PHYSICAL REVIEW B VOLUME 24, NUMBER 2 15 JULY 1981

Comments on the scaling theory of localization and conduction in disordered systems

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(Received 5 May 1980)

A simple demonstration of the Thouless scaling picture for the conductivity of noninteracting electrons in a disordered potential is presented. This picture, in the one-dimensional case, is shown to be equivalent to the formulation of Landauer. Possible modifications in the numerical procedures which have been used for effecting the scaling transformation are indicated. The dependence of the conductivity dividing the metallic and nonmetallic regimes on the microscopic length scale is obtained and shown to be in agreement with experiments on granular metals and with theoretical arguments by Adkins and Abeles and Sheng.

I. INTRODUCTION

The scaling theory[1-7] of localization[8,9] and conduction in disordered systems has gone recently through theoretical advances and numerical[3,5,10,11] investigations. However, in the delicate limiting case of two dimensions (2D) the latter results appear to not agree fully with the picture obtained from general simple scaling arguments. Experiments[12-14] on conduction in disordered thin wires and films are roughly consistent with the theoretical picture when inelastic scattering effects are taken into account.[2,15] On the other hand, the electron-electron interaction effects[1,9,16-18] may well be relevant and, in fact, lead to predictions that are also consistent with experiments. It has been suggested[19] that localization-related ideas may be relevant to explaining the unusual conduction properties of 3D amorphous and granular[20] metals. Very early, Landauer[21] suggested an interesting picture of conduction in disordered 1D (one-dimensional) systems, relating the conductance to the quantum-mechanical transmission through the system and obtaining the exponential increase of resistance in 1D. This picture was recently generalized and used to construct a novel scaling theory for 1D conduction in Ref. 7, remedying the delicate problem[21] of the anomalous distribution of conductances and resistances in 1D.

In this Comment, we first present in Sec. II a simple demonstration of the Thouless scaling theory for conductance in a degenerate noninteracting electron gas in a disordered potential. This concentrates on the lifetime, τ_L, of an electron in one block. The lifetime τ_L, which is easily calculable[22] for large blocks using the Fermi golden rule, immediately yields the interblock conductance and a definition of the appropriate coupling constant (or "Thouless number") which is somewhat different from the one which has been used. This indicates some possible modifications in the numerical procedure for effecting the scaling renormalization-group (RG) transformation. For weak interblock tunneling, τ_L is easily related to the interblock transmission coefficient. This correspondence and a scaling argument yielding the full equivalence to the Landauer[21] result for strong transmission in one dimension is presented in Sec. III. Difficulties[23] in defining the conductivity and diffusion coefficient[21] for high transmission are shown to be properly handled by the above procedure. Finally, the scaling picture is applied to estimate the resistivity ρ_m separating the metallic from the nonmetallic regime in granular metals. The estimates for ρ_m and its grain-size dependence are different from those obtained from a simple Yoffe-Regel criterion[9,24] but agree both with experiment and with the result of qualitative physical arguments by Adkins[25] and Abeles.[26]

II. A DERIVATION OF THE THOULESS SCALING PICTURE OF CONDUCTANCE IN A DISORDERED SYSTEM

Consider the macroscopic system to be divided into equal blocks of cubical shapes (in d dimensions) having a linear size L. L is taken to be larger enough in size than the microscopic length, a, so that the density of states at the Fermi energy $N_L(\epsilon=\epsilon_F)$ is large. This includes the case where L is on the order of the appropriate correlation length, which is the case of interest. The typical energy-level separation[1] at the Fermi surface

$$w_L \equiv 1/N_L(\epsilon_F)\,, \tag{1}$$

is thus, taken to be small. How small w_L has to be will become clear below. w_L also represents the order of magnitude of the mismatch in energy levels ϵ_i and ϵ_j of neighboring blocks at ϵ_F. These states $|i\rangle$ and $|j\rangle$ are connected by a matrix element v_{ij}. In the usual microscopic Anderson model[8] with one state per site, v_{ij} is the typical matrix element "V". In our case, however, there

are many states in adjacent blocks that can interact. In fact, for large L these states form a quasicontinuum, which is what we shall assume. The transition probability per unit time, or the inverse lifetime τ_L of an electron against going from the block i to a neighboring block j is given by the Fermi golden rule[22]:

$$\tau_L^{-1} = \frac{2\pi}{\hbar}\overline{|v^2|}N_j(\epsilon_F) = \frac{2\pi}{\hbar}\overline{|v^2|}/w_j , \tag{2}$$

where $\overline{|v^2|}$ is an appropriate average of $|v_{ij}|^2$. The condition for using (2) is that L be large enough so that the time $\tau_{\min}$ beyond which the transition probability increases linearly with the time is much smaller than the time $\tau_{\max}$ limiting the validity of the first-order perturbation theory, i.e., $\tau_{\min} \ll \tau_{\max}$. To estimate $\tau_{\max}$, we note that two conditions must be satisfied to ensure the validity of the golden rule. The depletion of the ground state must be small, so that $\tau_{\max} \lesssim \tau_L$ and the number of final states that are significantly populated after a time $\tau_{\max}$ should be larger than unity. The latter condition leads to $\hbar/(\tau_{\max} w_L) \gtrsim 1$. Thus, $\tau_{\max}$ is given by the smaller of τ_L and $\hbar/w_L$. On the other hand, $\tau_{\min}$ is usually given by the inverse of the bandwidth or the Fermi energy. To obtain[22] the interblock conductance, G_L, we note that if a small voltage V is applied between the blocks, $eVN_i(\epsilon_F)$ states in the ith block would decay into the block j, each with a lifetime τ_L. The interblock conductance is thus given by

$$G_{ij,L} = \frac{e^2 N_i(\epsilon_F)}{\tau_L} = \frac{2\pi e^2}{\hbar}\frac{\overline{|v^2|}}{w_i w_j} = \frac{2\pi e^2}{\hbar}\overline{|v^2|}N_i(\epsilon_F)N_j(\epsilon_F) , \tag{3}$$

where the validity of the golden rule was assumed to get the second equality. Equation (3) is the well known formula for the conductance of a tunnel junction.[22] The dimensionless conductance[6,7] $g_L \equiv G_L/(e^2/\pi\hbar)$ is thus given by

$$g_{L,ij} = 2\pi^2\frac{\overline{|v^2|}}{w_i w_j} = \frac{\pi\hbar}{\tau_L w_i} \tag{4}$$

as defined by Thouless[1,2] and used by Abrahams *et al.*[6] In terms of g_L we find

$$\tau_{\max} = \begin{cases} \tau_L , & g_L \gtrsim 1 \\ \tau_L g_L , & g_L \ll 1 . \end{cases} \tag{5}$$

It follows that for a block of N atoms, the validity of the golden rule at an intermediate time range requires $N \gg 1$ for $g_L \ll 1$ and $N \gg g_L$ for $g_L \gtrsim 1$. From Eq. (4), g_L is indeed found to be given by the ratio of the two energies $\pi\hbar/\tau_L$ and w_L. However, τ is related to the inverse of the typical energy $\overline{|v_L^2|}/w_L$, *not* to $\overline{|v_{ij}|}$. This suggests a procedure for numerically effecting the scaling transformation by computing the scaling of $\overline{|v_L^2|}/w_L$. In general, this procedure might not be equivalent to the one used in Refs. 10 and 11 where $\overline{|v_{ij}|}$ has been used as an interblock coupling parameter. On the other hand, rough arguments can be constructed in lowest-order perturbation theory to indicate that $\hbar/\tau_L$ is indeed of the same order of magnitude as the shift in the energy levels, ΔE_s, of a given block due to its coupling to a neighboring one. A similar boundary condition shift, $\Delta E_s'$, has been used by Thouless and co-workers[1-3] in Ref. 10. From the results of Ref. 10, it appears that $\Delta E_s/w$ scales differently from $\overline{|v|}/w$. Since $\overline{|v|^2}/w$ gives g directly and is related to ΔE_s, our point here is that it is a good candidate for use in further systematical numerical studies.

III. THE EQUIVALANCE OF LANDAUER'S PICTURE IN ONE DIMENSION

As early as in 1960 and 1969, Landauer[21] demonstrated the general result that the conductance through any one-dimensional system is given by

$$g = \frac{t}{r} , \tag{6}$$

where t and r are, respectively, the transmission and the reflection coefficients ($t + r = 1$) of the given system imagined as linking two ideal electrodes. This has been generalized to many channels by Anderson *et al.*[7] For weak coupling through the system, $t \ll 1$, $r \simeq 1$, this will reduce to

$$g = t \quad (t \ll 1) . \tag{7}$$

Let us first show that the formulation which concentrates [Eq. (4)] on the lifetime τ_L is indeed equivalent to Eq. (7) for weak coupling. The strong-coupling case will be treated later. Here we consider in the simplest one-dimensional case[22] a single block of size L connecting two ideal electrodes, each being also of the same size. The conductance of this arrangement must be equal to g_L. The electron wave packet spends a time $2L/v_F$ for traversing each electrode back and forth and it hits the barrier $v_F/2L$ times per unit time. For a small transmission probability t, it takes, on the average, t^{-1} trials to cross the barrier.[22] Thus, the characteristic lifetime against crossing the intervening block, or barrier, is

$$\tau = 2L/(v_F t) . \tag{8}$$

Using Eqs. (4), (8), and $N(\epsilon_F) = 2L/(\pi\hbar v_F)$, we obtain Eq. (7). We note that for t which is not very small, Landauer's result (6) replaces the weak coupling t in Eq. (7) by $t/(1-t)$. This looks like a possible result of summing to all orders in t. It is difficult

to generalize the argument leading to (8) to a finite t. However, we shall now present a rough, simple scaling-type argument for replacing t by $t/(1-t)$ for t close to unity.

We take a single block of size L, the transmission through which t_1 is very strong,

$$r_1 = (1 - t_1) \ll 1 . \tag{9}$$

Let us try to couple n such blocks in series, so that the total transmission coefficient, t_n, is small and we can use (7). A key observation here is that the wave aspects of the problem should be lost if the phases ϕ_{ij} picked up by the wave traveling between the obstacles are random, span many intervals of 2π, and can be averaged upon.[7] That this observation must be true should, in fact, be clear for any classical situation of light passing through two absorbers where the interference effects are destroyed by random dephasing. Anderson *et al.*,[7] in fact, proved that this follows from the expression derived by Landauer[21] for transmission of an electron through two absorbers when the phase in between them is random enough. They based a new scaling theory of $d = 1$ conduction on this fundamental property. According to the above, once the distance across which the phase is randomized is much less than L, it follows that

$$\overline{\ln t_n} = n \ln t_1 . \tag{10}$$

Let us now choose n to be so large that $\overline{\ln t_n} = C$, where C is a negative constant whose absolute value is of the order of unity, say $C = -1$ ($1/e$ of the total transmission):

$$n = C/\ln t_1 \simeq O(1)/r_1 . \tag{11}$$

The conductance of the system of n blocks is, using the weak-coupling rough approximation, with $t_n = \exp(\overline{\ln t_n})$,

$$g_n \simeq t_n = \exp(-C) . \tag{12}$$

If n is not too large (the demarcation range is precisely the localization range, in this case), Ohm's law is valid and

$$g_n = g_1/n \tag{13}$$

$$g_1 = n g_n \simeq \frac{O(1)}{r_1} = \frac{O(1) t_1}{r_1} , \tag{14}$$

which is Eq. (6) up to a constant of order unity. This is, of course, equivalent to the observation that the scaling theory[7] built on Eq. (6) yields the correct Ohm's-law addition of resistances in series, for small resistances. Note that this procedure also can be regarded as settling a question of principle. For a single junction with $t_1 \simeq 1$ between two ideal electrodes, one may argue that the meaning of the appropriate diffusion coefficient and conductance is not obvious. The whole system is, by definition, effectively shorter than a mean free path and the electron motion through it is almost "ballistic." Equation (13) can be viewed as an operational definition of g_1, designed such that by connecting n g_1's in series, one will obtain the correct g_n.

IV. APPLICATION OF THE SCALING THEORY TO GRANULAR METALS. THE DEPENDENCE OF THE CONDUCTIVITY SEPARATING THE METALLIC FROM THE NONMETALLIC REGIMES ON GRAIN SIZE

In Ref. 19, it was suggested that the scaling theory of localization may be relevant to explaining the conductivity of highly disordered metals. A case of interest here is granular metals[20] where metallic grains with typical diameters, d, ranging from a few to ten thousand atomic spacings are separated by insulating barriers. When the normal-state resistivity, ρ, of such a material increases above a characteristic value ρ_m, the behavior of the material becomes semiconducting (i.e., ρ decreases with temperature). For an average grain size of a few hundred Å, ρ_m is typically on the order of 10^{-2} Ω cm and it increases with d. The above value of ρ_m corresponds to an effective mean free path of $l \sim 10^{-1}$ Å $\ll a < d$, and the increase of ρ_m with d would amount to a *decrease* of the corresponding mean free path. Both of these facts would not agree with a minimum metallic conductivity based on the Yoffe-Regel criterion[9,24] with either $l \sim a$ or $l \sim d$. On the other hand, the scaling theory would predict that ρ_m is determined by an intergrain conductance on the order of $e^2/\hbar$. According to this, one obtains in 3D:

$$\rho_m \sim 10^4 d , \tag{15}$$

in units of Ω cm, which agrees with both the order of magnitude and grain-size dependence of ρ_m. In fact, Adkins[25] and Abeles and Sheng[26] have obtained Eq. (15) from different qualitative physical arguments and found it to be in good agreement with experiment. The appropriate estimate in 2D films would also roughly agree with experiments.[14,27]

In conclusion, our remarks in this note further substantiate the scaling theory of localization[1,2,6] and its equivalence to the Landauer picture. The predictions of this theory with respect to ρ_m in granular metals are consistent with experiment, without the necessity of taking into account electron-electron interactions. The latter interaction, as well as the electron-lattice one are certainly relevant. It would be extremely useful if describing such interactions with an inelastic mean free path that determines the length scale, as sug-

gested by Thouless[2] (see also Ref. 15), would yield a qualitatively correct picture of conduction in highly disordered and granular metals.

ACKNOWLEDGMENTS

Useful discussions with D. J. Bergman, E. Domany, G. Deutscher, Y. Gefen (Feigenblatt), D. Horn, R. Landauer, D. Prober, and M. Strongin are gratefully acknowledged. The author is grateful to C. M. Soukoulis for discussions on Ref. 23. This research was partially supported by the U.S.-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

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