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Fluid-magnet universality: Renormalization-group analysis of ϕ^5 operators

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The question of a possible difference between the universality classes of fluids and Ising-like magnets is addressed by perturbation theory and the renormalization group. The most dangerous possibility is that of an ϕ^5 addition to the usual ϕ^4 theory. We show that no ϕ^5 fixed point exists in the framework of an expansion around $d=\frac{10}{3}$. Further we show that to $O(\epsilon_4^2)$, $\epsilon_4 \equiv 4-d$, the ordinary ϕ^4 fixed point is stable against the perturbations that mix with ϕ^5. Two new correction-to-scaling exponents are found. One of the exponents, Δ_5, is poorly determined with a range of values from 0.5 to 1.0 compatible with the $O(\epsilon_4^2)$ result. However, its positivity rules out a separate fluid fixed point, indicating fluid-magnet asymptotic universality. The second exponent, Δ_3, can be determined exactly: $\Delta_3 = 1-\alpha-\beta$. This implies the universal existence of a contribution to the fluid diameter scaling like the internal energy.

I. INTRODUCTION

Recently, Valls and Hertz[1] suggested that the widely, if tentatively, held notion that fluid systems had the same asymptotic critical behavior as uniaxial magnets[2] was open to question. They based their analysis on the fact that for d, the dimension of space, less than $\frac{10}{3}$ (not $\frac{10}{3}-\frac{5}{3}\eta$ as stated in Ref. 1), the ϕ^5 term in a Landau-Ginzburg-Wilson effective Hamiltonian becomes relevant at the Gaussian or trivial fixed point. This opens the possibility of a new fixed point with new critical-point exponents that would be distinct from those at symmetric magnetic systems

In general, there are two generic ways new fixed points can appear as some parameter such as the number of dimensions or field components is varied: (a) splitting off from existing fixed points; and (b) appearing in pairs in any region. In the first case, the signature is the approach to marginality of some operator representing a perturbation on an existing fixed point.[3] The classic example is the splitting off of the Wilson-Fisher fixed point (ϕ^4) from the Gaussian as d goes below four. It should be emphasized that "marginality" is a necessary but not sufficient condition. There are fewer examples of the second[4] case. We only wish to remark that a systematic theory of the renormalization group flows in the neighborhood (where the fixed points are to appear) is needed for credible description of such behavior.

For the fluid case, within the context of a perturbative one-component field theory, we know of no scheme by which a pair of fixed points can appear in a controlled fashion. However, we can say something about the possibility of new fixed points due to possibility (a). Following the philosophy of Valls and Hertz, we will consider the effects of the ϕ^5 operator, which is expected in a Landau-Ginzburg-Wilson Hamiltonian with no $\phi \rightarrow -\phi$ symmetry, on the two known fixed points: Gaussian and Wilson-Fisher.

The nature of this perturbation on the first is exactly known: marginal in $d=\frac{10}{3}$ and relevant for $d<\frac{10}{3}$. Thus, there is a *possibility* of a "ϕ^5 fixed point" in an ϵ_5 expansion ($\epsilon_5 \equiv 5-3d/2$). However, in Sec. II, we will show that this possibility is not realized: To first order in ϵ_5, no fixed point associated with a pure ϕ^5 theory exists. Lacking a fixed point at this order, there can be none at all, *within the* *framework of the ϵ_5 expansion.* This property persists for all odd ($\geqslant 3$) powered single-component field theories. (For multicomponent theories, tensorial couplings may alter this conclusion.)

The effect of ϕ^5 on the Wilson-Fisher fixed point may be calculated in the well-known ϵ_4 expansion ($\epsilon_4 \equiv 4-d$). The $O(\epsilon_4)$ result is contained in the general result of Wegner[5]: It is strongly irrelevant for all $d<4$. In Sec. III, we present results to $O(\epsilon_4^2)$. No longer an "eigenoperator," this perturbation mixes nontrivially with $\phi^2\nabla^2\phi$. After diagonalization, both eigenperturbations are found to be still irrelevant at $d=3$, thus ruling out the possibility of a "fluid fixed point" splitting off from the usual ϕ^4 one. These two results allow us to conclude that fluid systems belong to the Ising universality class.

The two eigenoperators represent two distinct sources of asymmetric contributions. Recently, Vause and Sak,[6] working to $O(\epsilon_4)$, calculated the effects on the equation of state of the ϕ^5 term only; the other eigenoperator was not included. As the exponent calculations in Sec. III indicate and as shown in detail in a separate paper,[7] the operator omitted from Ref. 6 generates precisely those terms usually associated with the "revision" of the temperaturelike variable to include both true temperature and chemical-potential dependence.[8] Thus this operator guarantees the universal occurrence of a $|t|^{1-\alpha}$ singularity in the fluid diameter where $t \propto T-T_c$, α is the specific-heat exponent and the diameter is defined by $\rho_d \equiv (\rho_{\text{liquid}}+\rho_{\text{gas}}-\rho_c)/2$. In a precisely symmetric theory, the diameter would be zero. In general, analytic terms are possible leading to the law of the rectilinear diameter, $\rho_d \propto t$. The two eigenoperators discussed in Sec. III lead to two singular contributions $|t|^{\beta+\Delta_3}$ and $|t|^{\beta+\Delta_5}$ where β is the coexistence curve exponent and Δ_3 and Δ_5 are the correction-to-scaling exponents calculated here, with $\Delta_3 = 1-\alpha-\beta$ exactly.

II. ϕ^5 FIELD THEORY

As discussed in Sec. I, the appropriate approach to a ϕ^5 fixed point is an expansion around the Gaussian or free field theory in ϵ_5, $\epsilon_5 \equiv 5-3d/2$. We will show in renormalized field theory that no fixed point exists in such an expansion. The method is described in detail in Brézin *et al.*[9] and Amit.[10] To study the critical (massless) theory in this case, we must require the vanishing of the renormalized vertex functions, $\Gamma_R^{(N)}(p=0)=0$, $N=1,\ldots,4$. These conditions are met for $N=1$, 2, and 3 by the usual requirements of a second-order transition and the appropriate choice of critical parameters (temperature, pressure, and density). Although no external mechanism is available to keep $\Gamma^{(4)}=0$, we impose this condition to seek a ϕ^5 fixed point. If successful, the stability of such a point against ϕ^4 insertions would have to be explored.

We therefore consider an interaction Hamiltonian $g_0\int\phi^5(x)$. As is usual, we define a dimensionless coupling constant u_0 by $g_0 = u_0\mu^{\epsilon_5}$ where μ is an arbitrary mass scale. The μ independence of the bare theory is expressed in the following renormalization-group equations for the renormalized N-point vertex functions $\Gamma_R^{(N)}$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(u)\frac{\partial}{\partial u}-\frac{N}{2}\eta(u)\right]\Gamma_R^{(N)}=0 \ . \tag{2.1}$$

The functions β and η are functions of the renormalized dimensionless coupling constant u

$$\eta = \beta\frac{\partial}{\partial u}\ln Z \ , \tag{2.2a}$$

$$-\beta = \epsilon_5\left[\frac{\partial \ln u_0}{\partial u}\right]^{-1} \ , \tag{2.2b}$$

where Z is the wave-function renormalization constant. The renormalized vertex functions are defined by

$$\Gamma_R^{(N)}(u,\mu) = Z^{N/2}\Gamma_B^{(N)}(u_0) \ , \tag{2.3}$$

and are finite, even in the limit of infinite cutoff ($\Lambda\rightarrow\infty$) or at the borderline dimension, $\epsilon_5=0$.

At the lowest order, the precise renormalization scheme is not important. We can choose, e.g., the following conditions of renormalization:

$$\partial_{k^2}\Gamma_R^{(2)}(k=0)=1,\ \ \Gamma_R^{(5)}|_{\text{sp}} = u\mu^{\epsilon_5} \ , \tag{2.4}$$

where sp stands for symmetry point: $\vec{k}_i\cdot\vec{k}_j = (5\delta_{ij}-1)\mu^2/6$. Writing the relevant bare functions as

$$\Gamma_B^{(2)} = k^2\left[1-\frac{Au_0^2}{\epsilon_5}+O(u_0^4)\right] \ , \tag{2.5a}$$

$$\Gamma_B^{(5)} = u_0\mu^{\epsilon_5}\left[1+\frac{Cu_0^2}{\epsilon_5}+O(u_0^4)\right] \ , \tag{2.5b}$$

we have

$$u_0 = u\left[1-\left(C+\frac{5A}{2}\right)\frac{u^2}{\epsilon_5}+\cdots\right] \ , \tag{2.6a}$$

$$Z = 1+(Au^2/\epsilon_5)+\cdots \ , \tag{2.6b}$$

where A and C are contributions associated with the diagrams in Fig. 1.

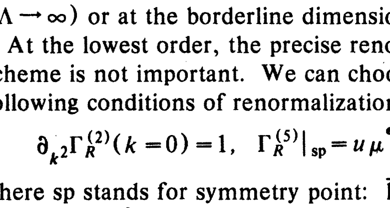

FIG. 1. The leading order diagrams of the two-point and five-point vertex functions are shown for the ϕ^5 theory and $\epsilon \ll 1$. ($\epsilon = \epsilon_5 \equiv 5-\frac{3}{2}d$.)

From Eq. (2.2) and these expression, we get

$$\beta(u) = -\epsilon_5 u - (2C + 5A)u^3 \ , \tag{2.7a}$$

$$\eta(u) = -2Au^2 \ . \tag{2.7b}$$

Evaluating the diagrams we have $A = -\frac{1}{720}$, $C = \frac{115}{72}$ so that

$$\beta(u) = -\epsilon_5 u - \frac{51}{16} u^3 \ . \tag{2.7c}$$

We see that there is no nontrivial fixed point at this order. Higher-order terms (e.g., u^5) can never produce a fixed point in the ϵ_5 expansion. Even if the signs were favorable, solving $\beta(u^*) = 0$ formally will produce $u^* \sim O(1)$ with ϵ_5 corrections. Because $\beta(u)$ is expected to have only an asymptotic expansion, finite-order calculations are not likely to be believable. If a convincing method were found, it would have nothing to do with the ϵ_5 expansion.

This result for ϕ^5 is typical of all single-component odd-power field theories ϕ^{2p+1}. The above analysis can be repeated, requiring $\Gamma_{(N)} = 0$ for $N = 1, \ldots, 2p$. The critical dimension is now $2(2p+1)/(2p-1)$ and there is a possibility of a fixed point if $\epsilon_{2p+1} \equiv 2p + 1 - [d(2p-1)/2]$ is positive. The diagrams needed for the computation of an equivalent C in Eq. (2.5b) consist of triangles with sides of r_1, r_2, and r_3 lines, $r_1 + r_2 + r_3 = 2p + 1$ (cf. Fig. 2). Similarly, for A they are like Fig. 1 except with $2p$ legs. Only these diagrams give a pole in ϵ_{2p+1}. The condition for *no* fixed point is

$$2C + (2p+1)A > 0 \ . \tag{2.8}$$

On evaluating the integrals, this becomes

$$\sum_{r_i} \prod_{i=1}^{3} \frac{\Gamma((2p+1-2r_i)/(2p-1))}{(r_i!)^2 \Gamma(2r_i/(2p-1))} > \frac{6}{(2p+1)!2p!\Gamma((2p+1)/(2p-1))} \ . \tag{2.9}$$

In the Appendix we present a bound to show that this is satisfied for $p \geq 2$. These considerations do not preclude the existence of multicomponent odd fixed points. For such systems with tensor couplings the appropriate invariants will weight the various terms in Eq. (2.9) differently. Since not all the terms are positive, condition (2.8) may be violated, leading to a fixed point.[11]

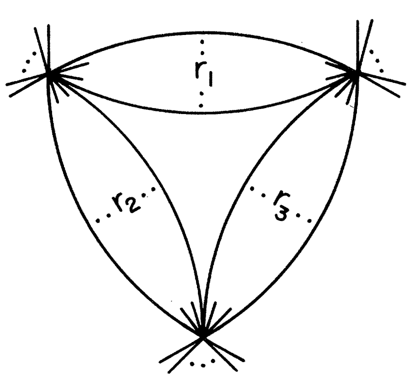

FIG. 2. The diagrams determining the constant C [cf. Eq. (2.8) in the text] for ϕ^{2p+1} theory consist of triangles each side of which have r_1, r_2, and r_3 legs with $r_1 + r_2 + r_3 = 2p + 1$.

Of course, Valls and Hertz recognize that a pure ϕ^5 theory has no fixed point. They propose that a quartic term of a particular nature may stabilize the quintic theory. They choose a form for Γ_4

$$\Gamma_4 = u_0 k_1^{\epsilon/3} k_2^{\epsilon/3} k_3^{\epsilon/3} \ , \tag{2.10}$$

where k_1 are the channel momenta and $\epsilon = \epsilon_4 = 4 - d$. They obtain the following form for Γ_5:

$$\Gamma_5 = v_5 + B v_5^3 (1 - Au_0)(-\ln r) \ , \tag{2.11}$$

where v_5 is the quintic coupling constant, A and B are constants, and r^{-1} is the susceptibility. If $Au_0 > 1$, this can indeed be exponentiated into

$$\Gamma_5 = v_5^* r^{[\epsilon_5 - (5/2)\eta]/(2-\eta)} \ , \tag{2.12}$$

thereby locating a fixed point value for v_5. However, another choice for Γ_4

$$\Gamma_4 = \frac{u_0(k_1^\epsilon + k_2^\epsilon + k_3^\epsilon)}{3} \ , \tag{2.13}$$

gives an additional term:

$$\Gamma_5 = v_5(1 + cu_0 \ln r) + B v_5^3 (1 - A' u_0)(-\ln r) \ . \tag{2.14}$$

This comes from the single loop diagram with one quartic and one quintic vertex and is present for all ϵ. In fact the first terms alone exponentiate to (at the same order)

$$\Gamma_5 = v_5 r^{-\lambda_5/(2-\eta)} \ , \tag{2.15}$$

where λ_5 is the anomalous dimension of the ϕ_5 insertion (cf. Sec. III). This represents the nontrivial ϕ^4 correction to the quintic eigenvalue. At higher order in u, operator mixing occurs but a similar result holds for the eigenoperator for all ϵ if the true Γ_4 is used. Thus, these diagrams can be ignored (and an expansion in ϵ_5 is possible) if and only if $\lambda_5 \sim 0$. This is a restatement of the general principle that a new fixed point of this character is associated with a marginal eigenvalue since the quintic eigenvalue is $\epsilon_5 + \lambda_5 \sim \lambda_5$. With such a term properly included as in Eq. (2.15) it is clear that no fixed point exists near the quartic fixed point as ϵ_5 small.

The ansatz Eq. (2.13) is suggested by the spherical model for which Γ_4 can be computed exactly:

$$\Gamma_4(p_1,p_2,p_3,p_4) = \frac{u}{1 + \frac{u}{u^*}\left[\prod(k_1) - 1\right]} \ , \tag{2.16}$$

where $k_1 = p_1 + p_2$ and $\prod$ is the 1-loop integral[12]

$$\prod = \int_0^1 d\alpha [\alpha(1-\alpha)k^2 + m^2]^{-\epsilon/2} \ . \tag{2.17}$$

This is obtained by noting that Γ_4's diagram expansion consists of a geometrically summable chain of 1-loop bubbles. For $m=0$ and $u=u^*$, $\Gamma_4 \sim u^* k_1^{\epsilon}$.

Equation (2.13) is only put forward as a counterexample to Eq. (2.10) which fails to represent the full Γ_4 in that it suppresses, for example, the one-loop contribution included in Eq. (2.14). Equation (2.13) picks up this contribution because it is zero only if *all* the momenta are zero. As noted above, the exact Γ_4 always gives rise to a term similar to Eq. (2.15) which is sufficient to spoil the analysis of Valls and Hertz as long as $\lambda_5 < 0$. This will be shown in the following section.

III. ODD PERTURBATIONS AT THE SYMMETRIC FIXED POINT

Having eliminated the possibility of a ϕ^5 fixed point separating from the Gaussian fixed point, we now turn to the stability of symmetric fixed point. In this section we will drop the subscript 4 and write $\epsilon = 4-d$.

A study of ϕ^5 perturbations on a ϕ^4 theory is entirely appropriate for fluid systems which lack the inversion symmetry of a magnetic system. Further, the requirements of a second-order transition only give $\Gamma^{(N)}=0$ for $N=1$, 2, 3 and thus we expect the presence of all the others. The identification of the order parameter for the fluid system is not simple[2,8,13] but whatever order parameter is chosen, the possibility of asymmetric terms is clearly important.

The first order in ϵ, the effects of ϕ^m perturbations at the symmetric fixed point are well known.[5] They are eigenperturbations (to lowest order) and anomalous dimensions can be obtained (to lowest nontrivial order) without considerations for "off-diagonal" corrections. At second order, the full problem of operator mixing must be analyzed. Anomalous dimensions will appear as eigenvalues of a matrix. For the ϕ^5 case, mixing occurs between $\int d^dx\,\phi^5(x)$ and $\int d^dx\,\phi^2(x)\nabla^2\phi(x)$.

Our notation will be the same as that of Ref. 11 with the operators to be inserted (at zero momentum) chosen to have the same naive dimension, $1-3\epsilon/2$:

$$A_5 \equiv \frac{1}{5!}\int d^dx\,\phi^5(x) \ , \qquad (3.1a)$$

$$A_3 \equiv -\frac{\mu^{-\epsilon}}{3!}\int d^dx\,\phi^2(x)\nabla^2\phi(x) \ . \qquad (3.1b)$$

These are the only operators that have the same naive dimension (at $d=4$) of $+1$, so that in an ϵ expansion, they are the only ones that mix.

Since the method given by Amit *et al.*[11] for analyzing nearly degenerate operators is well documented there, we will only give a brief indication of the calculation.

$\Gamma_a^{(N)}(a=3,5)$ denotes the N-point vertex function with the insertion of A_a. The multiplicative renormalizable pair of vertex functions we will need is

$$\Gamma^{\{3\}} \equiv \sum_{i=1}^{3} \partial/\partial k_i^2 \Gamma^{(3)}|_{\rm sp}$$

and

$$\Gamma^{\{5\}} \equiv \Gamma^{(5)}|_{\rm sp}$$

where the symmetry point (sp) is defined by $\vec{k}_i\cdot\vec{k}_j = (l_a\delta_{ij}-1)/(l_a-2)$; $l_a=3,5$. We find the dimensionless matrix $\hat{\Gamma}$ (cf. Ref. 9) at two loops to be given by

$$\hat{\Gamma}_5^{\{5\}} = 1-5au_0+(\tfrac{25}{4}a^2+20b)u_0^2 \ , \qquad (3.2a)$$

$$\hat{\Gamma}_5^{\{3\}} = 0+0-\tfrac{1}{2}c(1-\epsilon)u_0 \ . \qquad (3.2b)$$

$$\hat{\Gamma}_3^{\{5\}} = 0+10au_0^2-(15a^2+55b)u_0^3 \ , \qquad (3.2c)$$

$$\hat{\Gamma}_3^{\{3\}} = 1-\tfrac{1}{2}a(1-\tfrac{1}{2}\epsilon)u_0 + (\tfrac{1}{4}a^2+\tfrac{1}{2}b+\tfrac{3}{2}c)(1-\epsilon)u_0^2 \ . \qquad (3.2d)$$

The $O(u_0^3)$ two-loop terms in Eq. (3.2c) do not enter the calculations of eigenvalues but are needed to prove consistency to $O(\epsilon^2)$. Diagrams corresponding to Eq. (3.2) are given in Fig. 3. In Eq. (3.2) u_0 is the bare dimensionless coupling constant related to the bare coupling $g_0 = u_0\mu^{\epsilon}$. For details of the ϕ^4 theory see, for example, Ref. 10. The symmetry point used gives, to this order, the same in-

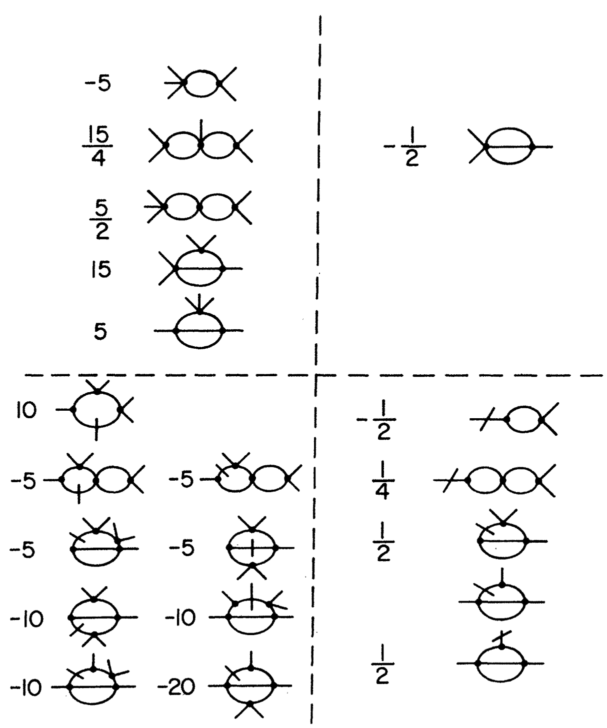

FIG. 3. The diagrams for the dimensionless matrix $\hat{\Gamma}_b^{\{c\}}$ are shown to two loops. A crossed line represents derivatives coming from A_3.

tegrals as in the ϕ^4 case[14]:

$$a=\frac{1}{\epsilon}(1+\frac{\epsilon}{2});\quad b=\frac{1}{2\epsilon^2}(1+\tfrac{3}{2}\epsilon);\quad c=-\frac{1}{8\epsilon}\ . \tag{3.3}$$

The eigenvalues of the anomalous dimension matrix γ are

$$\lambda_5=-\tfrac{10}{3}\epsilon+\tfrac{685}{324}\epsilon^2\ , \tag{3.4a}$$

$$\lambda_3=-\tfrac{4}{3}\epsilon+\tfrac{19}{162}\epsilon^2\ . \tag{3.4b}$$

Taking into account the naive dimension the conditions for irrelevance are $\omega_a=1-\frac{3}{2}\epsilon-\lambda_a>0$. To the order calculated we find at $\epsilon=1$

$$\omega_5=0.72\ , \tag{3.5a}$$

$$\omega_3=0.95\ , \tag{3.5b}$$

so that both perturbations are irrelevant at $d=3$. We note that Eq. (3.4b) is consistent with the result deducible from an analysis of the equation of motion[7]

$$\omega_3=\frac{d-2}{2}+\left(2-\frac{1}{\nu}-\tfrac{1}{2}\eta\right)\ . \tag{3.5c}$$

The correction-to-scaling exponents are $\Delta_i=\omega_i\nu$. Equation (3.5c) gives

$$\Delta_3=1-\alpha-\beta\ , \tag{3.6}$$

which is, of course, confirmed to $O(\epsilon^2)$ by the present calculation. This term is responsible for $|t|^{\beta+\Delta_3}(=|t|^{1-\alpha})$ singularities in the fluid diameter. The complete equivalence to revised scaling does not fall within the massless formalism used here but is given in Ref. 7.

The ϕ^5 exponent Δ_5 has the nearly useless expansion

$$\Delta_5=\tfrac{1}{2}(1+2\epsilon-\tfrac{31}{18}\epsilon^2)+O(\epsilon^3)\simeq 0.64\ . \tag{3.7}$$

This differs from the estimate $\Delta_5=\omega(\epsilon=1)\ \nu(\epsilon=1)\sim 0.46$. An accurate value for this exponent awaits more detailed study.[15] As an example

$$\omega_5=+\frac{1+(\frac{11}{6}+\frac{685}{594})\epsilon}{1+\frac{685}{594}\epsilon}\simeq +1.85\ . \tag{3.8}$$

This leads to $\Delta_5\sim 1.18$ while a similar Padé for Δ_5 itself gives $\Delta_5\sim 1.02$. Therefore anything in the range $0.5\leq\Delta\leq 1.0$ seems compatible with the present result.

This uncertainty in the exponent is transmitted to the equation of state. A phenomenological analysis including both effects has been given by Ley-Koo.[16] A detailed discussion of the renormalization-group calculation of the free energy and equation of state is deferred to Ref. 7.

We note finally that all other odd perturbation such as $\phi^7, \phi^9, \ldots$ are strongly irrelevant at $d=3$ at the Gaussian fixed point and, at first order in ϵ, become even more irrelevant at the Wilson-Fisher fixed point.[5] Therefore, we do not expect any such terms to affect the fluid-magnet universality indicated by the ϕ^5 calculations given here.

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APPENDIX

Here we supply some mathematical details for the analysis of a massless pure ϕ^m (m odd) theory. The interaction term is

$$\frac{g_0}{m!}\int d^dx\,\phi^m(x)\ . \tag{A1}$$

The critical dimension is

$$d^*=2m/(m-2)\ . \tag{A2}$$

We define ϵ by

$$\epsilon\equiv(\tfrac{1}{2}m-1)(d^*-d)\ , \tag{A3}$$

so that in ϵ expansions, $d=2(m-\epsilon)/(m-2)$ and g_0 has dimensions of $(\text{mass})^\epsilon$.

For the two-point function, we need to evaluate

$$D(p;m,d)=\int\left[\prod_{i=1}^{m-2}d^dk_i(2\pi)^{-d}(k_i)^{-2}\right]\left(p-\sum k_i\right)^{-2}\ .$$

Using coordinate space representation, this can be done exactly. The result is (S_d = volume of sphere in d dimensions):

$$D(p;m,d)=\left[\frac{S_d}{(2\pi)^d}\right]^{m-2}\left[\frac{1}{2}\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)\right]^{m-2}\left[\frac{\Gamma[(d/2)-1]\Gamma\{1-[(d-2)(m-2)/2]\}}{\Gamma[(d-2)(m-1)/2]}\right]p^{2+[(d-2)(m-2)-4]}\ . \tag{A4}$$

The contribution, including combinational factors, to $G_B^{(2)}$ is just

$$p^{-2}\frac{g_0^2}{\Gamma(m)}D(p)p^{-2}\ . \tag{A5}$$

Defining $u_0=\mu^{-\epsilon}g_0[S_d/(2\pi)^d]^{(m-2)/2}$, we write the result of the simple pole term (apart from a factor of p^{-2}):

$$-\frac{u_0^2}{\epsilon}\frac{(m-2)}{m!}\left[\frac{1}{2}\Gamma\left(\frac{2}{m-2}\right)\Gamma\left(\frac{m}{m-2}\right)\right]^{m-2}\ .$$

The factor A in Eq. (2.5a) is the coefficient of u_0^2/ϵ

in the formula [with $m=5$ for Eq. (2.5a)]. So

$$A=-\left[\frac{(m-2)}{m!}\frac{1}{2}\Gamma\left(\frac{2}{m-2}\right)\Gamma\left(\frac{m}{m-2}\right)\right]^{m-2} . \quad \text{(A6)}$$

The sign in front of Au_0^2/ϵ comes from $p^{(2)}=1/G^{(2)}$.

For the m-point function, only "triangle" graphs (Fig. 2) give rise to simple poles in ϵ. Each triangle graph has $r_i(i=1,2,3)$ legs in its three sides with total of $\sum r_i = m$ legs. The integral associated with such a side is just $D(k+q;r_i+1,d)$. Here k is a momentum around the triangular loop and q is some external momenta. Finally we must integrate over k. The pole term is independent of external momenta so that

$$\begin{aligned} I(r_1,r_2,r_3) &= \text{Res}\int \frac{d^dk}{(2\pi)^d} D(k;r_1+1,d)D(k+p;r_2+1,d)D(k+q;r_3+1,d) \\ &= \frac{S_d}{2(2\pi)^d} D(1;r_1+1,d)D(1;r_2+1,d)D(1;r_3+1,d) \ . \end{aligned} \quad \text{(A7)}$$

where $D(1;\cdot,\cdot)$ is a shorthand for the coefficient in (A4) and Res represents the residue. The weight of each of these (for $G^{(m)}$) is

$$m!\left(\frac{-1}{3!}\right)\left(\frac{1}{m!}\right)^3 T^3(m;r_1,r_2,r_3)r_1!r_2!r_3! \ , \quad \text{(A8)}$$

where T is the trinomial coefficient $m!/(r_1!r_2!r_3!)$. Making the usual absorption of spherical factors into u_0^2, we have

$$\Gamma^{(m)} = g_0\left\{1+\frac{u_0^2}{\epsilon(3!)}\left[\frac{1}{2}\Gamma\left(\frac{2}{m-2}\right)\Gamma\left(\frac{m}{m-2}\right)\right]^{m-2}\Gamma\left(\frac{2}{m-2}\right)T^2R_1R_2R_3\right\} \quad \text{(A9)}$$

so that the factor C for Eq. (2.5b) is

$$C=\frac{(\frac{1}{2}\Gamma\Gamma)^{m-2}}{3!m!}\Gamma\left(\frac{2}{m-2}\right)\sum T^2(m;r_1,r_2,r_3)R_1R_2R_3 \ , \quad \text{(A10)}$$

where

$$R_i=\Gamma\left(\frac{m-2r_i}{m-2}\right)\Big/\Gamma\left(\frac{2r_i}{m-2}\right) \ . \quad \text{(A11)}$$

From Sec. II, the condition for *no* fixed point is $2C>-mA$, i.e.,

$$\sum T^2(m;r_1,r_2,r_3)R_1R_2R_3 \overset{?}{>} \frac{6m}{\Gamma}\left(\frac{m}{m-2}\right) \ . \quad \text{(A12)}$$

Using the representation

$$\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Big/\Gamma\left(\frac{\nu-\mu+1}{2}\right)=\lim_{\alpha\to 0}\int_0^\infty 2e^{-\alpha x}J_\nu(2x)x^\mu dx \ , \quad \text{(A13)}$$

we can do the sum by the formula

$$\sum T^2 x^{2r_1}y^{2r_2}z^{2r_3}=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\frac{d\phi}{2\pi}(x+ye^{i\theta}+ze^{i\phi})^m(x+ye^{-i\theta}+ze^{-i\phi})^m \ . \quad \text{(A14)}$$

Repeated application of

$$|x+ye^{i\theta}|^2 \geq 2|x|\,|y|(1+\cos\theta) \ , \quad \text{(A15)}$$

gives a lower bound for Eq. (A14) in which the angular integrals

$$\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\frac{d\phi}{2\pi}(1+\cos\theta)^{m/2}(1+\cos\phi)^m \quad \text{(A16)}$$

appear. The integrands being positive, we can obtain a lower bound by

$$(1+\cos\theta) > \begin{cases} \frac{3}{2}, & -\frac{\pi}{3}<\theta<\frac{\pi}{3} \\ 0, & \text{otherwise} \ , \end{cases} \quad \text{(A17)}$$

so that Eq. (A16) is greater than

$$\frac{1}{9}\left(\frac{3}{2}\right)^{3m/2} . \tag{A18}$$

Performing the x,y,z integrals using Eq. (A13) will lead to

$$\sum T^2RRR > 3^{(3m-4)/2}\Gamma(\tfrac{1}{2})\Big/\Gamma\left[\frac{m}{m-2}+\frac{1}{2}\right] . \tag{A19}$$

So, the condition for no fixed point will be satisfied if

$$3^{(3m-6)/2}B\left[\frac{1}{2},\frac{m}{m-2}\right] > 2m . \tag{A20}$$

Since the beta function $B \geqslant \frac{16}{15}$ for $m \geqslant 3$, it is easy to check that the inequality is satisfied for $m \geqslant 5$.

[1]O. T. Valls and J. A. Hertz, Phys. Rev. B 18, 2367 (1978).
[2]See, for example, A. Levelt Sengers, R. Hocken, and J. V. Sengers, Phys. Today 30, 42 (1977).
[3]J. F. Nicoll, T. S. Chang, and H. E. Stanley, Phys. Rev. A 13, 1251 (1976).
[4]B. I. Halperin, T. C. Lubensky, and S-K. Ma, Phys. Rev. Lett. 32, 292 (1974); R. K. P. Zia and D. J. Wallace, J. Phys. A 8, 1495 (1975).
[5]F. J. Wegner, Phys. Rev. B 6, 1891 (1972). The applicability of Wegner's result to $O(\epsilon)$ has been pointed out by P. C. Hohenberg, in *Microscopic Structure and Dynamics of Liquids*, edited by J. Dupay and A. J. Dianous (Plenum, New York, 1978), pp. 333–365; J. F. Nicoll, Phys. Lett. 76A, 112 (1980), and has been rederived by C. Vause and J. Sak in Ref. 6.
[6]C. Vause and J. Sak, Phys. Lett. 77A, 191 (1980); Phys. Rev. A 21, 2099 (1980).
[7]J. F. Nicoll, Phys. Rev. A (in press).
[8]The term "revised scaling" is due to J. J. Rehr and N. D. Mermin, Phys. Rev. A 8, 472 (1973). For a recent review see J. V. Sengers and J. M. H. Levelt Sengers, in *Progress in Liquid Physics*, edited by C. A. Croxton (Wiley, Chichester, 1978), pp. 103–174.
[9]E. Brézin, J. C. LeGuillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), pp. 125–174.
[10]D. J. Amit, *Field Theory, the Renormalization-Group and Critical Phenomena* (McGraw-Hill, New York, 1978).
[11]D. J. Amit, D. J. Wallace, and R. K. P. Zia, Phys. Rev. B 15, 465 (1977).
[12]S-k. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, 1976), p. 308.
[13]L. Mistura, Nuovo Cimento 51, 125 (1979); 52, 277 (1979).
[14]Reference 10, pp. 234 and 235.
[15]High-order estimates of the sort described in J. C. LeGuillou and J. Zinn-Justin, Phys. Rev. Lett. 39, 95 (1977), and Phys. Rev. B 21, 3976 (1980), should provide more reliable results.
[16]M. Ley-Koo and M. S. Green, Phys. Rev. A 16, 2483 (1977). M. Ley-Koo, Ph.D. thesis (Temple University, Philadelpia, 1976) (unpublished); M. Ley-Koo and M. S. Green, Phys. Rev. A 23, 2650 (1981).