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Hall voltage dependence on inversion-layer geometry in the quantum Hall-effect regime

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A calculation of the Hall voltage is presented within a model of a finite two-dimensional inversion layer. An explicit form for the electric field is obtained and this is found to have a power-law singularity in the corners of the inversion layer. This singularity is most pronounced in the quantum Hall-effect regime where the Hall angle approaches $\pi/2$. The error in measuring the Hall voltage in this regime due to the shorting effect of the source and drain is calculated. This is found to be negligible at the level required for a new determination of the fine-structure constant and development of a new resistance standard using inversion-layer measurements in the quantum Hall-effect regime. Limitations of the model and other possible sources of error are briefly discussed.

I. INTRODUCTION

It is well known that the Hall coefficient of a two-dimensional electron gas (e.g., a MOSFET or heterojunction inversion layer) oscillates with magnetic field because of quantization of the Landau orbitals of the electrons.[1] For temperatures of a few kelvin and sufficiently strong magnetic field perpendicular to the electron gas, the carrier scattering rate between Landau levels becomes extremely low. When the Fermi level is then adjusted to fill i Landau levels, the Hall resistance goes through a plateau assuming the universal value[2] h/e^2i. This has recently led to a great deal of interest in the possibility of a high-precision measurement of the fine-structure constant and development of a new resistance standard.

In order to make an accurate measurement it is necessary to have a thorough understanding of all the possible sources of error in nonideal devices. Of particular concern in this paper is the error introduced by the finite length of the device. This question was investigated some years ago in connection with Hall-coefficient measurements in semiconductors.[3] However, these calculations did not anticipate the range of parameters necessary to take into account the extremely high Hall coefficient and extremely low resistivity possible in the inversion-layer systems. Furthermore the results of these earlier calculations are rather cumbersome and particularly difficult to evaluate in the limit of present interest. It is the purpose of this paper to present a simpler and more direct solution to the problem and to discuss the expected errors in typical inversion-layer geometries now in use.

II. INVERSION-LAYER MODEL

We will discuss a standard model used by previous authors.[3] The inversion-layer system is assumed to consist of a rectangle, shown in Fig. 1(a), within which the Hall coefficient R_H is constant. Electrodes with high carrier density (and therefore negligibly small Hall coefficient) are attached at opposite ends of the rectangle and represent the source and drain. The Hall probes, taken to be point contacts at arbitrary locations on the sides of the device, are assumed to draw no current. The boundary conditions for this configuration are that there be no current flow out the sides of the device and no electric field parallel to the ends. The assumption that the electrodes completely short out the Hall voltage at the ends slightly overestimates the error since the Hall coefficient in the electrodes, though small, is nonzero and has the same sign as in the inversion layer. The electric field and current density are assumed to be independent of the z

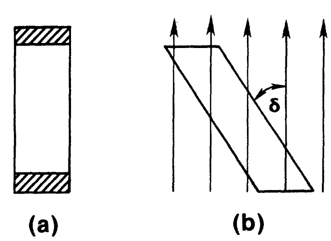

FIG. 1. (a) Geometry of the model inversion layer. The shaded regions represent the source and drain. (b) Parallelogram geometry. Notice that the boundary conditions are satisfied by a uniform electric field in the y direction.

coordinate which is along the thickness of the layer. Also, the magnetic field $\vec{H}$ is taken to point in the z direction so that we are dealing with a strictly two-dimensional problem. Then the fundamental constitutive equation for the system is

$$\vec{E}+\rho_H\vec{J}\times\hat{z}=\sigma^{-1}\vec{J}, \quad (2.1)$$

where $\rho_H=R_H H$ is the Hall resistivity, σ^{-1} is the Ohmic resistivity, and $\vec{J}$ is the current density. All quantities in this equation are in units appropriate to two dimensions (i.e., ohms per square for resistivity). Equation (2.1) implies that the current makes an angle δ with respect to the electric field. This Hall angle is given by

$$\tan\delta=\sigma\rho_H. \quad (2.2)$$

The resistivity σ^{-1} becomes very low in the interesting region where the Fermi energy lies in the gap between Landau levels so that δ is near $\pi/2$. It is therefore convenient to define the quantity ϵ via $\delta=\frac{1}{2}\pi(1-\epsilon)$. Current measurements[2,4] allow one to place an upper limit of 5×10^{-7} on ϵ.

One can replace the boundary condition $J=0$ on the sides by the condition that the electric field make an angle δ with respect to the sides. If the geometry of the device were that of the parallelogram shown in Fig. 1(b), then the boundary conditions would be satisfied by a uniform electric field $\vec{E}=E_0\hat{y}$. One can take advantage of this fact to obtain a solution to this two-dimensional problem by finding a conformal mapping which takes the rectangle into this parallelogram. Our method of using a direct conformal mapping is simpler than previous formulations which involved a pair of Schwarz-Christoffel transformations from the real axis to the boundary of the rectangle and the parallelogram.[3]

Under conditions where the currents are in steady state, $\vec{E}$ can be written as the gradient of a potential. Neglecting magnetic fields generated by the currents in the device, it then follows from Maxwell's equations and Eq. (2.1) that the potential in the inversion layer satisfies

$$\nabla^2\psi=0, \quad (2.3)$$

where ∇^2 is the two-dimensional Laplacian in the plane of the inversion layer. Therefore under these conditions a conformal mapping method can be used. It is important to realize that ψ is uniquely determined by Eq. (2.3) and the boundary conditions on the edges of the inversion layer and is not affected by the device configuration outside of the inversion layer. For instance, in a MOSFET the presence of the metal gate above the inversion layer has no effect on ψ under the conditions mentioned above. The metal gate does, however, affect the charge density in the device, which is determined by the three-dimensional Laplacian of ψ. Equation (2.3) does not imply a vanishing of the charge density in the inversion layer. This is determined by the discontinuity in the z component of the electric fields across the inversion layer.

A conformal mapping $W(Z)$ rotates local angles by an amount

$$\theta(Z)\equiv\arg\left(\frac{dW}{dZ}\right). \quad (2.4)$$

If we let

$$\frac{dW}{dZ}=\exp[f(Z)], \quad (2.5)$$

then

$$\theta(Z)=\operatorname{Im}f(Z). \quad (2.6)$$

The complex potential corresponding to the uniform electric field in the W plane (the plane of the parallelogram) is simply

$$V=\lambda W, \quad (2.7)$$

where λ is an arbitrary real constant which we take to be unity. The physical potential is given by the imaginary part of the complex potential. The complex electric field in the Z plane (the plane of the rectangle) is

$$-E=\frac{dV}{dZ}=\frac{dW}{dZ}=\exp[f(Z)]. \quad (2.8)$$

The physical field is related to the complex field via $E=E_y+iE_x$. Hence the transformation yields the desired electric field directly.

In order to map the rectangle into the parallelogram, $\operatorname{Im}f(Z)$ must be equal to δ on the sides of the rectangle and vanish on the ends. We therefore seek an analytic function whose imaginary part satisfies these boundary conditions. This problem is nothing more than the standard electrostatics problem of finding the potential inside a rectangle whose sides are at potential δ and whose ends are at ground. In the coordinate system shown in Fig. 2(a) one finds (neglecting an arbitrary real constant related to λ)

$$f(Z)=\sum_{n\,(\mathrm{odd})}\frac{4\delta}{n\pi}\left[\sinh\left(\frac{n\pi Z}{T}\right)\Big/\cosh\left(\frac{n\pi S}{T}\right)\right]. \quad (2.9)$$

Evaluation of this expression on the sides of the rectangle yields

$$\operatorname{Im}f(\pm S+iy)=\delta, \quad (2.10)$$

$$\operatorname{Re}f(\pm S+iy)=\pm[g(y)+h(y)], \quad (2.11)$$

where

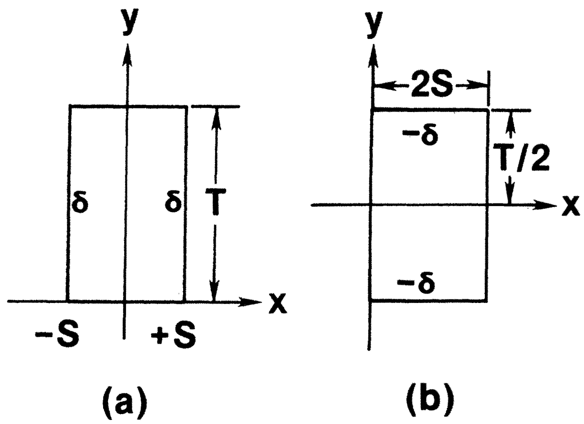

FIG. 2. Coordinate systems used in solving the boundary-value problem. (a) is discussed in Sec. II and (b) is discussed in Sec. III.

$$g(y) \equiv \sum_{n\,(\mathrm{odd})} \frac{4\delta}{n\pi} \cos\left(\frac{n\pi y}{T}\right) \tag{2.12}$$

and

$$h(y) \equiv -\sum_{n\,(\mathrm{odd})} \frac{4\delta}{n\pi} \left[1 - \tanh\left(\frac{n\pi S}{T}\right)\right] \cos\left(\frac{n\pi y}{T}\right). \tag{2.13}$$

Equation (2.12) may be explicitly evaluated yielding

$$g(y) = \frac{-2\delta}{\pi} \ln \tan\left(\frac{\pi y}{2T}\right). \tag{2.14}$$

Equation (2.8) becomes, for $Z = \pm S + iy$,

$$\frac{dV}{dZ} = \left[\tan\left(\frac{\pi y}{2T}\right)\right]^{\mp 2\delta/\pi} \exp[\pm h(y) + i\delta]. \tag{2.15}$$

This result shows that the electric field has a power-law singularity in the corners of the device. It is interesting to note that the singularity exponent is the same as that appearing in the many-body t matrix for the x-ray edge problem.[5] In both cases one is dealing with an analytic function whose phase changes suddenly from zero to δ. Such a function of necessity has a power-law singularity.

III. NUMERICAL EVALUATION

It is possible to determine the potential along the sides of the device (and hence the Hall voltage) by integrating the field in Eq. (2.15). However, the singularity in the corners causes numerical difficulties especially in the limit $2\delta/\pi \to 1$, which is the case of interest. This problem plagues the results of previous investigations as well.[3] A better approach is to integrate the field along a path directly across the middle of the device, thereby avoiding the corners. Equation (2.9) is not suitable for this case since it is rather slowly convergent, especially for high-aspect-ratio devices ($T/2S \gg 1$).

It is convenient to go to a new coordinate system shown in Fig. 2(b) and write $f(Z)$ as

$$f(Z) = i\delta + k(Z), \tag{3.1}$$

where $\mathrm{Im}\, k(Z)$ vanishes on the edges and equals $-\delta$ on the ends. One finds

$$k(Z) = -\sum_{n\,(\mathrm{odd})} \frac{4\delta}{n\pi} \left[\sinh\left(\frac{n\pi i Z}{2S}\right) \Big/ \cosh\left(\frac{n\pi T}{4S}\right)\right]. \tag{3.2}$$

This expression is rapidly convergent (for high-aspect-ratio devices) everywhere except in the corners. It becomes particularly simple on the center line ($y = 0$) where, by combining Eqs. (2.8), (3.1), and (3.2), one obtains for the two components of the electric field

$$E_y(x) = -\cos\xi(x), \tag{3.3}$$

$$E_x(x) = -\sin\xi(x), \tag{3.4}$$

where

$$\xi(x) \equiv \delta - \sum_{n\,(\mathrm{odd})} \frac{4\delta}{n\pi} \left[\sin\left(\frac{n\pi x}{2S}\right) \Big/ \cosh\left(\frac{n\pi T}{4S}\right)\right]. \tag{3.5}$$

The total current flowing through the device is given by

$$I = \int_0^{2S} dx\, J_y(x). \tag{3.6}$$

Using Eq. (2.1) this may be written

$$I = \int_0^{2S} dx \frac{\sigma}{1 + \tan^2\delta} (E_y + \tan\delta\, E_x). \tag{3.7}$$

The Hall voltage is

$$V = -\int_0^{2S} dx\, E_x(x), \tag{3.8}$$

and the Hall resistance is defined by

$$R = V/I. \tag{3.9}$$

For an infinitely long device

$$E_x/E_y = \tan\delta \tag{3.10}$$

for all x. The Hall resistance for an infinite device is therefore

$$R_\infty = -\frac{1}{\sigma}\tan\delta\,. \tag{3.11}$$

Combining Eqs. (3.7)–(3.11) yields

$$\frac{R}{R_\infty} = \frac{1+\tan^2\delta}{\tan\delta}\left(\int_0^{2S} dx\, E_x\right)\Big/\left(\int_0^{2S} dx\,(E_y + \tan\delta E_x)\right). \tag{3.12}$$

Evaluation of Eq. (3.12) shows that $R/R_\infty \sim 1-\epsilon$. Numerical results for the geometry of Fig. 3 are displayed graphically in Fig. 4. Equation (2.15) gives the electric field along the edge of the device. The parallel component E_y is found to be of order ϵ. Hence an experimental measurement of V_{xx} (see Fig. 3), the voltage drop between two probes on the same side of the device, allows a determination of ϵ. The magnitude of V_{xx} for the geometry of Fig. 3 is displayed in Fig. 4 as a function of ϵ.

IV. DISCUSSION

The smallest attainable value of ϵ is not known but preliminary measurements by the NBS-NRL (National Bureau of Standards-Naval Research Laboratory) group[4] place an upper limit of 5×10^{-7} on ϵ. This means that the error (within this model) due to the shorting effect at the ends is extremely small. This is readily seen in the results for a device with an aspect ratio of 6.5 as shown in Fig. 4. One implication of this small value of ϵ is that the Hall voltage V_H will be equal to the voltage applied between the source and drain, V_{SD}, also to within a very small error. This is not inconsistent with the fact that V_{XX} is nearly zero because there is a surface-charge layer between the end electrode and the inversion region. Almost all the voltage drop occurs at these interfaces (within this model).

The results of our model scale with the size of the inversion layer and hence only the aspect ratio enters the calculations. However, some of the physical assumptions built into the derivation of our equations break down under such a scaling. In particular, we implicitly assume that the electrons scatter before reaching the drain so that it is meaningful to speak of a Hall angle. If the size of the system were scaled down, for a given value of V_{SD}, the system may eventually reach a hot electron regime. The scale at which this occurs may be quite large because of the extremely low scattering rate.

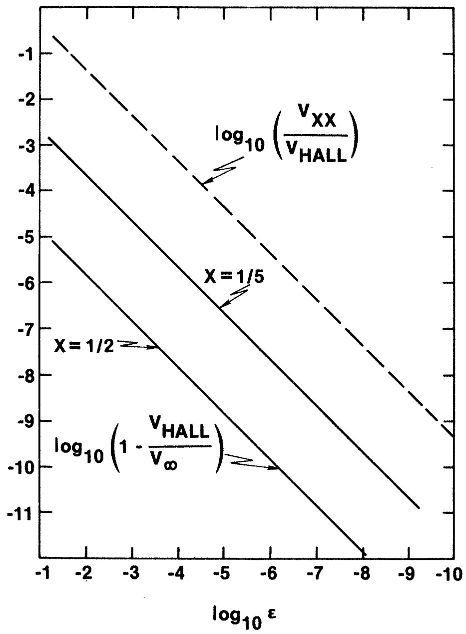

FIG. 4. Error in the measured Hall voltage due to the shorting effect for a device with aspect ratio 6.5. $V_{HALL}=IR$ is the measured Hall voltage. $V_\infty = IR_\infty$ is the ideal Hall voltage for an infinitely long device. The lower solid line is for Hall probes at the midpoint of the device. The upper solid line is for Hall probes at the $\frac{1}{5}$ point as in the NRL geometry. The dashed line gives V_{xx}/V_{HALL} for the NRL geometry (shown in Fig. 3). The quantity ϵ is defined by $\delta = \frac{1}{2}\pi(1-\epsilon)$.

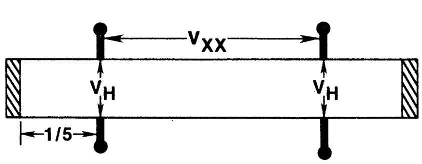

FIG. 3. Geometry of the NRL device. The aspect ratio is 6.5 and the Hall probes are located $\frac{1}{5}$ of the way along each side.

Our model also assumes that the Hall angle is uniform over the inversion layer. This results in a large discontinuity in the electric field lines at the conducting end electrodes in the $2\delta/\pi \to 1$ limit, giving the power-law singularity in the corners of the device. This situation of a high corner field density has been illustrated by Kawaji.[6] In an actual device, δ will be nonuniform near the electrodes due to the high field density, the finite conductivity of the electrodes, and rounding of the corners of the device. Therefore the assumption of a constant δ becomes less accurate in the case of very-large-aspect ratio where the field singularity is more severe. However, as mentioned in Sec. II, our model overestimates the error in this respect. The neglect of the magnetic fields generated by the currents in the device also becomes less accurate for large-aspect ratio.

Our model assumes no current flow through the Hall probes. Any such current flow could lead to serious measurement errors because the output impedance of the device is so high ($R_{\rm out} = h/e^2 i$).

Other complications may arise in a small device due to such things as the finite size of the cyclotron orbit and fringing fields at the edge of the gate which makes the boundaries of the inversion layer ill-defined. It is hoped that nonuniformity in the direction normal to the plane (such as variations in oxide-barrier thickness) will not be important due to the insensitivity of the Hall resistance to the gate voltage at the Hall step. Another difficulty is the possibility of localization of some of the carriers by the random impurity potential. A physical argument has often been presented to demonstrate that localized states will not affect the Hall resistance; however, this has not been shown in detail.[7] R. E. Prange[8] has recently shown that this is true for the special case of a single delta-function scatterer.

For the purpose of a new determination of the fine-structure constant and development of a new resistance standard, it is desirable to reduce errors in the Hall-resistance measurement to the 10^{-8} level or lower. We conclude that, for inversion layers described by the present simple model, the error due to shorting of the Hall field at the ends of a finite device is negligible. From Fig. 4 one sees that for a device with aspect ratio 6.5 and $\epsilon = 5 \times 10^{-7}$, the error due to the shorting effect is approximately 3×10^{-10} for Hall probes at the center of the device.

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