

Definition and measurement of the electrical and thermal resistances

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An ideal potentiometer and a thermometer are defined in order to show the experimental situations leading to Landauer's formula for the electrical resistance and to the corresponding expression for the thermal resistance.

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Recently, Economou and Soukoulis (ES),¹ and thereafter Fisher and Lee (FL),^{2,3} derived from the Kubo formula an expression for the dc electrical conductance, G , of a one-dimensional chain for noninteracting (spinless) electrons $G = (e^2/h)T$, where T is the transmission coefficient. This result contradicts Landauer's⁴ formula (LF) for the electrical conductance, $G = (e^2/h)T/(1 - T)$. This disagreement is disconcerting since LF was used in recent treatments^{5,6} of the electrical conductance of one-dimensional chains. We want here to rederive LF from an experimental point of view since the exact experimental conditions leading to LF have remained unclear. We stress that the electrical resistance is defined in terms of chemical potential differences, rather than voltage differences. Alternative and independent discussions of LF have recently been given in Refs. 7-10.

In the derivation of LF in Ref. 5 the difference between the chemical potentials of the reservoirs at the ends of the sample was directly related to the net current through the sample. The chemical potential, however, is not well defined when a net current is flowing through the walls of a reservoir. To avoid this problem, we introduce an ideal potentiometer (Fig. 1). The in- and outgoing currents are equal leading to a well-defined chemical potential μ . The operational definition of the potentiometer is given by the S matrix

$$S = \begin{pmatrix} r_\alpha & t_{\alpha\beta} & t_{\alpha\gamma} \\ t_{\beta\alpha} & r_\beta & t_{\beta\gamma} \\ t_{\gamma\alpha} & t_{\gamma\beta} & r_\gamma \end{pmatrix}, \tag{1}$$

where $|t_{\alpha\gamma}| = |t_{\beta\gamma}| = \sqrt{\epsilon}$ and $|r_\gamma| = 1 - \epsilon$, $\epsilon \ll T$. The subscripts α , β , and γ refer to the left and right transport channels and to the connection to the reservoir, respectively, and ϵ is a controllable design parameter, the strength of coupling of the potentiometer. For the following discussion we do not need to specify the remaining elements of the S matrix except for requiring that the unitarity conditions are fulfilled and that $|t_{\alpha\beta}| = 1 + O(\epsilon)$, $|r_\alpha| = |r_\beta| = O(\epsilon^n)$, with $n \geq \frac{1}{2}$. Since the reservoir has a finite width,¹¹ the

currents to and from the reservoir are determined by the phase-averaged transmission and reflection coefficients. The difference between the outgoing current j_A from the reservoir A (Fig. 2) with chemical potential μ_A and the current at equilibrium ($\mu_A = 0$) is

$$j_A = \frac{e}{2} \int_{-\infty}^{+\infty} [f(E - \mu_A) - f(E)]v(E)N(E) dE, \tag{2}$$

where $v(E)$ is the velocity, $N(E)$ is the density of states per unit energy and length, and $f(E)$ is the Fermi distribution function. The energy is measured relative to the chemical potential of the sample at equilibrium. Since $v = \partial E / \partial p$ and $N(E) = (1/\pi\hbar) \times (\partial p / \partial E)$, Eq. (2) can be simplified to

$$j_A = \frac{e}{h} \int_{-\infty}^{+\infty} [f(E - \mu_A) - f(E)]dE \approx \frac{e}{h}\mu_A. \tag{3}$$

Our aim is to relate j_A and j_B to the net current through the sample j . The ingoing currents to the reservoirs A and B consist of the transmitted parts of $j_1 = (e/h)\mu_1$ and $j_2 = (e/h)\mu_2$ and of the reflected

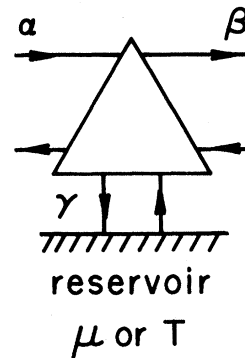


FIG. 1. Ideal potentiometer (thermometer). The in- and outgoing electrical (thermal) currents of the reservoir are equal leading to a well-defined chemical potential (temperature). Since the reservoir has a finite width, an average over the phases of the reflection and transmission coefficients for the in- and outgoing currents has to be taken.

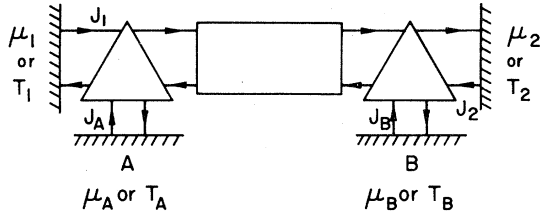


FIG. 2. Measurement of the resistance. The currents j_1 and j_2 are impressed from left and right, respectively. The potentiometers (thermometers) divert a small part of the current from the channel into the reservoirs at A and B . The difference between the currents j_A and j_B determines the potential (temperature) difference.

parts of j_A and j_B . The transmission amplitude from the left ingoing channel to the reservoir, $t_{1A}(E)$, including terms to order $\sqrt{\epsilon}$, is

$$\sqrt{\epsilon} e^{i\phi_1(E)} + \sqrt{\epsilon} |r(E)| e^{i\phi_2(E)}, \quad (4)$$

where $r(E)$ is the reflection amplitude of the sample. Owing to the finite width of the reservoir an average over the relative paths of the directly transmitted wave to the reservoir and of the reflected wave from the sample has to be taken; i.e., T_{1A} is given by its phase average,

$$\langle T_{1A}(E) \rangle = \epsilon [1 + |r(E)|^2]. \quad (5)$$

The ingoing currents at A originating from the external currents j_1 and j_2 are

$$\epsilon \frac{e}{h} \mu_1 \int_{-\infty}^{+\infty} [1 + |r(E)|^2] \left[-\frac{\partial f}{\partial E} \right] dE \quad (6)$$

and

$$\epsilon \frac{e}{h} \mu_2 \int_{-\infty}^{+\infty} |t(E)|^2 \left[-\frac{\partial f}{\partial E} \right] dE. \quad (7)$$

Since $j_A(1-2\epsilon)$ is the part of the current from the reservoir at A that is reflected back, the condition for zero net flow through the walls of the reservoir is

$$j_A = j_A(1-2\epsilon) + \epsilon \frac{e}{h} \mu_1 \int_{-\infty}^{+\infty} [1 + |r(E)|^2] \left[-\frac{\partial f}{\partial E} \right] dE \\ + \epsilon \frac{e}{h} \mu_2 \int_{-\infty}^{+\infty} |t(E)|^2 \left[-\frac{\partial f}{\partial E} \right] dE. \quad (8)$$

The difference in chemical potential between A and B , $\delta\mu_{AB}$, can then be related to j ,

$$\delta\mu_{AB} = \frac{h}{e} \frac{\int_{-\infty}^{+\infty} |r(E)|^2 \left[-\frac{\partial f}{\partial E} \right] dE}{\int_{-\infty}^{+\infty} |t(E)|^2 \left[-\frac{\partial f}{\partial E} \right] dE} j \quad (9)$$

by using Eqs. (3) and (8) and the corresponding expressions for the reservoir at B . Since $(-\partial f/\partial E) = \delta(E)$ at $T=0$ we have rederived Landauer's⁴ formula for the electrical resistance¹²

$$R_{el} = \frac{\delta\mu_{AB}}{ej} = \frac{h}{e^2} \frac{|r(0)|^2}{|t(0)|^2}. \quad (10)$$

The expression of ES¹ and FL^{2,3} is dubious since it relates the difference in chemical potential over the sample to the end reservoirs which are not in internal equilibrium. In order to get a well-defined potential we need to apply measurement reservoirs, for which the zero net flow condition is applied rigorously. This corresponds to a four-probe experiment, where the potentiometers do not discriminate between the left- and right-going flows.

We can use a similar procedure to derive the formula for the thermal resistance. The potentiometer is replaced by a thermometer defined by the same scattering matrix (1). The temperature is determined by the Fermi distribution function, for which the in- and outgoing heat currents are equal at the reservoir. The temperature is the only variable since the chemical potential is constant in a heat conduction experiment. When the temperature of the unperturbed sample is T_0 , the extra heat current from a reservoir at temperature T is

$$j_{th} = \frac{1}{h} \int_{-\infty}^{+\infty} [f_T(E) - f_{T_0}(E)] E dE \\ \approx \frac{\pi^2 k_B^2 T_0}{3h} (T - T_0). \quad (11)$$

Since $j_{A,th}(1-2\epsilon)$ is the part of the heat current from the reservoir at A that is reflected back, the condition for zero net heat flow through the walls of the reservoir is

$$j_{A,th} = j_{A,th}(1-2\epsilon) \\ + \epsilon \frac{T_1 - T_0}{hT_0} \int_{-\infty}^{+\infty} [1 + |r(E)|^2] E^2 \left[-\frac{\partial f}{\partial E} \right] dE \\ + \epsilon \frac{T_2 - T_0}{hT_0} \int_{-\infty}^{+\infty} |t(E)|^2 E^2 \left[-\frac{\partial f}{\partial E} \right] dE. \quad (12)$$

If we exchange the subscripts 1 and 2 in Eq. (12), we obtain the equation determining $j_{B,th}$. By using Eqs. (11) and (12) we readily derive the thermal resistance of the sample

$$R_{th} = \frac{3h}{\pi^2 k_B^2 T_0} \frac{\int_{-\infty}^{+\infty} |r(E)|^2 E^2 \left[-\frac{\partial f}{\partial E} \right] dE}{\int_{-\infty}^{+\infty} |t(E)|^2 E^2 \left[-\frac{\partial f}{\partial E} \right] dE}. \quad (13)$$

The Wiedemann-Franz law

$$\frac{R_{el}}{R_{th}} = \frac{\pi^2}{3} \left(\frac{k_B}{e} \right)^2 T_0, \quad (14)$$

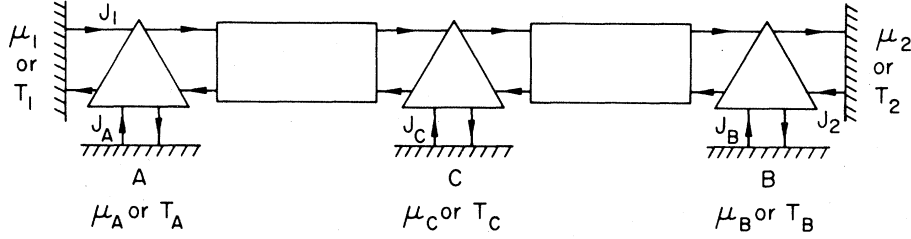


FIG. 3. Extra potentiometer (thermometer) on the sample. The resistance between two points in the channel depends in general on the scattering properties of the total system; i.e., no "local" resistance can be defined.

is valid when $|r(E)|^2 \approx |r(0)|^2$ for energies near the Fermi level. Since the reflection coefficient is a rapidly varying function of energy in one dimension, we do not expect the Wiedemann-Franz law to be satisfied even for elastic scattering. This is in contradiction to the usual relaxation-time approximation for elastic scattering.

We now apply a potentiometer (thermometer) to the sample (Fig. 3). The transmission amplitude for electrons from the left to the reservoir at C is to order $\sqrt{\epsilon}$,

$$t_{1C}(E) = \sqrt{\epsilon} \frac{|t_1(E)| e^{i\psi_1(E)} [1 + |r_2(E)| e^{i\psi_2(E)}]}{1 - |r_1(E)| |r_2(E)| e^{i\psi_3(E)}}. \quad (15)$$

$$\frac{\delta\mu_{AC}}{\delta\mu_{AB}} = \frac{1}{2} \left\{ 1 + \left[\int_{-\infty}^{+\infty} \frac{|r_1(E)|^2 - |r_2(E)|^2}{1 - |r_1(E)|^2 |r_2(E)|^2} \left(-\frac{\partial f}{\partial E} \right) dE / \int_{-\infty}^{+\infty} |r(E)|^2 \left(-\frac{\partial f}{\partial E} \right) dE \right] \right\}. \quad (17)$$

Similarly, the ratio of the temperature difference between A and C to the total temperature difference, T_{AB} , for thermal conduction is

$$\frac{\partial T_{AC}}{\partial T_{AB}} = \frac{1}{2} \left\{ 1 + \left[\int_{-\infty}^{+\infty} \frac{|r_1(E)|^2 - |r_2(E)|^2}{1 - |r_1(E)|^2 |r_2(E)|^2} E^2 \left(-\frac{\partial f}{\partial E} \right) dE / \int_{-\infty}^{+\infty} |r(E)|^2 E^2 \left(-\frac{\partial f}{\partial E} \right) dE \right] \right\}. \quad (18)$$

The measured resistances for sections of the chain add linearly. Because of long-range phase coherence for each energy, however, they are determined by all scatterers in the sample, not just those between the measurement points. The "classical" case of local resistances corresponds to a complete phase randomization of the reflection coefficient for each energy, i.e., $|r(E)|^2 = \langle |r(E)|^2 \rangle$. Only when the inelastic

The subscripts 1 and 2 refer to the scatterers to the left and to the right of the measurement point C. Since the reservoir has a finite width the current from the left channel to the reservoir at C is determined by the phase-averaged transmission coefficient

$$\langle T_{1C}(E) \rangle = \epsilon \frac{|t_1(E)|^2 [1 + |r_2(E)|^2]}{1 - |r_1(E)|^2 |r_2(E)|^2}. \quad (16)$$

$\langle T_{1C}(E) \rangle$ is given by Eq. (16) with the subscripts 1 and 2 exchanged. Since $\epsilon \ll |t(E)|^2$, the applied potentiometer (thermometer) does not change the transmission coefficient of the sample. The potential difference between A and C, $\delta\mu_{AC}$, in terms of $\delta\mu_{AB}$, is then

scattering is strong enough to destroy phase coherence between the measurement points can local resistances be defined.

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- ¹¹The finite width of the connection to the measurement reservoir means, of course, that it has many, say, N channels. Thus we have $|t_{\alpha\gamma i}| = |t_{\beta\gamma i}| = \sqrt{\epsilon/N}$ and $|r_{\gamma i}| = 1 - \epsilon/N$ for each channel γ_i . Since Eq. (3) is $j_A = (Ne/h)\mu_A$, the relevant Eqs. (9), (13), (17), and (18) are independent of N . Hence we limit ourselves to a single-channel connection to the measurement reservoir.
- ¹²We expect corrections to the relation $\delta\mu = e\delta V$ to be of the order λ_D/L , where δV is the difference in the average electrostatic potential between the measurement reservoirs (voltage difference at zero current through the sample have already been subtracted), λ_D is the Debye screening length, and L is the linear dimension of the reservoir. Since the reservoirs have to be macroscopic, these corrections are negligible.