Derivation of extended scaling relations between critical exponents in two-dimensional models from the one-dimensional Luttinger model

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The extended scaling relations between the critical exponents of the 8-vertex model can be derived by a mapping of this model onto the Luttinger model. The equivalence of this method to the one that connects the 8-vertex model to the Gaussian model is discussed. The Luttinger model is equivalent to the Gaussian model. Its operators are identified as vortex and spin-wave operators. The spin-wave operator $\cos 4\phi$ is present in the critical 8-vertex Hamiltonian via an umklapp process. This explains the Kosterlitz-Thouless transition in the 6-vertex model, and resolves questions concerning the validity of the lattice continuum limit in the treatment by Luther and Peschel.

INTRODUCTION

From the Baxter solution[1] it is known that the critical exponents in the 8-vertex model vary continuously. The solution gives the exponent of the energy operator (the specific heat). It has been conjectured, mainly on numerical evidence, that the exponents of different operators satisfy extended scaling relations, such as[2,3]

$$x_T^{8V} x_T^{AT} = 1, \quad x_E^{8V} = \frac{1}{4} x_T^{8V} .$$

x_T^{8V} is the correlation function exponent of the energy operator, x_E^{8V} that of the electrical field, and x_T^{AT} that of the crossover operator (which in the Ashkin-Teller language becomes an energy operator).

Recently these relations are derived by exploring the relationship of the 8-vertex model to the generalized Villain model.[4,5] This is a Gaussian model with interactions that introduce vortex and spin-wave excitations. Kadanoff and Brown[6] used the operator expansion method. Knops[7] on the other hand showed that the 8-vertex model can be imbedded in the generalized Villain model. After a few initial renormalization transformations, the 6-vertex model (i.e., the critical line in the 8-vertex model) will be described by the exact soluble Gaussian model with spin-wave interactions of type $\cos 4\phi$. The presence of $\cos 4\phi$ is necessary to describe the infinite-order phase transition in the 6-vertex model.

The approach by Knops appears to be similar to an earlier one by Luther and Peschel.[8] The Luttinger model (= massless Thirring model) is a one-dimensional quantum-field model of massless fermions.[9] In Sec. I it will be shown that this model is the quantum-field version of the two-dimensional Gaussian model. The *XYZ* model (chain of Heisenberg spins) is in the same way the counterpart of the 8-vertex model.[10] Luther and Peschel showed that the *XXZ* model (which is equivalent to the 6-vertex model) maps, in the limit of small lattice constant, onto the Luttinger model. They derived the relation $x_E^{8V} = \frac{1}{4} x_T^{8V}$. Luther and Peschel's and Knops's results should be the same. They overlooked however the presence of the spin-wave operator. By a more careful review of their calculation it will be shown in Sec. II that this operator is hidden in the $\sigma_i^z \sigma_{i+1}^z$ operator as an umklapp process. After this adjustment the results agree with those of Knops. The *XXZ* model (6-vertex model) is equivalent to a massive-Thirring model (sine-Gordon model).

In order to be able to compare the two methods, we must first establish in detail the relationship between the Luttinger model and the Gaussian model (Sec. I). The generalized Villain model not only describes the critical behavior of the 8-vertex model, but also that of virtually all other 2D systems, e.g., the planar model (its Kosterlitz-Thouless transition describes superfluid He films), the sine-Gordon model, the discrete-Gaussian model (its roughening transition describes the growth of a crystal surface), and the 2D Coulomb gas. In the past these models have already been shown to be related to the massive-Thirring model.[11–13] This is a Luttinger model with extra fermion interactions that make the model massive. Because of the more fundamental nature of the Gaussian model, these equivalencies become very simple in the presentation of Sec. I. The transfer matrix of the Gaussian model is equal to the infinitesimal time-evolution operator of the Luttinger model. Moreover we can identify all vortex and spin-wave operators with fermion operators of the Luttinger model.

Kadanoff and Brown[6] and Knops[7] showed how to identify the operators of the 8-vertex model to those of the Gaussian model, without referring to the 1D quantum-field models. In Sec. II we will first translate the 8-vertex operators into the Pauli-spin operators of the *XXZ* model. Next, we will extend the Luther and Peschel method, and identify each of them to a fermion operator in the Luttinger language.

Some of the identifications that are reported in this paper are new. It is shown for example that a direct electrical field in the 6-vertex model (i.e., a transverse field in the *XXZ* model) corresponds to the operator $\partial\phi/\partial\vec{r}$ (the density operator) in the Gaussian (Luttinger) language. This leads to the remarkable result that the 6-vertex model in a direct electrical field can be used as a model to describe a commensurate-incommensurate transition (Sec. II).

This does not mean that the quantum-field method is more powerful. The same results can be obtained by the operator product expansion method or by the renormalization method of Knops. There was some hope that the quantum-field models might give more insight in the operators that do not fit in the present scheme, e.g., the magnetic field operator of the 8-vertex model and the Potts operators.[6,7,14]

This is not the case, since the set of fermion operators of the Luttinger model are precisely the same as that of the Gaussian model. The quantum-field method discussed in this paper appears to be a third equivalent way of showing the relationship between the spin models and the Gaussian model, and of deriving the extended scaling relations for the critical exponents of the spin models.

I. EQUIVALENCE OF THE LUTTINGER MODEL TO THE GAUSSIAN MODEL

The Luttinger model was introduced in the context of a one-dimensional electron gas (of spinless electrons).[9] Figure 1(a) shows schematically the dispersion relation in a tight-binding approximation. Extra interactions will only influence the low-lying excitations, i.e., the states at the Fermi surface. A truncation of the dispersion relation far away from k_F is not expected to change the physics, while it may simplify calculations. Identify the states around $+k_F$ with type-"1" particles (moving to the right) and those around $-k_F$ with type-"2" particles (moving to the left)

$$\psi_1(k)=\psi(k+k_F),\quad \psi_2(k)=\psi(k-k_F) \ . \tag{1.1}$$

Then we can represent the Hamiltonian by the linearized form

$$H_0=\sum_k vk[\psi_1^\dagger(k)\psi_1(k)-\psi_2^\dagger(k)\psi_2(k)] \ . \tag{1.2}$$

This is the diagonal part of the Luttinger Hamiltonian. The particles are fermions; they satisfy anticommutation relations

$$\begin{aligned}&\{\psi_i(k),\psi_j(l)\}=\{\psi_i^\dagger(k),\psi_j^\dagger(l)\}=0 \ ,\\ &\{\psi_i^\dagger(k),\psi_j(l)\}=\delta_{kl}\delta_{ij} \ .\end{aligned} \tag{1.3}$$

In real space the model is considered to be continuous and periodic over a length L. Then strictly speaking the energy levels are not bounded from below ($-\infty < k < \infty$). This means that there is no

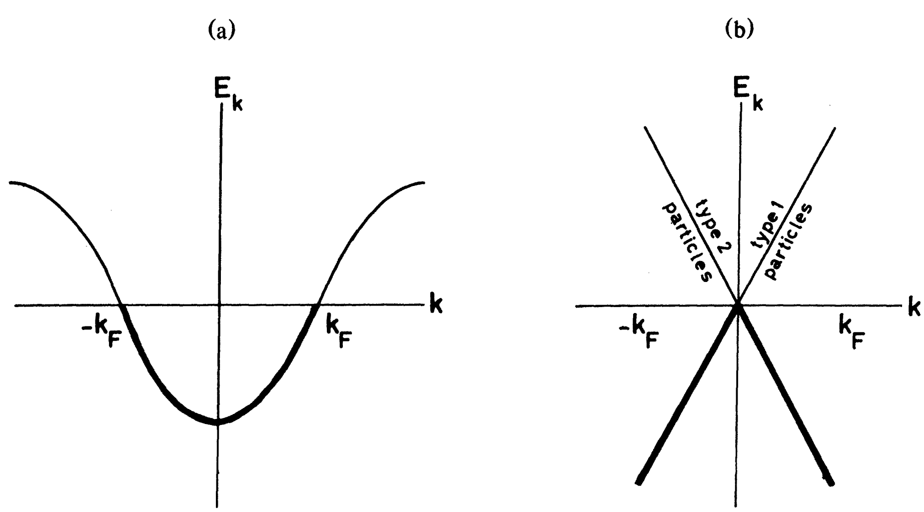

FIG. 1. Dispersion relations for an (a) electron gas in a tight-binding approximation and the *XX* model, and for (b) the Luttinger model.

ground state. In order to keep the model physically meaningful we have to use a cutoff for large k. That is, we visualize that our model is really placed upon a lattice with a small lattice constant. When in the ground state of the lattice model all states with $\epsilon_k < 0$ are filled, the continuum limit leads to a Dirac sea that is filled up to $\epsilon_k = 0$.

Consider the momentum representation of the density operators

$$\rho_i(r) = \psi_i^\dagger(r)\psi_i(r) - A \quad . \tag{1.4}$$

Since

$$\psi_i(r) = L^{-1/2}\sum_k e^{ikr}\psi_i(k) \quad , \tag{1.5}$$

one finds that $\rho_i(k)$ is given by

$$\rho_i(k) = \int_0^L dr\, e^{ikr}\rho_i(r) = \sum_l \psi_i^\dagger(k+l)\psi_i(l) - LA\,\delta_{k,0} \quad . \tag{1.6}$$

The constant A is chosen such that the expectation value of ρ_i with respect to the ground state is zero. A is only finite when a cutoff procedure is used ($A = 1/2\pi\alpha = 1/2L$ number of states).

Mattis and Lieb[15] showed that, in the above-mentioned continuum limit, $\rho_1(k)$ and $\rho_2(k)$ have a boson character

$$[\rho_1(-k),\rho_1(l)] = [\rho_2(k),\rho_2(-l)] = \frac{kL}{2\pi}\delta_{k,l} \quad . \tag{1.7}$$

They are raising and lowering operators of H_0

$$\begin{aligned} [H_0,\rho_1(k)] &= \nu k\rho_1(k) \quad , \\ [H_0,\rho_2(k)] &= -\nu k\rho_2(k) \quad . \end{aligned} \tag{1.8}$$

Therefore the fermion operators in H_0 can be replaced by these boson operators

$$\begin{aligned} H_0 &= \frac{2\pi\nu}{L}\sum_{k>0}[\rho_1(k)\rho_1(-k) + \rho_2(-k)\rho_2(k)] + E_0 \\ &= \frac{2\pi\nu}{L}\sum_{\substack{\text{all}\\ k}}\tfrac{1}{2}[\rho_1(k)\rho_1(-k) + \rho_2(-k)\rho_2(k)] + \tfrac{1}{2}E_0 \quad . \end{aligned} \tag{1.9}$$

In the Luttinger model a nondiagonal boson pair interaction is added.

$$H_L = H_0 + \frac{2\pi\lambda}{L}\sum_{k>0}[\rho_1(k)\rho_2(-k) + \rho_2(k)\rho_1(-k)] \quad . \tag{1.10}$$

The diagonalization is simple. We use a canonical transformation $e^{iS(\phi)}$. Define $S(\phi)$ to be

$$S(\phi) = \frac{2\pi i}{L}\phi\sum_{k>0}\frac{\rho_1(k)\rho_2(-k) - \rho_2(k)\rho_1(-k)}{k} \quad . \tag{1.11}$$

Its effect on the density operators is given by

$$e^{iS(\phi)}[\rho_2(k) \pm \rho_1(k)]e^{-iS(\phi)} = e^{\pm\phi}[\rho_2(k) \pm \rho_1(k)] \quad . \tag{1.12}$$

When

$$\frac{\lambda}{\nu} = -\tanh 2\phi \tag{1.13}$$

this brings H_L in the diagonal form H_0 with an effective $\tilde{\nu} = \nu\cosh^{-1}2\phi$. The diagonalization is only possible for $|\lambda/\nu| < 1$.

In addition to the fermion and boson representation of H_L there exists also a free scalar-field representation.[12,16] Define for this purpose $\Theta_+(r)$ and $\Theta_-(r)$ via a type of Jordan-Wigner transformation

$$\Theta_\pm(r) = 2\pi i\int_0^r dx\,[\rho_2(x) \pm \rho_1(x)] \quad . \tag{1.14}$$

Their derivatives are the density operators

$$\frac{\partial}{\partial r}\Theta_\pm(r) = 2\pi i[\rho_2(r) \pm \rho_1(r)] \quad . \tag{1.15}$$

Notice that the periodic boundary condition $\Theta_\pm(r) = \Theta_\pm(r+L)$ is ensured via the normal ordering of the $\rho_i(r)$ (the constant A).

In its real-space representation H_L can now be rewritten

$$H_L = -\frac{1}{2\pi}\int dr\left[\tfrac{1}{4}(\nu+\lambda)\left(\frac{\partial\Theta_+}{\partial r}\right)^2 + \tfrac{1}{4}(\nu-\lambda)\left(\frac{\partial\Theta_-}{\partial r}\right)^2\right] \quad . \tag{1.16}$$

Here we dropped the constant $\frac{1}{2}E_0$. Next define a momentum operator p and position operator q

$$p(r) = \frac{i}{2\pi}\frac{\partial\Theta_-}{\partial r}, \quad q(r) = \frac{i}{2}\Theta_+ \quad . \tag{1.17}$$

It is easy to check via Eq. (1.7) that p and q indeed satisfy canonical commutation relations

$$\begin{aligned} [p(r),q(s)] &= -i\delta(r-s) \quad , \\ [p(r),p(s)] &= [q(r),q(s)] = 0 \quad . \end{aligned} \tag{1.18}$$

H_L reduces in these operators to the free scalar-field Hamiltonian:

$$H_L = \int dr\left[\frac{\pi}{2}(\nu-\lambda)p^2(r) + \frac{1}{2\pi}(\nu+\lambda)\left(\frac{\partial q}{\partial r}\right)^2\right] \quad . \tag{1.19}$$

The importance of this representation is, that it makes the equivalence of H_L to the Gaussian model obvious.

Consider a two-dimensional square lattice with at the sites $\vec{r}$ variables $\phi(\vec{r})$ that can take all real values. The $\phi(\vec{r})$ interact via nearest-neighbor

Gaussian interactions

$$H_{\mathcal{G}} = -\sum_{\vec{r}} \tfrac{1}{2} K_x [\phi(x,y) - \phi(x+s,y)]^2 + \tfrac{1}{2} K_y [\phi(x,y) - \phi(x,y+s)]^2 \ . \qquad (1.20)$$

In the limit of small lattice constant s the partition function reads

$$Z_{\mathcal{G}}(K_x,K_y) = \sum_{\{\phi(\vec{r})\}} \exp\left\{-\int dy \int dx \left[\tfrac{1}{2} K_x \left(\frac{\partial \phi}{\partial x}\right)^2 + \tfrac{1}{2} K_y \left(\frac{\partial \phi}{\partial y}\right)^2\right]\right\} . \qquad (1.21)$$

When the y direction is interpreted as a time axis, the K_y term represents the kinetic energy $p^2/2K_y$ (with p the momentum). The partition function is equal to the trace over the time-evolution operator for complex times t

$$Z_{\mathcal{G}}(K_x,K_y) = \mathrm{tr}_\phi \exp\left[-i \int dt\, H_L(\nu,\lambda)\right] \qquad (1.22)$$

of a free quantum-field Hamiltonian. (For a more detailed proof in the context of a Landau-Wilson Hamiltonian see Scalapino and Sears.[17]) So the transfer matrix of the Gaussian model is the infinitesimal time-evolution operator of H_L, with the identification

$$\nu + \lambda = \pi K_x, \quad (\nu - \lambda)\pi K_y = 1 \ . \qquad (1.23)$$

Despite its almost trivial nature, the Gaussian model plays an important role in our understanding of critical phenomena in two dimensions. The model is critical for all K_x, K_y. Its critical exponents vary continuously with the parameter ϕ

$$\phi = -\tfrac{1}{4} \ln(\pi^2 K_x K_y) = \tfrac{1}{2} \operatorname{arctanh}\left(-\frac{\lambda}{\nu}\right) . \qquad (1.24)$$

In order to move away from criticality and see the singularities in the thermodynamic properties, one needs to add interactions to $H_{\mathcal{G}}$ that introduce spin-wave and vortex excitations. This leads to the generalized Villain model. Important models such as the planar model, the discrete Gaussian model, the two-dimensional Coulomb gas and the 8-vertex model (including the Ashkin-Teller model) can be imbedded in the generalized Villain model.[5–7] Under a renormalization transformation their critical points flow towards the Gaussian model, which is exactly scale invariant. Properties such as the extended scaling relations between critical exponents are easily obtained and understood in the context of $H_{\mathcal{G}}$.

In the Luttinger language, the critical nature of $H_{\mathcal{G}}$ is reflected in the absence of a gap in the energy spectrum (it is linear); the model is massless. When the H_L interactions are added that are the equivalents of the spin-wave and vortex interactions, the model becomes massive. Stated more precisely: Introduce in $H_{\mathcal{G}}$ an extra interaction

$$H = H_{\mathcal{G}}(K_x,K_y) + \frac{u_i}{(2\pi\alpha)^2} \int d\vec{r}\, O_i(\vec{r}) \ . \qquad (1.25)$$

$O_i(\vec{r})$ is a local operator and u_i its conjugate field. Let the free energy show a singularity with respect to u_i

$$f \sim |u_i|^{2/y_i} \ . \qquad (1.26)$$

The critical exponent y_i can be obtained from a correlation function. At a critical point the correlation length diverges, i.e., the correlation functions show a power-law decay

$$\mathcal{G} = \langle O_i(\vec{r}+\vec{r}\,')O_i(\vec{r}\,')\rangle_{u_i=0} \sim r^{-2x_i} \ . \qquad (1.27)$$

Scaling implies that $x_i + y_i = 2$. From Eq. (1.22) it follows that in the Luttinger model $\mathcal{G}$ corresponds to a time-dependent correlation function in the ground state $|0\rangle$. For equal times

$$\mathcal{G} = \langle 0|O_i(r+r')O_i(r')|0\rangle \sim r^{-2x_i} \qquad (1.28)$$

corresponds to a correlation between two operators in the same row of the two-dimensional lattice.

The ground-state energy E_0 of the quantum-field model

$$H = H_L(\nu,\lambda) + \frac{u_i}{(2\pi\alpha)^2} \int dr\, O_i(r) \qquad (1.29)$$

is equal to the free energy of the classical model and therefore shows the same singularity with respect to u_i. The gap between E_0 and the energy of the first excited state is proportional to the inverse of the correlation length

$$\Delta \sim \xi^{-1} \sim |u_i|^{1/y_i} \ . \qquad (1.30)$$

First we discuss the spin-wave operators. Consider the partition function[5,18]

$$Z = \sum_{\{\phi(\vec{r})\}} \exp\left[-\sum_{\langle \vec{r},\vec{r}\,'\rangle} \frac{K}{2}[\phi(\vec{r}) - \phi(\vec{r}\,')]^2\right] \prod_{\vec{r}} \sum_{N(\vec{r})} \exp[ip\phi(\vec{r})N(\vec{r})]\, u_p^{N^2(\vec{r})} \ . \qquad (1.31)$$

The integer variables $N(\vec{r}) = 0, \pm 1, \pm 2, \ldots$ are just as the $\phi(\vec{r})$ located at the lattice sites $\vec{r}$. When the fugacity u_p is small, the $N(\vec{r})$ will only take the values $0, \pm 1$. Then, only spin-wave excitations with spin-wave number p are present. Since

$$\sum_{N(\vec{r})} \exp[ip\phi(\vec{r})N(\vec{r})]u_p^{N^2(r)} \simeq \exp\{2u_p\cos[p\phi(\vec{r})]\} \quad (1.32)$$

we have effectively included an external field that favors the values $\phi(\vec{r})=2\pi(n/p)(n=0, \pm1, \pm2, \ldots)$. The resulting system is called the sine-Gordon model. In the limit $u_p \to 1$, where the $\phi(r)$ are restricted to the values $2\pi(n/p)$ one obtains the discrete Gaussian model. When on the other hand we carry out in Eq. (1.31) the trace over all $\phi(\vec{r})$ configurations, the model becomes the Coulomb gas.

The operator

$$O_N(\vec{r})=\exp[iN\phi(\vec{r})] \quad (1.33)$$

generates a spin-wave excitation with spin-wave number N at site $\vec{r}$. The corresponding quantum-field operator is [see Eq. (1.17)]

$$O_N(r)=\exp[-\tfrac{1}{2}N\Theta_+(r)] \ . \quad (1.34)$$

For $N=\pm2$ this is the boson representation of the operator

$$\begin{aligned}\psi_1^\dagger(r)\psi_2(r)&=(2\pi\alpha)^{-1}O_2(r) \ ,\\ \psi_2^\dagger(r)\psi_1(r)&=(2\pi\alpha)^{-1}O_{-2}(r) \ .\end{aligned} \quad (1.35)$$

Just as in the case of H_0, which we can represent by both Eq. (1.2) and Eq. (1.9) this is not an operator identity. However, as first shown by Luther and Peschel,[19] they satisfy the same equations of motion and commutation relations, and hence have (apart from short-distance, cutoff effects) the same correlation functions.

What we have found now is the well-known equivalence of both the sine-Gordon model[12] and Coulomb gas[11] to the massive-Thirring model

$$H=H_L(\nu,\lambda) + \frac{u_2}{2\pi\alpha}\int dr\,[\psi_1^\dagger(r)\psi_2(r)+\psi_2^\dagger(r)\psi_1(r)] \ . \quad (1.36)$$

The fermion representation for integer values of N, other than 2, is obtained via the contraction of $\frac{1}{2}N$ different O_2 operators, e.g.,

$$\begin{aligned}O_4(r)&=\lim_{\epsilon\downarrow0}O_2(r)O_2(r+\epsilon)\\ &=(2\pi\alpha)^2\lim_{\epsilon\downarrow0}[\psi_1^\dagger(r)\psi_2(r)\psi_1^\dagger(r+\epsilon)\psi_2(r+\epsilon)]\\ &=(2\pi\alpha)^2[\psi_1^\dagger(r)\psi_2(r)]_\epsilon^2 \ .\end{aligned} \quad (1.37)$$

In the Luttinger language it is natural also to consider the operators

$$O_M(r)=\exp[-M\Theta_-(r)] = \begin{cases}(2\pi\alpha)^M[\psi_1(r)\psi_2(r)]_\epsilon^M, & M>0\\ (2\pi\alpha)^{-M}[\psi_2^\dagger(r)\psi_1^\dagger(r)]_\epsilon^{-M}, & M<0 \ .\end{cases} \quad (1.38)$$

These are the operators that generate vortex excitations in the Gaussian language, as we will see below.

Consider the partition function of the Villain model[4,5]

$$Z=\sum_{\{\Theta,M,n\}}\exp\left[\sum_{\langle\vec{r},\vec{r}'\rangle}-\frac{K}{2}[\Theta(\vec{r})-\Theta(\vec{r}')-2\pi n(\vec{r},\vec{r}')]^2\right] \times\prod_{\vec{R}}O_{qM}(\vec{R})u_q^{M^2(\vec{R})} \ , \quad (1.39)$$

with $0\le\Theta(\vec{r})<2\pi$ and $M(\vec{R}),n(\vec{r},\vec{r}')=0,\pm1,\pm2,\ldots$. The $\Theta(r)$ are located at the lattice sites $\vec{r}$, the integer variables $M(\vec{R})$ at the sites $\vec{R}$ of the dual lattice, and the $n(\vec{r},\vec{r}')=-n(\vec{r}',\vec{r})$ are bond variables. The vortex operator $O_{qM}(\vec{R})$ restricts the values of the four bond variables around $\vec{R}$.

$$O_{qM}(\vec{R})=\delta(qM(\vec{R}),n(\vec{r}_1,\vec{r}_2)+n(\vec{r}_2,\vec{r}_3)+n(\vec{r}_3,\vec{r}_4)+n(\vec{r}_4,\vec{r}_1)) \ . \quad (1.40)$$

For $u_q=0$ all $M(\vec{R})$ are zero, and no vortices are allowed. The bond variables can be rewritten then as site variables $n(\vec{r},\vec{r}')=n(\vec{r})-n(\vec{r}')$ and the model reduces via $\phi(\vec{r})=\Theta(\vec{r})+2\pi n(\vec{r})$ to the Gaussian model. For small u_q, $M(\vec{R})$ can take only the values $0, \pm1$. When $M(R)=1$, $O_{qM}(\vec{R})$ generates a vortex with vorticity q at site $\vec{R}$; along a closed path around R the sum over the difference variables $\Theta(\vec{r})-\Theta(\vec{r}')$ will add up to $q2\pi$. In the limit $u_q\to1$ the $n(\vec{r},\vec{r}')$ become unrestricted, and for $q=1$ give rise to an interaction that is periodic over 2π (the planar model).

The combined vortex and spin-wave operator

$$O_{N,M}(\vec{r})=O_N(\vec{r})O_M(\vec{r}) \quad (1.41)$$

generates a spin-wave excitation with spin-wave number N and a vortex excitation with vorticity M at site $\vec{r}$ (in the continuum limit the difference between $\vec{r}$ and $\vec{R}$ can be neglected). In the Luttinger language this becomes the operator

$$O_{N,M}(r)=\exp\left[-\frac{N}{2}\Theta_+(r)-M\Theta_-(r)\right] \ . \quad (1.42)$$

The simplest way of proving the identification of the vortex operators is obtained via studying the symmetries of both models. The Gaussian model is known to have a duality transformation.[5,18] It exchanges high and low temperatures. Moreover it maps the Villain model [Eq. (1.39)] and the sine-Gordon model [Eq. (1.31)] into each other. It maps vortex operators into spin-wave operators:

$$\pi K_x \leftrightarrow \frac{1}{\pi K_y}\ , \quad O_{N,M} \leftrightarrow O_{2M,N/2}\ . \tag{1.43}$$

In the Luttinger language, the same effect is obtained by changing the sign of $\rho_1(r)$. This implies $\lambda \rightarrow -\lambda$ and $\Theta_+ \leftrightarrow -\Theta_-$ and therefore proves the identification of the vortex operators.

The Gaussian model is also invariant under the dilatation $\phi(\vec{r}) \rightarrow e^{+\mu}\phi(\vec{r})$, $K_i \rightarrow e^{-2\mu}K_i$. Now the spin-wave and vortex operators are mapped according to $O_{N,M} \rightarrow O_{e^{\mu}N, e^{-\mu}M}$. In the Luttinger language this is obtained by the canonical transformation $e^{iS(\mu)}$ that we used before to diagonalize H_L.

For completeness, in the boson representation the single-fermion operators Ψ_i are given by

$$\psi_1 = (2\pi\alpha)^{-1/2}O_{-1,1/2},\quad \psi_1^\dagger = (2\pi\alpha)^{-1/2}O_{1,-1/2}\ ,$$
$$\psi_2 = (2\pi\alpha)^{-1/2}O_{1,1/2},\quad \psi_2^\dagger = (2\pi\alpha)^{-1/2}O_{-1,-1/2}\ . \tag{1.44}$$

As pointed out before, many properties of the Gaussian and Luttinger model with small vortex and spin-wave interactions can be understood from the power-law decay of the correlation functions. Their calculation is simple. In the Gaussian language they are extensively discussed by Kadanoff *et al.*[18] For the Luttinger model their calculation is described by Mattis and Lieb[15] (for the fermion representation) and by Luther and Peschel[8] for the boson representation.

Consider the equal time multipoint correlation function between n spin-wave and m vortex operators

$$\mathcal{G} = \langle 0|\prod_{i=1}^{n} \exp[-(N_i/2)\Theta_+(r_i)] \prod_{k=1}^{m} \exp[-M_k\Theta_-(r_k)]|0\rangle\ . \tag{1.45}$$

First transform to the diagonal form of H_L. Equations (1.11)–(1.14) give, when we define $x = e^{-\phi}$

$$\mathcal{G} = \langle 0|\prod_{i=1}^{n} \exp[-(N_i/2x)\Theta_+(r_i)] \prod_{k=1}^{m} \exp[-xM_k\Theta_-(r_k)]|0\rangle\ . \tag{1.46}$$

Split each vortex and spin-wave operator in an operator e^A that generates excitations and e^B that annihilates them

$$e^{\Theta_\pm(r)} = e^{A_\pm(r)}e^{B_\pm(r)}e^{C_\pm(r)}\ , \tag{1.47}$$

with

$$\begin{aligned} A_\pm(r) &= -\frac{2\pi}{L}\sum_{k>0}\frac{1}{k}[-\rho_2(-k)(e^{ikr}-1) \pm \rho_1(k)(e^{-ikr}-1)]\ ,\\ B_\pm(r) &= -\frac{2\pi}{L}\sum_{k>0}\frac{1}{k}[\rho_2(k)(e^{-ikr}-1) \mp \rho_1(-k)(e^{ikr}-1)]\ ,\\ C_\pm(r) &= -\tfrac{1}{2}[A_\pm(r),B_\pm(r)] + \frac{2\pi i}{L}r[\rho_2(0) \pm \rho_1(0)]\ . \end{aligned} \tag{1.48}$$

The commutator results from the relation $e^{A+B} = e^A e^B e^{-[A,B]/2}$. This relation is valid when $[A,B]$ commutes with both A and B. The commutator between an A and B operator is given by

$$\begin{aligned} [B_+(r),A_+(s)] = [B_-(r),A_-(s)] &= \ln\left[\frac{g(r-s,1)g(0,1)}{g(r,1)g(-s,1)}\right]\ ,\\ [B_+(r),A_-(s)] = [B_-(r),A_+(s)] &= \ln\left[\frac{h(r-s,1)h(0,1)}{h(r,1)h(-s,1)}\right]\ , \end{aligned} \tag{1.49}$$

with

$$g(r,a) = \exp\left[-2a\frac{2\pi}{L}\sum_{k>0}\frac{\cos(kr)-1}{k}\right]\ ,\quad h(r,a) = \exp\left[-2ia\frac{2\pi}{L}\sum_{k>0}\frac{\sin kr}{k}\right]\ . \tag{1.50}$$

The A and B operator are introduced because of their simple impact on the ground state: $B_\pm|0\rangle = 0$ and $\langle 0|A_\pm = 0$, while by definition $\langle 0|\rho_i(0)|0\rangle = 0$. The trick is to move all $e^B(e^A)$ to the right (left). A commutator results for every time an e^A and e^B operator are exchanged. So the correlation function factorizes into pair contributions

$$\mathcal{G}=\prod_{\substack{\text{spin-wave}\\ \text{pairs } i,j}} g\left[r_i-r_j,\frac{1}{4x^2}N_iN_j\right]\prod_{\substack{\text{vortex}\\ \text{pairs } k,l}} g(r_k-r_l,x^2M_kM_l)\prod_{\substack{\text{spin-wave-vortex}\\ \text{pairs } i,k}}[h(r_k-r_i,\tfrac{1}{2}N_iM_k)h(r_i,\tfrac{1}{2}N_iM_k)h(-r_k,\tfrac{1}{2}N_iM_k)]^{\delta}$$

$$\text{with } \delta=\text{sgn}(r_i-r_k) \ . \quad (1.51)$$

The boundary terms $g(r,1)$ from Eq. (1.49) cancel each other, since the total vorticity and spin-wave number are zero.

$$\sum_{i=1}^{n} N_i=\sum_{k=1}^{m} M_k=0 \ .$$

Both g and h are only properly defined with a cutoff in k. The usual procedure is to replace g by

$$g(r,a)=\exp\left[-2a\frac{2\pi}{L}\sum_{k>0}\frac{\cos(kr)-1}{k}e^{-\alpha|k|}\right]$$
$$=\left[\frac{r^2+\alpha^2}{\alpha^2}\right]^a\simeq\left|\frac{r}{\alpha}\right|^{2a} \ . \quad (1.52)$$

A cutoff independent description is obtained by introducing a reference length r_0 ($\gg\alpha$). Equation (1.50) implies

$$g(r,a)=g(r_0,a)\left|\frac{r}{r_0}\right|^{2a} \ , \quad h(r,a)=h(r_0,a) \ . \quad (1.53)$$

For $r<r_0$ the results are cutoff dependent, while at $r=0$, $g(0,a)=h(0,a)=1$. The vortex–spin-wave pairs in $\mathcal{G}$ only give rise to an extra multiplicative constant, because h is distance independent.

The pair-correlation functions give the critical exponents $x_{N,M}$ of the $O_{N,M}$ operators

$$\langle 0|O_{N,M}(r_1)O_{-N,-M}(r_2)|0\rangle\sim\left|\frac{r_0}{r_1-r_2}\right|^{2x_{N,M}} \ , \quad (1.54)$$

with

$$x_{N,M}=\left[\frac{N^2}{4x}+M^2x^2\right] \ . \quad (1.55)$$

The $x_{N,M}$ satisfy extended scaling relations, e.g.,

$$x_{0,1}x_{2,0}=1, \quad x_{1,0}=\tfrac{1}{4}x_{2,0} \ . \quad (1.56)$$

These relations can also be obtained as direct results of the above discussed symmetry relations that map the various $O_{N,M}$ onto each other.

The critical exponents vary continuously with ϕ, and can take all values $0\le x_{N,M}\le\infty$. Only for $x_{N,M}<2$ the operator $O_{N,M}$ is relevant, and generates a gap. For $x_{N,M}>2$ it is irrelevant; the model remains critical (massless).

The phase diagram of the generalized Villain model is well known.[5] For the case that only a spin-wave interaction is added

$$H=H_L(\nu,\lambda)+\frac{u_p}{(2\pi\alpha)^2}\int dr\, O_{p,0}(r) \quad (1.57)$$

(using the Luttinger language) it is shown in Fig. 2 ($\pi K=e^{-2\phi}$).

The value of x_p is sufficient for a zeroth-order approximation of the renormalization equations. When $x_p>2$ ($x_p<2$) one flows towards (away) from the Luttinger model (which itself is scale invariant). In the shaded domain the model remains critical (massless), while in the nonshaded domain it is massive. An equivalent statement is, that in the (scaling) limit $\alpha\downarrow 0$, one is allowed to neglect $O_{p,0}$ as long as $x_p>2$. Equation (1.54) implies that $O_{p,0}$ is proportional to α^{x_p}. So the second term in Eq. (1.57) is proportional to α^{x_p-2}. Along path 1 the gap vanishes with the exponent $y_p=2-x_p$ [see Eq. (1.30)]. Along path 2 the transition is of infinite order. It is the Kosterlitz-Thouless transition.[20] In this case the gap vanishes as

$$\Delta\sim\zeta^{-1}\sim\exp(-b|K-K_A|^{-1/2}) \ . \quad (1.58)$$

Suppose that we have been able to show that a specific model can be imbedded in the generalized-Villain model, and that its "critical" Hamiltonian can be described by Eq. (1.57) with $p=4$. Let in the

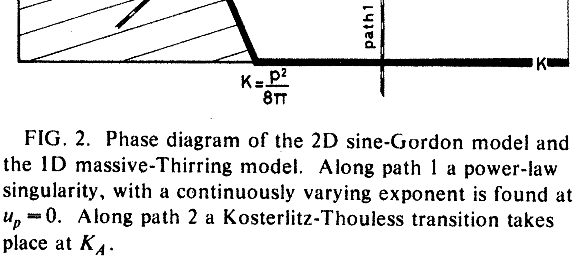

FIG. 2. Phase diagram of the 2D sine-Gordon model and the 1D massive-Thirring model. Along path 1 a power-law singularity, with a continuously varying exponent is found at $u_p=0$. Along path 2 a Kosterlitz-Thouless transition takes place at K_A.

language of the model λ/ν be the ratio between two coupling constants and u_p, such that one is restricted to path 2. Further let the operator $O_{0,1}$ be identified as the energy operator O_T and $O_{0,1/2}$ as an electrical field O_E.

From Fig. 2 we see that this model has a critical line with continuously varying exponents, that satisfy the extended scaling relation $x_E = \frac{1}{4}x_T$. Moreover the critical line is found to stop at K_A where it shows a Kosterlitz-Thouless transition. Beyond this point the free energy can remain regular or show a first-order transition with respect to O_T and O_E. This depends on the fixed point to which the points in the nonshaded domain flow. The properties of this fixed point lie beyond the Gaussian analysis.

As we will see in Sec. II the situation sketched in Fig. 2 is precisely that for the 8-vertex model.

Finally we have to discuss the density operators themselves. Equation (1.15) yields that in the Gaussian language these are the gradients of the $\phi(r)$ fields. Kadanoff[18] named them $F_{N,M}$ operators. Define

$$F_{N,M}(r) = \left[\frac{\partial \Theta_+}{\partial r}\right]^N \left[\frac{\partial \Theta_-}{\partial r}\right]^M . \tag{1.59}$$

Their critical exponents do not change along the critical line; the diagonalization of H_L only multiplies $F_{N,M}$ by a constant. The values of $x^{(F)}_{N,M}$ are again easily calculated; one finds

$$\langle 0|F_{N,M}(r_1)F_{N,M}(r_2)|0\rangle = c(r_1 - r_2)^{N+M} , \tag{1.60}$$

with

$$c(r) = -\frac{2\pi}{L}\sum_{k>0} 2k\cos(kr) . \tag{1.61}$$

The same cutoff procedure as in Eq. (1.52) yields

$$c(r) = \frac{\partial}{\partial r}\frac{-2r}{r^2+\alpha^2} \simeq \frac{2}{r^2} . \tag{1.62}$$

Our definition of $F_{N,M}$ actually slightly differs from that of Kadanoff. In his definition also terms like $(\partial^{n_1}/\partial r^{n_1})(\partial\Theta_+/\partial r)^{n_2}$ (with $n_1 + n_1 = N$) are included. These gradient operators have the same exponents $x = N + M$ since their correlation functions are derivatives of the ones in Eq. (1.60).

The low-lying $F_{N,M}$ operators require some extra comments. $F_{0,1}$ and $F_{1,0}$ have a relevant exponent $x = 1$ but do not generate a gap. When these interactions are added to H_L, the new Hamiltonian can be brought back to the original form via a translation $\rho_i \to \rho_i + b_i$. $F_{1,0}$ and $F_{0,1}$ are so-called redundant operators since the volume integral over their correlation function, leading to the susceptibility vanishes.

$F_{1,1}$, $F_{2,0}$, and $F_{0,2}$ have a marginal exponent $x = 2$. H_L exists out of these operators. Their marginality could be expected, since this is a necessary condition for the existence of a critical line with continuously varying critical exponents.

II. MAPPING OF THE 8-VERTEX AND XYZ MODEL ONTO THE LUTTINGER MODEL

In its Ising-spin representation the 8-vertex model consists of two Ising models with nearest-neighbor interactions. The spins $T(i)$ of the second model are located at the sites of the dual lattice of the first model [with spins $S(i)$]. The two models are coupled via a four-spin interaction

$$H_{8V} = \sum_{\langle ijkl\rangle} J_S S(i)S(j) + J_T T(k)T(l) + K S(i)S(j)T(k)T(l) . \tag{2.1}$$

The summation runs over all basic squares of the composite lattice. A critical line is located at $J_S = J_T$, $e^{2K}\sinh 2J = 1$. Its critical exponents vary continuously with the parameter $\phi = \tanh 2K$. Allowing $J_S \neq J_T$ corresponds to moving away from the critical line in a crossover direction. A duality transformation on the S_i spins converts the model into the Ashkin-Teller model. In that language the crossover operator $T(k)T(l) - S(i)S(j)$ becomes the energy operator. In the model solved by Baxter J_S must be equal to J_T. They are however allowed to be anisotropic. The Hamiltonian of the XYZ model

$$\begin{aligned} H_{XYZ} = -\sum_i [\sigma^x(i)\sigma^x(i+1) + \sigma^y(i)\sigma^y(i+1)] \\ + \lambda\sigma^z(i)\sigma^z(i+1) \\ + \gamma[\sigma^x(i)\sigma^x(i+1) - \sigma^y(i)\sigma^y(i+1)] \end{aligned} \tag{2.2}$$

is obtained from the transfer matrix by taking the logarithmic derivative with respect to the anisotropy parameter V, in the limit of extreme anisotropy.[10] This implies that the ground-state energy E_0 is equal to the derivative of the free energy with respect to V. The free energy of the 8-vertex model is regular with respect to the anisotropy.[1] So this is another redundant operator. E_0 has singularities with the same exponents as the 8-vertex model. For more details the reader is referred to Baxter's paper[10] and the Appendix.

The eigenspectrum of the XYZ model follows directly from the solution of the 8-vertex model.[10,21] Its phase diagram is shown in Fig. 3.

The model is critical (massless) at the five heavily lined curves. They are equivalent: (1) maps onto (3) via $\sigma^y \to -\sigma^z$, $\sigma^z \to \sigma^y$; the lines (1) and (4) are mapped onto each other when the same procedure is applied at the sites with even values of i and $\sigma^y \to \sigma^z$, $\sigma^z \to -\sigma^y$ at the uneven ones. In the standard identification, $\gamma = 0$ is the 6-vertex model and line (4) is the critical line that one encounters in the Ising

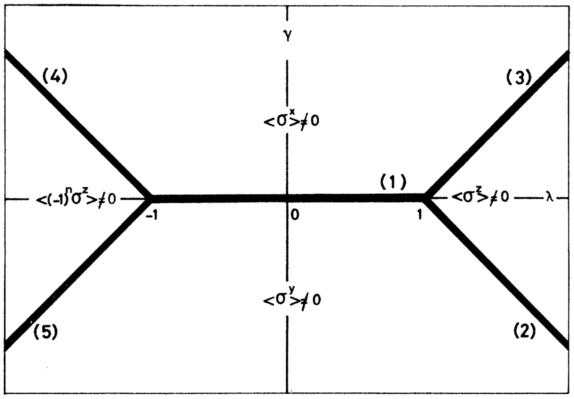

FIG. 3. Phase diagram of the XYZ model.

representation of the 8-vertex model. The XXZ model ($\gamma=0$) will be chosen as the basis for our further discussion. For $|\lambda|<1$, where the model is massless, a gap develops with respect to γ,

$$\Delta \sim \zeta^{-1} \sim |\gamma|^{1/y_T^{8V}} , \tag{2.3}$$

$$y_T^{8V} = 2 - x_T^{8V} = \frac{2}{\pi} \arccos(-\lambda) . \tag{2.4}$$

γ is directly related to a change of the temperature in the 8-vertex model. So $\sigma^x(i)\sigma^x(i+1)-\sigma^y(i)\sigma^y(i+1)$ is identified as the energy operator O_T^{8V}.

Many other operators can be added to H_{XXZ}. A list is given in Table I. Of course we have no Baxter solution to tell us whether these operators give rise to a gap, nor what are the critical exponents of such possible gaps. All these operators, however, have in the 8-vertex language a well-known meaning, and for most of them an extended scaling relation is proposed (see Table II). The details of the identification to 8-vertex operators can be found in the Appendix. We will be able to identify each of them with a spin-wave, vortex, or density operator in the Luttinger model. As a result their extended scaling relations will follow directly from Eq. (1.55).

The XXZ model becomes massive at $|\lambda|>1$. For $\lambda>1$ the model is frozen in the ferroelectrical ground state $\langle\sigma^z\rangle=1$. The transition at $\lambda=1$ is first order. Also in the Luttinger model this is a boundary of the critical line. From Eqs. (1.55) and (1.24) it follows that $0<x_{N,M}<\infty$. In the XXZ model x_T^{8V} is equal to zero at $\lambda=1$.

The other limit $\lambda=-1$ is of more interest. An infinite-order transition takes place into the antiferroelectrical order state. Here $x_T^{8V}=2$. This does not correspond to a natural boundary of the Luttinger model. Here the presence of the spin-wave operator $O_{4,0}$ (i.e., the notion that we are moving along path 2 in Fig. 2) becomes important.

As a first step towards the Luttinger model the

TABLE I. Identification of 8-vertex operators in the XYZ-model language.

Name	Identification in 8-vertex model	Identification in XYZ model Pauli-spin representation	Fermion representation
O_E^{6V}	Direct electrical-field 6-vertex model	$\sigma^z(n)$	$\rho(n)$
		$i[\sigma^y(n)\sigma^x(n+1)-\sigma^x(n)\sigma^y(n+1)]$	$\frac{1}{2}[a^\dagger(n)a(n+1)+a(n)a^\dagger(n+1)]$
O_S^{6V}	Staggered electrical-field 6-vertex model	$(-1)^n\sigma^z(n)$	$(-1)^n a^\dagger(n)a(n)$
O_T^{8V}	Energy operator 8-vertex model	$\sigma^x(n)\sigma^x(n+1)-\sigma^y(n)\sigma^y(n+1)$	$\frac{1}{2}[a^\dagger(n)a^\dagger(n+1)-a(n)a(n+1)]$
O_T^{AT}	Energy operator Ashkin-Teller model	$(-1)^n[\sigma^x(n)\sigma^x(n+1)+\sigma^y(n)\sigma^y(n+1)$ $+\lambda\sigma^z(n)\sigma^z(n+1)]$	$(-1)^n\{\frac{1}{2}[a^\dagger(n)a(n+1)-a(n)a^\dagger(n+1)]$ $+\lambda a^\dagger(n)a(n)a^\dagger(n+1)a(n+1)\}$
O_E^{8V}	Direct electrical-field 8-vertex model	$\sigma^y(n)$	$\frac{1}{2i}a^\dagger(n)\exp\left[\sum_{m=1}^{n-1}\frac{1}{2}i\pi[2\rho(m)+1]\right]+\text{c.c.}$
O_S^{8V}	Staggered electrical-field 8-vertex model	$(-1)^n\sigma^y(n)$	$\frac{(-1)^n}{2i}a^\dagger(n)\exp\left[\sum_{m=1}^{n-1}\frac{1}{2}i\pi[2\rho(m)+1]\right]+\text{c.c.}$
O_E^{AT}	Electrical-field Ashkin-Teller model	$\prod_{m=1}^{n-1} i2\sigma^z(m)=\exp\left[\sum_{m=1}^{n-1} i\pi\sigma^z(m)\right]$	$\exp\left[\sum_{m=1}^{n-1} i\pi\rho(m)\right]$

TABLE II. Identification of 8-vertex operators in Luttinger language.

8-vertex language (compare Table I) Name	Identification	Luttinger language Name	Fermion representation	Extended scaling relation
O_E^{6V}	Direct electrical-field 6-vertex model	$\frac{is}{2\pi}F_{1,0}$ $\frac{s}{2\pi}F_{0,1}$	$s[\rho_2(r)+\rho_1(r)]$ $\frac{s}{i}[\rho_2(r)-\rho_1(r)]$	$x_{1,0}^{(F)}=x_{0,1}^{(F)}=1$ (Redundant)
O_S^{6V}	Staggered electrical-field 6-vertex model	$O_{2,0}+O_{-2,0}$	$s[\psi_1^\dagger(r)\psi_2(r)+\psi_2^\dagger(r)\psi_1(r)]$	$x_{2,0}\times x_{0,1}=1$
O_T^{8V}	Energy operator 8-vertex model	$\frac{1}{i}[O_{0,1}-O_{0,-1}]$	$\frac{s}{i}[\psi_1(r)\psi_2(r)-\psi_2^\dagger(r)\psi_1^\dagger(r)]$	
O_T^{AT}	Energy operator Ashkin-Teller model	$\frac{1}{i}[O_{2,0}-O_{-2,0}]$	$\frac{s}{i}[\psi_1^\dagger(r)\psi_2(r)-\psi_2^\dagger(r)\psi_1(r)]$	$x_{2,0}\times x_{0,1}=1$
O_E^{8V}	Direct electrical-field 8-vertex model	$O_{0,-1/2}+O_{0,1/2}$	$\sqrt{s}\,\psi_1^\dagger(r)\exp[\frac{1}{2}\Theta_+(r)]+\text{c.c.}$	$x_{0,1/2}=\frac{1}{4}x_{0,1}$
O_S^{8V}	Staggered electrical-field 8-vertex model	$O_{-2,-1/2}+O_{2,1/2}$	$\sqrt{s}\,\psi_2^\dagger(r)\exp[\frac{1}{2}\Theta_+(r)]+\text{c.c.}$	$x_{2,1/2}=\frac{1}{4}x_{0,1}+\frac{1}{x_{0,1}}$
O_E^{AT}	Electrical-field Ashkin-Teller model	$O_{-1,0}$	$\exp[\frac{1}{2}\Theta_+(r)]$	$x_{1,0}=\frac{1}{4}x_{2,0}$

Pauli-spin operators must be transformed into fermion operators via a Jordan-Wigner transformation

$$\sigma^\dagger(i)=\sigma^x(i)+i\sigma^y(i)=a^\dagger(i)\exp\left[i\pi\sum_{j=1}^{i-1}a^\dagger(j)a(j)\right], \tag{2.5}$$

$$\sigma^z(i)=a^\dagger(i)a(i)-\tfrac{1}{2}\ . \tag{2.6}$$

The XXZ Hamiltonian then reads

$$H_{XXZ}=-\sum_{i=1}^{N}\tfrac{1}{2}[a^\dagger(i)a(i+1)-a(i)a^\dagger(i+1)] + \lambda[a^\dagger(i)a(i)-\tfrac{1}{2}][a^\dagger(i+1)a(i+1)-\tfrac{1}{2}]\ . \tag{2.7}$$

The results for the other operators is given in Table I. After a Fourier transformation

$$a(k)=N^{-1/2}\sum_{j=1}^{N}e^{-ikr_j}a(j) \tag{2.8}$$

one obtains

$$H_{XXZ}=-\sum_k\cos(ks)\left[a^\dagger(k)a(k)+\frac{\lambda}{N}\rho(k)\rho(-k)\right]. \tag{2.9}$$

Here $\rho(k)$ is the Fourier transform of the density operator $\rho(i)=a^\dagger(i)a(i)-\frac{1}{2}$. Just as in Sec. I, ρ_i is defined such that its expectation value is zero: $\langle\sigma^z(i)\rangle=\langle\rho(i)\rangle=0$.

For $\lambda=0$ the XYZ model reduces to the XY model. In the 8-vertex language this is the free-fermion model (two decoupled Ising models). As $\gamma=0$ its Hamiltonian is already diagonal

$$H_{XX}=-\sum_k\cos(ks)a^\dagger(k)a(k)\ . \tag{2.10}$$

Since $\langle\sigma^z(i)\rangle=0$ we have a half-filled band.

This is the situation of Fig. 1(a). We will replace the dispersion curve by that of the Luttinger model, Fig. 1(b).

First we must convince ourselves that the diagonal Luttinger Hamiltonian H_0 [Eq. (1.2) with $\nu=1$] still describes the XY model correctly. The replacement of H_{XX} by H_0 involves two steps. First H_{XX} is rewritten in terms of the type-"1" and type-"2" particles. This replaces the lattice by a cutoff procedure in k space. Second the continuum limit $s\downarrow 0$ is taken. This neglects the short-range (lattice) effects, and is equivalent to going to the scaling limit. The effect is comparable to that of a few initial renormalization transformations.

Define the operators a_1 and a_2 via a translation in

k space

$$a(k)=a_1(k-k_F)=a_2(k+k_F) \ , \qquad (2.11)$$

i.e., the real space

$$a(i)=e^{ik_Fr_i}a_1(i)=e^{-ik_Fr_i}a_2(i) \ . \qquad (2.12)$$

The summation in H_{XX} can then be restricted to $|k|<k_F$.

$$H_{XX}=\sum_{k<k_F}\sin(ks)[a_1^\dagger(k)a_1(k)-a_2^\dagger(k)a_2(k)] \ . \qquad (2.13)$$

Let $\psi_1(x)$ and $\psi_2(x)$ be the field operators of Sec. I. They are defined on the continuum $0\leq x<sN$ (with periodic boundary conditions). The a_i can be replaced by $a_i(j)\to\sqrt{s}\,\psi_i(x_j)$. This gives

$$H_{XX}=\sum_k\sin(ks)[\psi_1^\dagger(k)\psi_1(k)-\psi_2^\dagger(k)\psi_2(k)] \ . \qquad (2.14)$$

The lattice is now represented by a cutoff in k space. Equation (2.14) suggests a sharp cutoff. When however the lattice constant is small enough, also the softer cutoff of Eq. (1.52) (with $\frac{1}{2}\pi\alpha=s$) can be employed.

Expansion of $\sin ks$ gives

$$s^{-1}H_{XX}=\sum_k\left[k-\frac{1}{3!}k^3s^2+\cdots\right] \times[\psi_1^\dagger(k)\psi_1(k)-\psi_2^\dagger(k)\psi_2(k)] \ . \qquad (2.15)$$

In the limit $s\downarrow 0$ only the term proportional to k remains. This is the diagonal part H_0 of the Luttinger Hamiltonian. The higher-order terms correspond (in real space) to higher-order gradients of the $\psi_i^\dagger\psi_i$ operator. They are all irrelevant under a renormalization transformation.

The effect of the procedure on the correlation functions can be illustrated by the density operator $\rho(i)=a^\dagger(i)a(i)-\frac{1}{2}$. Its correlation function in the ground state of H_{XX} is easily calculated, via the Fourier transform of $a(i)$ and the property $a^\dagger(k)|0\rangle=0$ for $|k|<k_F$. One finds

$$\mathcal{G}=\langle 0|\rho(n+i)\rho(i)|0\rangle =\frac{1}{4\pi^2}\left[\frac{s}{r}\right]^2[2\cos(2k_Fr)-2] \qquad (2.16)$$

with $r=ns, \quad k_F=\pi/2s$.

The alternating piece is due to the staggered electrical-field operator O_S and the rest to the direct electrical-field operator O_E (both in the 6-vertex language identification, see Table I). As we will see $\rho(i)$ is replaced by $O_E+\cos(2k_Fr)O_S$ with

$$O_E(r)=s[\rho_2(r)+\rho_1(r)]=-\frac{is}{2\pi}F_{1,0}(r) \ , \qquad (2.17)$$

$$O_S(r)=s[\psi_1^\dagger(r)\psi_2(r)+\psi_2^\dagger(r)\psi_1(r)] =\frac{s}{2\pi\alpha}[O_{2,0}(r)+O_{-2,0}(r)] \ . \qquad (2.18)$$

The interactions

$$\sum_i\sigma^z(i)=\sum_k a^\dagger(k)a(k)-\tfrac{1}{2} \ ; \qquad \sum_i(-1)^i\sigma^z(i)=\sum_k a^\dagger(k)a(k-2k_F) \qquad (2.19)$$

will only influence the states near the Fermi surface. But we must be careful to take into account all processes that take place there. The operators a_1 and a_2 allow many different ways of representing the same operator

$$a^\dagger(i)a(i)=a_1^\dagger(i)a_1(i)=a_2^\dagger(i)a_2(i) =(-1)^ia_1^\dagger(i)a_2(i)=(-1)^ia_2^\dagger(i)a_1(i) \ . \qquad (2.20)$$

Choose the representation that brings all processes in the interval $|k|<k_F$.

$$\sum_i\sigma^z(i)=\sum_{k<k_F}a_1^\dagger(k)a_1(k)+a_2^\dagger(k)a_2(k)-1 \ , \qquad (2.21)$$

$$\sum_i(-1)^i\sigma^z(i)=\sum_{k<k_F}a_2^\dagger(k)a_1(k)+a_1^\dagger(k)a_2(k) \ . \qquad (2.22)$$

Each process at the Fermi surface now takes place at small k. So we can safely change the dispersion rela-

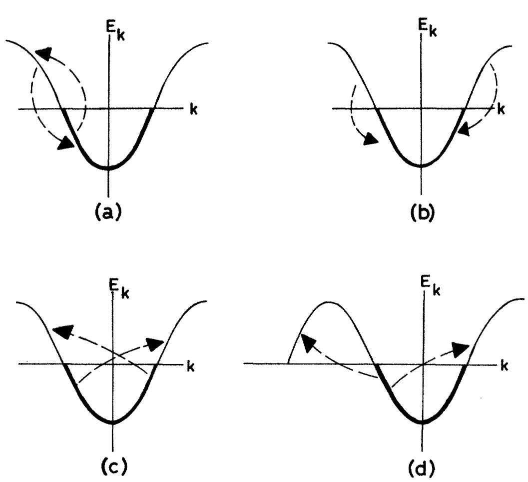

FIG. 4. Four different types of processes at the Fermi surface, generated by the $\sigma^z(n)\sigma^z(n+1)$ interaction in the XXZ model. (c) is called backwards scattering. (d) is known as umklapp.

tion at large k. When we again replace the $a_j(i)$ by $\sqrt{s}\,\psi_j(r_i)$ we obtain Eqs. (2.17) and (2.18).

In the new language the correlation function of Eq. (2.16) reads

$$\mathcal{G} = -\frac{s^2}{4\pi^2}\langle 0|F_{1,0}(r+r_i)F_{1,0}(r_i)|0\rangle + \frac{s^2}{4\pi^2\alpha^2}\cos(2k_F r)\langle 0|O_{2,0}(r+r_i)O_{-2,0}(r_i)|0\rangle \ . \tag{2.23}$$

G is known from the results of Sec. I. In the cutoff-independent domain $r > r_0$ the result is the same as before.

Consider the $\rho(i)\rho(i+1)$ interaction term in H_{XXZ}. It gives rise to four different processes at the Fermi surface, and therefore to four different contributions. Remember that $\rho(k)$ has the effect of generating an excitation with energy k [see Eq. (1.6)]. In Fig. 4 every arrow represents the action of $\rho(k)$. The small-k processes (a) and (b) lead to diagonal and nondiagonal Luttinger interactions

$$-\frac{\lambda}{L}\sum_k \cos(ks)[\rho_1(k)\rho_1(-k)+\rho_2(k)\rho_2(-k) + 2\rho_1(k)\rho_2(-k)] \ . \tag{2.24}$$

The $k \simeq \pm 2k_F$ process (c) is known as backward scattering. It gives the contribution

$$\lambda\int dr\,[\psi_2^\dagger(r)\psi_1(r)\psi_1^\dagger(r+s)\psi_2(r+s)+\text{c.c.}] \ . \tag{2.25}$$

This operator is a contraction of the two spin-wave operators $O_{2,0}(r)O_{-2,0}(r+s)$ [compare Eq. (1.57)]. The composite operator preserves the number of type-"1" and type-"2" particles, and can be expanded in the density operators as

$$-\lambda\int dr\left[2\rho_2(r)\rho_1(r) + s\left(\rho_2(r)\frac{\partial\rho_1}{\partial r}+\frac{\partial\rho_2}{\partial r}\rho_1(r)\right)+\cdots\right] \ . \tag{2.26}$$

Figure 4(d) represents the umklapp process. Its contribution is a contraction of two alike $O_{\pm 2,0}$ spin-wave operators

$$\lambda\int dr\,[\psi_2^\dagger(r)\psi_1(r)\psi_2^\dagger(r+s)\psi_1(r+s)+\text{c.c.}] \ . \tag{2.27}$$

In Sec. I [Eq. (1.37)] we have identified this as the spin-wave operator $O_{4,0}+O_{-4,0}$. This contribution is neglected by Luther and Peschel.

The final result for H_{XXZ} is

$$s^{-1}H_{XXZ} = \sum_k\left\{k[\psi_1^\dagger(k)\psi_1(k)-\psi_2^\dagger(k)\psi_2(k)] - \frac{\tilde\lambda_1}{L}[\rho_1(k)\rho_1(-k)+\rho_2(k)\rho_2(-k)] - \frac{4\tilde\lambda_2}{L}\rho_1(k)\rho_2(-k)\right\} + \frac{\tilde\lambda_3}{s^2}\int dr\,[O_{4,0}(r)+O_{-4,0}(r)] \ . \tag{2.28}$$

Only the term generated by the umklapp process can lead to a gap. The other operators are always massless.

In the 2D Gaussian language Eq. (2.28) becomes a sine-Gordon model (Sec. I). It is the same model that Knops found after a few initial renormalization transformations.

Knops was able to rewrite the 6-vertex model directly into the language of the generalized Villain model. He found a Gaussian next-nearest-neighbor interaction, and a modified form for the nearest-neighbor coupling. His statement that this modification is irrelevant, i.e., that it renormalizes to the conventional Gaussian model, is equivalent to the statement that we are allowed to expand the $\sin(ks)$ in Eq. (2.14), the $\cos(ks)$ in Eq. (2.24), the backward scattering term in Eq. (2.25), and only take into account their leading contributions. This is motivated by the knowledge that at the fixed line (the Luttinger model) these neglected operators have an irrelevant scaling index.

Next to these massless interactions Knops found that the 6-vertex Hamiltonian contains the staggered energy operator of the Ashkin-Teller model. Via the renormalization transformation it generates the $\cos 4\phi$ operator. The staggered O_T^{AT} itself is redundant in the Gaussian limit.

From Table I we see that in the Pauli-spin language H_{XXZ} indeed can be presented as the staggered O_T^{AT}. Also in our treatment it only becomes apparent later, during the replacement of the lattice by the cutoff in k space, that $O_{4,0}$ is still hidden in this operator (via the umklapp process).

The renormalization equations are known only around the Luttinger model for small u_i.[5,9,22,23] They give rise to the phase diagram of Fig. 2. It is likely that its topology is correct also for larger values of u_i (no new fixed points). Within this assumption, the mapping of the XXZ model onto Eq. (2.28), and the equivalent procedure by Knops in 2D is exact.

Since we do not know the explicit renormalization equations that map H_{XXZ} onto Eq. (2.28), we do not know the details of the analytic relations between the renormalized coupling constants $\tilde\lambda_i$ and the parameter λ in H_{XXZ}. Without the Baxter solution, which gives $x_T^{8\text{V}}(\lambda)$, we would have lost control over the precise dependence of the critical exponents on λ.

From Eq. (2.28) we can now understand the critical behavior of the XXZ model. We are dealing with path 2 in Fig. 2. The massless domain of H_{XXZ}, $-1 < \lambda < 1$ corresponds to $K < K_A$. These critical points flow towards the Gaussian model. So we know their critical exponents exactly. Each 8-vertex operator in Table I can be identified with a vortex, spin-wave, or density operator. O_E^{6V} and O_S^{6V} are already discussed above. The procedure for the other operators is straightforward. The results are given in Table II. Most operators contain products of Pauli spins at different sites. This implies that next to the dominant operator given in Table II they also contain less relevant operators such as gradients.

The extended scaling relations for the 8-vertex model follow directly from Eq. (1.55). Also, we know a large set of correlation functions for the 8-vertex model at T_c in the scaling limit [Eq. (1.51)].[8,18]

The infinite-order transition in the XXZ model, from the massless into the antiferroelectrical ordered domain, takes place at $\lambda = -1$. It occurs through a Kosterlitz-Thouless mechanism, caused by the excitations of the umklapp operator. We know that at $\lambda = -1$ the critical exponent x_T^{8V} is marginal. O_T^{8V} is identified as the vortex operator $O_{0,1}$ (see Table II). Indeed the extended scaling relation $x_{0,1}x_{4,0} = 4$ implies that also $x_{4,0}$ is marginal at $\lambda = -1$, i.e., that the excitations of $O_{4,0}$ become relevant.

In this paper we have discussed models in 2D statistical mechanics and also in 1D quantum-field theory that can be mapped exactly onto each other or that via a renormalization transformation are equivalent. I would like to finish with another example.

Recently the sine-Gordon model

$$H = -\sum_{\langle \vec{r}, \vec{r}' \rangle} \frac{K}{2}[\phi(\vec{r}) - \phi(\vec{r}') - E]^2 + u_4 \sum_{\vec{r}} \cos[4\phi(\vec{r})] \tag{2.29}$$

has been studied in the context of the commensurate-incommensurate transition.[24–26] Consider particles that are located at a surface, and that are coupled to each other via springs. The $\cos 4\phi$ term represents the substrate potential. The displacements $\phi(x,y)$ are measured with respect to the substrate. E is the misfit parameter. The model is simplified, since instead of 2 there is only 1 displacement component.

E couples to the operators $F_{1,0}$ and $F_{0,1}$ (Sec. I). From Tables I and II it follows that this corresponds to a direct electrical field in the 6-vertex model. This model has been solved exactly and is discussed by Lieb and Wu[27] (see Fig. 32). The antiferroelectrical domain corresponds to the commensurate phase and the massless domain to the incommensurate phase. At the commensurate-incommensurate transition the specific heat only diverges at the incommensurate site. Its exponent is $\alpha = \frac{1}{2}$. The netto polarization of a row of arrows corresponds to the incommensurate order parameter, i.e., the number of domain walls. It vanishes with the exponent $\beta = \frac{1}{2}$. The 6-vertex model has also been used before as a crystal-growth model, showing a roughening transition. It is the body-centered solid-on-solid (BC-SOS) model.[28] In this language the misfit regions (the domain walls) correspond to steps on the crystal surface. The misfit parameter is a field acting on the boundaries of the lattice, favoring a netto height difference. It acts as the fugacity for a step on the surface.

In order to obtain the critical properties of Eq. (2.29) it is (as one would expect) not necessary to go to the more complex level of the spin models. The same results were already obtained from its Luttinger representation.[25,26]

It is however instructive to find another example showing the central role that the 6-vertex model plays in the description of two-dimensional critical phenomena.

Note added in proof. After this work was completed I received a report of work prior to publication by Black and Emery, independently pointing out the importance of the umklapp process in the 6-vertex model.

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APPENDIX

In this appendix the identification of the 8-vertex operators with those of the XYZ model (Table I) is discussed.

The transfer matrix of the 8-vertex model is equal to the trace of a product of R_0 matrices.[1]

$$T(\{\alpha\}, \{\alpha'\}) = \mathrm{Tr}_\lambda \prod_{i=1}^{N} R_0(\alpha_i, \lambda_{i+1} | \lambda_i, \alpha_i') \ . \tag{A1}$$

Each R_0 represents the contribution of one vertex in the row.

$$R_0(\alpha, \lambda' | \lambda, \alpha') = \sum_{j=1}^{4} P_j \sigma^j_{\alpha\lambda'} \sigma^j_{\lambda\alpha'} \ . \tag{A2}$$

$\alpha_i = \pm 1$ ($\lambda_i = \pm 1$) denotes the direction of the arrow at a vertical (horizontal) bond. The σ^j are the Pauli matrices ($\sigma^x, \sigma^y, \sigma^z, \frac{1}{2}1$). The Boltzmann weights P_j are related to the more usual weights a, b, c, and d as: $P_1 = 2(b+d)$, $P_2 = 2(b-d)$, $P_3 = 2(a-c)$, and

$P_4=2(a+c)$. The XYZ Hamiltonian

$$H_{XYZ}=-\sum_{i=1}^{N} p_1\sigma_i^x\sigma_{i+1}^x+p_2\sigma_i^y\sigma_{i+1}^y+p_3\sigma_i^z\sigma_{i+1}^z+\tfrac{1}{4}p_4 \tag{A3}$$

is obtained by taking the logarithmic derivative of T with respect to V in the extreme anisotropy limit.[10] V is one of the three parameters in Baxter's elliptic-function representation of the Boltzmann weights. V can be interpreted as the lattice anisotropy; the two other parameters as the temperature t and the universality-class parameter λ. The extreme anisotropy limit is in the Boltzmann weight language obtained at the point $a=c=1$, $b=d=0$. At this point all equi-(λ,t) curves come together. The p_j in H_{XYZ} are the tangents of these curves at this point,

$$p_j=\left(\frac{\partial P_j}{\partial V}\right)_{\substack{a=c=1\\ b=d=0}} . \tag{A4}$$

This provides a graphical method for constructing the phase diagram of the XYZ model. The five different critical lines in Fig. 4 are due to the five different critical planes in the 8-vertex model that meet at $a=c=1$, $b=d=0$, i.e., the planes $d=0$, $a\pm b\pm d=c$. Notice that in the 6-vertex model $(d=0)$, P_1 is equal to P_2. So γ in Eq. (2.2) is zero.

For most operators in Table I it is sufficient to realize that σ_i^z represents α_i. In the Ising representation $(-1)^i\sigma_i^z$ represents the product of two nearest-neighbor spins S_iT_i in the same row. We can also obtain the identification of the operators by a straightforward generalization of Baxter's method for deriving H_{XYZ}. When all eight possible vertex states are allowed to have different Boltzmann weights $(\omega_i, i=1-8)$ the transfer matrix includes cross terms:

$$R=R_0+i(Q_1\sigma^y_{\alpha\lambda'}\sigma^x_{\lambda\alpha'}-Q_2\sigma^x_{\alpha\lambda'}\sigma^y_{\lambda\alpha'})+(Q_3\sigma^4_{\alpha\lambda'}\sigma^z_{\lambda\alpha'}+Q_4\sigma^z_{\alpha\lambda'}\sigma^4_{\lambda\alpha'}) , \tag{A5}$$

with

$$\begin{aligned} Q_1&=\omega_3-\omega_4-\omega_7+\omega_8 ,\\ Q_2&=\omega_3-\omega_4+\omega_7-\omega_8 ,\\ Q_3&=\omega_1-\omega_2+\omega_5-\omega_6 ,\\ Q_4&=\omega_1-\omega_2-\omega_5+\omega_6 . \end{aligned} \tag{A6}$$

Taking again the logarithmic derivative leads to the following extra terms in the quantum-field Hamiltonian:

$$H=H_{XYZ}-\sum_{i=1}^{N}\tfrac{1}{2}(q_4\sigma_i^z+q_3\sigma_{i+1}^z)+i(q_1\sigma_i^y\sigma_{i+1}^x-q_2\sigma_i^x\sigma_{i+1}^y) , \tag{A7}$$

with

$$q_j=\left(\frac{\partial Q_j}{\partial V}\right)_{\substack{a=c=1\\ b=d=0}} . \tag{A8}$$

This result enables us to identify the operators of Table I. Apply an electrical field to the 6-vertex model that makes $\omega_1\neq\omega_2$. This gives a transverse field interaction $\sum_i\sigma_i^z$ in the XYZ model. A staggered electrical field in the 6-vertex model acts on the weights ω_5 and ω_6. It leads to a staggered transfer field $\sum_i(-1)^i\sigma_i^z$, since $Q_3(i)=-Q_4(i)=-Q_3(i+1)=Q_4(i+1)$.

A (staggered) electrical field in the 8-vertex model gives the same result. However, one then discusses the critical behavior along critical line (4) in Fig. 4. The transformation $\sigma^y\to-\sigma^z$, $\sigma^z\to\sigma^y$ for even values of i and $\sigma^y\to-\sigma^z$, $\sigma^z\to\sigma^y$ at the uneven ones that maps line (4) onto (1) gives

$$O_S^{8V}=(-1)^i\sigma_i^y,\quad O_E^{8V}=\sigma_i^y . \tag{A9}$$

In a previous paper I discussed the relationship between the 8-vertex, 6-vertex, and Ashkin-Teller model.[14] The energy operator of the Ashkin-Teller model is shown there to correspond to the field that makes the weight a and b staggered. This leads to the identification

$$O_T^{AT}=(-1)^i(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\lambda\sigma_i^z\sigma_{i+1}^z) . \tag{A10}$$

It was also shown there that in the 6-vertex model the energy operator of the Ashkin-Teller model coincides with the magnetic field operator S_i of the 8-vertex model; i.e., use the Ising-spin language of the 8-vertex model and discuss the behavior of the operator S_i along line (1), which in that language is located in a zero-temperature direction. A spin-spin correlation function in the 8-vertex model can be represented by a string of electrical field operators:

$$\langle S_{i+n}S_i\rangle=\langle(S_{i+n}T_{i+n-1})(T_{i+n-1}S_{i+n-2})\cdots(T_{i+1}S_i)\rangle=\langle 0|\prod_{j=i}^{i+n-1}2i\sigma_j^z|0\rangle . \tag{A11}$$

This gives the electrical-field operator of the Ashkin-Teller model

$$O_E^{AT}=\prod_{j=1}^{i-1}(2i\sigma_j^z)=\exp\left[i\pi\sum_{j=1}^{i-1}\sigma_j^z\right] . \tag{A12}$$

The magnetic field operator of the 8-vertex model can be identified as

$$O_H=\prod_{j=1}^{i-1}(2\sigma_j^y)=\exp\left[\frac{i\pi}{2}\sum_{j=1}^{i-1}(2\sigma_j^y-1)\right] . \tag{A13}$$

Unfortunately the Jordan-Wigner transformation does not leave us with a simple form for this operator.

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