nickel, in *Proceedings of the 14th Conference on Statistical Physics* (North-Holland, Amsterdam, 1981).
[39]G. F. Mazenko and O. T. Valls, Phys. Rev. B (in press).