Scaling theory of Anderson localization: A renormalization-group approach*

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A position-space renormalization-group method, suitable for studying the localization properties of electrons in a disordered system, was developed. Two different approximations to a well-defined exact procedure were used. The first method is a perturbative treatment to lowest order in the intercell couplings. This yields a localization edge in three dimensions, with a fixed point at the band center ($E=0$) at a critical disorder $\sigma_c \simeq 7.0$. In the neighborhood of the fixed point the localization length L is predicted to diverge as $L \sim (\sigma - \sigma_c + \beta E^2)^{-\nu}$. In two dimensions no fixed point is found, indicating localization even for small randomness, in agreement with Abrahams, Anderson, Licciardello, and Ramakrishnan. The second method is an application of the finite-lattice approximation, in which the intercell hopping between two (or more) cells is treated to infinite order in perturbation theory. To our knowledge, this method has not been previously used for quantum systems. Calculations based on this approximation were carried out in two dimensions only, yielding results that are in agreement with those of the lowest-order approximation.

I. INTRODUCTION

The nature of electronic eigenstates in disordered systems has been the subject of intense study ever since the problem was first proposed by Anderson.[1] The problem arose in connection with the diffusion of electrons when disorder was introduced in an otherwise perfectly crystalline solid. Such disorder can be caused, for example, by the presence of impurities in a metal or a semiconductor. A pertinent question, that has a direct bearing on physical observables, such as the conductivity, is whether or not the eigenstates are localized. One example of localized states is the impurity states in a semiconductor; an impurity produces bound states which fall off exponentially away from the impurity. These levels, which may either accept from or provide an electron to the system, differ in energy from impurity to impurity (even when they are of the same type) because of their random environment.

However, when there is a dense concentration of impurities, the overlap between neighboring states can be large. In this case, if the energies do not vary too wildly, quantum-mechanical tunneling can cause the states to spread over the whole system so that the theory of metallic conduction becomes applicable.[2,3]

Similar behavior also occurs in other disordered systems such as random alloys,[4] semiconducting glasses,[2] metal-oxide-semiconductor field-effect transistor (MOSFET),[5] dirty metals,[6] etc. All these systems have one feature in common, namely, that the disorder can localize some or all of the quantum-mechanical states, and important physical quantities depend mainly on the localization properties. With this in mind, we address the question of localization in a specific context, namely, the Anderson model,[1] which can be considered a model for amorphous semiconductors and has been most extensively studied.

The Anderson model describes noninteracting electrons on a lattice in terms of the Hamiltonian

$$H = \sum_{\vec{r}} \epsilon_{\vec{r}} |\vec{r}\rangle\langle\vec{r}| + \frac{1}{2} \sum_{\vec{r}\vec{r}'}{}' V_{\vec{r}\vec{r}'} (|\vec{r}\rangle\langle\vec{r}'| + |\vec{r}'\rangle\langle\vec{r}|) \ , \tag{1.1}$$

where the state $|\vec{r}\rangle$ corresponds to a single atomic orbital of energy $\epsilon_{\vec{r}}$ localized at the site $\vec{r}$. $V_{\vec{r}\vec{r}'}$ is the overlap between different orbitals, enabling electrons to hop from site to site, and is taken to be nonzero only when $\vec{r}$, $\vec{r}'$ are nearest neighbors. Disorder can be introduced by taking either or both $\epsilon_{\vec{r}}$ and $V_{\vec{r}\vec{r}'}$ to be random variables. The case that received the most attention[1-4,7] is the problem of diagonal disorder in which the hopping elements $V_{\vec{r}\vec{r}'}$ are assumed to be constant and the site energies $\epsilon_{\vec{r}}$ are chosen independently from a rectangular distribution of width W.

The dimensionless quantity $\sigma = W/V$ is a measure of the degree of randomness in the system. The lim-

iting cases $\sigma=0$ and ∞ can be trivially solved. For $\sigma=0$ the states are infinitely extended plane waves, whereas for $\sigma=\infty$ the eigenstates are given by the orbitals $|\vec{r}\rangle$ and therefore are completely localized. Between these two limits there must be a transition from extended to localized states. This transition has been named the Anderson transition and is reminiscent of phase transitions in magnetic systems. The manner in which such phase changes take place usually depends on the spatial dimensionality of the system.

The main theoretical questions one may ask about the Anderson model concern (i) the density of states, (ii) the existence of localized and extended states, and (iii) their effect on physical observables. Except in one dimension[11] and the unphysical case of a Cayley tree[8] no exact analysis of the model exists. Numerous theoretical calculations[3] based on approximate methods yielded conflicting results and generated considerable controversy about almost all aspects of the model. The qualitative picture that emerges from earlier studies can be summarized as follows: (i) For a rectangular distribution of site energies with mean zero and width W, the spectrum of the Hamiltonian (1.1) forms a continuum within a band extending from $-ZV-\frac{1}{2}W$ to $ZV+\frac{1}{2}W$ when Z is the coordination number (number of nearest neighbors) of the lattice. Lifshitz[9] has argued that near the band edges the density of states $n(E)$ vanishes exponentially as $\exp(-\text{const}/|E-E_0|)$ where $E_0=ZV+\frac{1}{2}W$. (ii) In one dimension all states are exponentially localized.[10,11] An eigenstate of the Hamiltonian (1.1) corresponding to eigenvalue E can be written as

$$|\psi\rangle_E=\sum_{\vec{r}} a_{\vec{r}E}|\vec{r}\rangle \ . \tag{1.2}$$

By exponential localization we mean that the magnitude $|a_{\vec{r}E}|$ falls off exponentially away from a "center of localization" $\vec{r}_0$ as $|a_{rE}| \sim \exp(-|\vec{r}-\vec{r}_0|/L)$. The localization length L, averaged over the ensemble, is a function of energy and the degree of disorder, σ. (iii) In higher dimensions ($d>1$) in addition to localized states there may exist a region of extended states, i.e., states which have nonzero amplitudes over the whole lattice. On physical grounds we expect that localized and extended states will not coexist in energy, since the slightest perturbation will cause the former to delocalize by mixing with the corresponding extended state of the same energy. Moreover, as we shall see in Sec. III, the extended states should occur near the band center ($E=0$). (iv) Suppose σ_0 is the randomness at which the band center becomes extended. As σ is reduced more and more states become extended. We can then talk about regions of localized and extended states. Mott[12] argued that there exists a sharp boundary $E_c(\sigma)$ that separates these two regions. This situation is shown in Fig. 1.

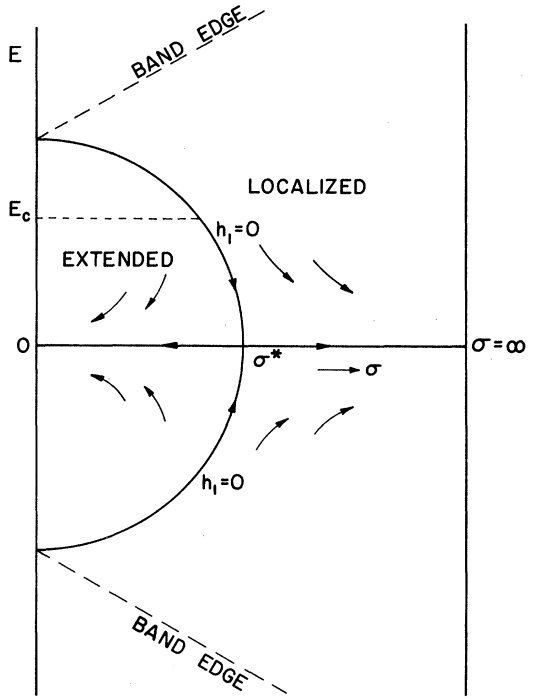

FIG. 1. Schematic phase diagram in the σE plane. The dotted line shows the band edge and E_c is the mobility edge. The arrows show the flows generated by the RG transformation. The line $h_1=0$ is the "critical surface" separating localized from extended states.

The localization length L increases as the "mobility edge" E_c is approached from the localized regime and is expected to diverge at E_c as

$$L\sim|E-E_c|^{-\nu'} \ , \tag{1.3}$$

where the "critical exponent" ν' depends on the dimensionality d. Alternatively, if $\sigma_c(E)$ is the randomness at which a given state of energy E becomes extended, then as σ is approached from above, the localization length is expected to diverge as

$$L\sim(\sigma-\sigma_c)^{-\nu} \tag{1.4}$$

where the exponent ν is of the same order as ν', but the exact values may not be equal.

Physical observables which depend only on the density of states (e.g., specific heat) are not sensitive to localization. On the other hand, properties such as the conductivity, which depends on the mobility of electrons, are expected to show very different behavior depending on whether the Fermi energy E_F is in the extended or in the localized region. For example, the dc conductivity at zero temperature is essentially determined by the states at the Fermi energy E_F. As long as E_F is in the extended region, the conductivity is metallic. When E_F crosses E_C into the localized re-

gime, the electrons are trapped, and the dc conductivity should be zero. According to Mott[2,13] as long as E_F is in the localized regime the conductivity C remains zero; at $E_F = E_C$, C jumps discontinuously to a finite value C_0. C_0 is therefore called the "minimum metallic conductivity."

Due to the lack of any exact information, numerous approximate methods have been devised to analyze the Anderson model. There are some excellent reviews[2–4,13,14] on the subject.

The bulk of the theoretical effort has been directed toward (i) establishing the existence of a mobility edge; (ii) understanding the nature of the transition and calculating the values of the critical parameters $(\nu, \nu', E_C, \sigma_C)$; and (iii) testing the validity of the concept of minimum metallic conductivity.[2,13] The results to date have been conflicting and controversial. It is generally believed that all states are localized in one dimension[10,11] and for $d > 2$ there is a mobility edge. In two dimensions, although earlier analytical and numerical work predicted a transition, recent numerical work by Licciardello and Thouless,[15] and renormalization-group (RG) studies[16–18] indicate that, as in one dimesnion, all states may be localized.[19] However, others[20,21] have predicted existence of a localization edge in two dimensions. There are similar disagreements regarding the existence of a minimum metallic conductivity and the values of the critical parameters. Table I gives a summary of the available information in two and three dimensions.

We have developed a scaling theory of localization in the spirit of the position-space RG ideas of Niemeijer and van Leeuwen.[22] Our method is similar to the one proposed by Wegner[23] where he considers a RG transformation of the matrix elements of the Hamiltonian (1.1). The basic idea is to study the localization length L characteristic of a state vector of energy E, in two and three dimensions. To this end we have performed a sequence of calculations, which can be viewed as a set of systematically improving approximations to an exact treatment contained in the general formalism. In addition to diagonal randomness, we have also considered off-diagonal randomness of the type in which the signs of the hopping elements $V_{\vec{r}\vec{r}'}$ are random. Our procedure is based on a two-parameter RG, which are σ and E.

The results can be summarized as follows: In three dimensions, (i) we find a localization edge. The "critical behavior," characterized by $1.25 < \nu < 1.75$, is governed by a fixed point of our RG transformation located at $E = 0$, $\sigma_c \simeq 7.0$. (ii) This fixed point is stable, implying that the localization length L diverges with the same exponents along the entire localization edge $E_c(\sigma)$. (iii) On the basis

TABLE I. Summary of the available results for the Anderson model. The abbreviation MMC stands for minimum metallic conductivity, asterisk represents results obtained for the diamond lattice.

Authors	Ref.	2D σ_C	2D ν	2D ν'	3D σ_C	3D ν	3D ν'	MMC
Anderson	1,46	28			62		0.60	
Ziman	30	22			22			
Herbert and Jones	47				$20 \sim 40$			
Licciardello and Economou	36	7.2			14.5 8.2*			
Abram and Edwards, Abram	48,49			0.75			0.60	
Edwards and Thouless	25	$5 \sim 6$			15*			Yes
Weaire and Srivastava	50	6			15 8.0*			
Yoshino and Okazaki	51	6.5	0.80	2.0				
Licciardello and Thouless	15,26	6.1		>1.0				Yes
Prelovsek	52			$0.7 \sim 1.2$				No
Schönhammer and Brenig	37				24			
Stein and Krey	21,29	6.5 ± 0.5	0.80 ± 0.05	1.30 ± 0.10	8.0 ± 0.5*	0.66 ± 0.05	1.25 ± 0.10	Yes
Schuster	53						0.670 ± 0.006	
Aharony and Imry	54						0.59	
Freed	55						$\frac{2}{3}$	
Abrahams *et al.*	16	No mobility edge				$\nu = \nu'$	<1.0	No
Lee	20	~6.0	>1.0					Yes
McMillan	18	No mobility edge					0.807	No
Pollit	44			~1.0				

of a simple scaling argument, we predict a parabolic "phase boundary" $E_c^2(\sigma) \propto (\sigma_c - \sigma)$. (iv) In two dimensions, we have not found a fixed point; which implies that all states are localized, in agreement with Abrahams *et al.*[16]

The organization of this paper is as follows. In Sec. II we discuss the general philosophy of RG approach to localization, some recent scaling theories, and some numerical RG calculations of localization and their relation to our method. The general formalism on which our method is based is presented in Sec. III. Explicit approximate recursion relations, derived using first-order-perturbation theory, are given in Sec. IV; the numerical method and results of this approximation are given in Sec. V. Section VI contains an infinite-order-perturbative formalism and the resulting numerical recursion relations (for two dimensions only). Our findings and the physical conclusions to which they lead are summarized in Sec. VII.

II. SCALING THEORIES OF LOCALIZATION

Renormalization-group techniques were applied, with quite remarkable success, in theoretical studies of critical phenomena and the Kondo problem.[24] The basic ingredients of a systematic RG procedure can be summarized as follows. One starts with a system with a large number (N) of degrees of freedom, characterized by some Hamiltonian H. The property that one wants to calculate is then related to that of a new system, with $N' = Nb^{-d} < N$ degrees of freedom. The new system is characterized by a new Hamiltonian H'; in addition, all lengths L associated with the original problem get rescaled by b, the space rescaling factor, i.e., $L' = L/b$. The mapping $H \rightarrow H'$ constitutes an RG transformation or recursion relation.

While in almost all cases of interest carrying out an exact transformation is practically impossible, it is important to at least define an underlying *exact procedure*. Once this is done, approximate derivations of the transformation can be actually performed. In Sec. III we will present a procedure designed to study the problem of localization for the Anderson model, that incorporates the above-mentioned ingredients of an RG method.

Various other workers have constructed and carried out RG procedures to tackle the localization problem, some of which we proceed to review, in order to point out the differences and similarities with respect to our approach. We start by discussing the numerical finite-lattice calculations of Thouless and co-workers,[15,25–27] which provided the basis for later RG studies. They considered changes in the energy levels of a d-dimensional hypercube of linear size l caused by a change in the boundary conditions. For an exponentially localized state the change in the energy ΔE of a particular level should be exponentially small ($\sim e^{-\alpha l}$) when the boundary conditions are changed from periodic to antiperiodic. On the other hand, for an extended state this change is expected to be much larger, roughly of the order of the spacing between the levels.

Thouless and co-workers therefore studied the variation of the dimensionless quantity

$$g(l) = \overline{\Delta E} \Big/ \left| \frac{dE}{dN} \right| , \tag{2.1}$$

with the size l. Here $\overline{\Delta E}$ is the geometric mean of the energy shift and (dE/dN) is the mean spacing between the levels. By treating the change in the boundary conditions as a perturbation they relate $g(l)$ to the conductance of the system.

Since $dE/dN \sim l^{-d}$, and for localized states $\Delta E \sim e^{-\alpha l}$, $g(l)$ is predicted to decrease exponentially with l. For extended states the system exhibits metallic conductivity, and therefore the conductance should vary as $g(l) \sim l^{d-2}$. (This has been pointed out by Abrahams *et al.*; see below.) The l dependence of $g(l)$ was studied for various energies; preliminary calculations[25,26] with lattices of size $l \lesssim 14$ in $d=2$ and $l \lesssim 7$ in $d=3$ indicated the existence of mobility edges in both two and three dimensions. However, when l was increased to about 30 for two dimensions, $g(l)$ was found to decrease with l even for small $\sigma = W/V$. Licciardello and Thouless[15] therefore tentatively suggested that all states may be localized in two dimensions.

Note that these studies were based on direct calculations with large finite systems. More recently, Abrahams *et al.*[16] presented a scaling theory based on the above studies. They consider the size dependence of the conductance $g(l)$ for hypercubes of size l. For a given randomness the conductance is a function of σ as well as the size of the cube. Suppose a large hypercube of size bl is formed by putting together b^d smaller ones, each of size l. Each of these small cubes can be considered a single "atom" containing many levels ($\sim l^d$). The larger cube then consists of b^d "atoms," and its conductance will therefore be a function of the size b and the interaction parameters between the adjacent atoms as well as the energy levels of the individual atoms. Abrahams *et al.*[16] argue that the conductance of the large hypercube is a function of $g(l)$ only.

From the above assumption it follows that there is a RG equation for $g(bl)$ given by

$$g(bl) = f(g(l), b) \ . \tag{2.2}$$

Notice that Eq. (2.2) implies that the l dependence of $g(bl)$ comes entirely through the l dependence of $g(l)$.

Writing $b = 1 + \epsilon$, where ϵ is infinitesimal, we can obtain a differential recursion relation for g by ex-

panding both sides of Eq. (2.2) about $b=1$. This gives

$$\frac{d\ln g}{d\ln l}=\frac{1}{g}\frac{\partial f}{\partial b}\bigg|_{b=1}\equiv\beta_d(g) \quad . \tag{2.3}$$

Note that g^{-1} is a measure of the "degree of randomness" σ in the system. If $\beta(g^*)=0$, g^* is a fixed point of Eq. (2.3). If this fixed point is unstable, the system scales to large values of g for $g>g^*$ and to small values for $g<g^*$. Therefore, if such is the case, a localization edge is predicted.

For small g, $g\sim e^{-al}$, which yields

$$\lim_{g\to 0}\beta_d(g)=\ln(g/g_d) \quad , \tag{2.4}$$

where g_d is a constant. On the other hand, as $g\to\infty$ (zero randomness), $g\propto l^{d-2}$ yields

$$\lim_{g\to\infty}\beta_d(g)=d-2 \quad . \tag{2.5}$$

To obtain $\beta_d(g)$ for intermediate values Abrahams *et al.* assume that β should not have a built-in singularity since it represents the behavior of finite blocks. Moreover, since more randomness means more localization β should be monotonic in g. This smoothness assumption automatically precludes the possibility of a minimum metallic conductivity. Using perturbation theory they proceed to connect the asymptotics (2.4) and (2.5). Their conclusions can be summarized as follows: (i) For $d=3$ there is a fixed point for finite g; the divergence of the localization length is characterized by the exponent $\nu'<1$. (ii) For $d=2$, β is always negative for finite g and goes smoothly to zero as $g\to\infty$, implying that all states are localized.

The above results follow from some plausibility arguments based on physical grounds. Without any actual calculations the theory could not provide any answers as to the values of critical randomness σ_c and the exponents.

In order to check the qualitative arguments presented above, Lee[20] developed a numerical position-space renormalization-group procedure. He studied the 2D problem, for states at the band center, by two different ways. First, he generates an ensemble of 4×4 random lattices, for which the eigenstates are calculated; for the six states with energies closest to the band center the amplitudes on the cell boundary are stored. Then an ensemble of systems that constitute 2×2 lattices of such cells is generated. An effective Hamiltonian is constructed in each corresponding space of 24 states. Any two sets of neighboring (original cell) states are connected by 36 hopping elements. Lee chooses the geometrical mean of these to represent an effective hopping. The ratio v of the average effective hopping to the width of the eigenvalue distribution is used as a measure of the randomness. As this procedure is repeated for increasing size (generating 2×2 systems of the new "cell states") the scaling of v with l is studied. Lee does find a fixed point, and therefore a mobility edge, in 2D.

His second method uses a similar strategy in order to calculate directly the variable $g(l)$ investigated by Abrahams *et al.* Results based on this procedure indicate that $\beta(g)$ vanishes for some finite g^*, and stays near zero for $g>g^*$. On this basis Lee conjectures that $\beta(g)$ may be nonanalytic, with $\beta<0$, for $g<g^*$ and $\beta=0$ for $g>g^*$, which contradicts the assumption made by Abrahams *et al.*[16]

Finally, we briefly discuss another recent numerical calculation by Stein and Krey.[21] They also consider the diagonalization of the Anderson Hamiltonian for large ($\sim 10^4$ lattice sites) but finite lattices. By using the Lanczos recursion method[28,29] they convert the two- and the three-dimensional problem into an equivalent one-dimensional chain with on-site and nearest-neighbor interactions. The ratio of these two interactions is used as a localization parameter (W/V). The scaling properties of this parameter are then studied by eliminating alternate sites. An important feature of their method is that the energy enters as a second scaling variable. They find localization edges in both two and three dimensions. More importantly, their calculated values of the exponents ν and ν' are found to differ by a factor $\phi\simeq\frac{5}{3}$ $(\nu'=\phi\nu)$. Usually two different critical exponents imply two independent relevant scaling "fields." Stein and Krey therefore suggested that the Anderson model may be "multicritical."

The numerical works of Lee and Stein and Krey have in common an important feature. Both can be viewed as rather sophisticated methods to find eigenvalues and eigenvectors for sizable but finite lattices. If one wishes to study larger lattices, the amount of computer storage space must be increased. Therefore both methods differ from the renormalization-group approach in critical phenomena, where lattices of increasing size are treated exactly on the same footing. Lee's formalism does not include scaling of the energy, and therefore he does not study the question whether the divergence of the localization length is characterized by the same exponent all along the mobility edge.

Both studies keep track, to some extent, of randomness of the off-diagonal hopping elements, induced by the renormalization group.

It should also be noted, that Stein and Krey apply renormalization-group methods on the linear-chain problem only, *not* directly on the d-dimensional lattice. Since one expects that the physical effects of randomness will manifest in the effective 1D chain that is derived, the most important part of the physics is not treated by a renormalization-group procedure, but rather, incorporated in a numerical finite-lattice calculation.

Our method, as discussed in Sec. III, does follow in spirit the renormalization-group ideas that were used

successfully in studies of critical phenomena. Lattices of increasing size *are* treated on the same footing. Also, our formalism can be clearly viewed as an approximation to an exact theory; it contains energy scaling and was applied both in two and three dimensions.

III. GENERAL RENORMALIZATION-GROUP PROCEDURE

In this section we assume that an RG procedure can be constructed to treat the Anderson model. We define the appropriate parameter space in which the RG transformation operates, and discuss various features that can be expected. This is done using only some symmetry properties of the model and the general properties of RG transformations. Later (in Sec. IV) we actually construct a procedure that has all the properties assumed in this section, and that contains the ingredients essential to qualify as an RG procedure, that were mentioned in the first paragraph of Sec. II.

The most direct way to study localization is to solve the eigenvalue problem for the Hamiltonian (1.1). A normalized state vector of energy E can be written

$$|\psi\rangle = \sum_{\vec{r}} a_{\vec{r}} |\vec{r}\rangle, \quad H|\psi\rangle = E|\psi\rangle \ . \tag{3.1}$$

We note that dividing the eigenvalue equation by V sets the energy scale of the problem, but does not change the amplitudes $a_{\vec{r}}$. Therefore, without loss of generality we can take $V=1$. In what follows we shall consider only hypercubic lattices.

If a state is localized, a localization length L can be defined in a variety of ways. Since we are dealing with a random system, only statistically averaged quantities make sense. Hence, the localization length L is assumed to be averaged over the ensemble and is expected to have the following properties. (i) L depends only on the distribution $\rho(\epsilon)$ of the site energies and on the eigenvalue E. That is, the ensemble average will be over all states of the same energy E belonging to different members of the ensemble. In practice, the spectrum forms a dense continuum and $L(E)$ would then be defined as the localization length corresponding to the interval $E, E+dE$. If, in addition to the site energies, hopping elements are also random, then L may be a function of their distribution as well.

The distribution $\rho(\epsilon)$ can be parametrized by a set of moments:

$$(\sigma_n)^n \equiv \langle \epsilon^n \rangle = \int_{-\infty}^{\infty} \epsilon^n \rho(\epsilon)\, d\epsilon \tag{3.2}$$

with the normalization

$$\int_{-\infty}^{\infty} \rho(\epsilon)\, d\epsilon = 1 \ . \tag{3.3}$$

Then

$$L = L(\sigma_1, \sigma_2, \ldots, E) \ . \tag{3.4}$$

(ii) If $\rho(\epsilon)$ is chosen to be symmetric, i.e., if $\rho(\epsilon)=\rho(-\epsilon)$, then all the odd moments vanish. It is easy to show (see Sec. IV) that a local gauge transformation $|\vec{r}\rangle \rightarrow -|\vec{r}\rangle$ leaves the spectrum as well as the amplitudes of the eigenvectors (apart from a trivial change in sign) invariant. If we perform such a transformation on every other site orbital the result will be to change the hopping element V to $-V$. Let us consider a particular configuration of the site energies, $\{\epsilon\}$, giving rise to $H(\epsilon)$ with eigenstates $|\psi\rangle$ corresponding to eigenvalues E. The Hamiltonian obtained by flipping the signs of all ϵ will be denoted H'. The eigenstates of H', e.g., $|\psi'\rangle$, can be obtained from $|\psi\rangle$ by flipping the sign of the amplitude on every other site. The corresponding eigenvalue will be $-E$. Since such a sign flip of amplitudes does not change the localization length associated with $|\psi\rangle$, and since the configuration $\{\epsilon\}$ occurs with the same probability as $\{-\epsilon\}$, we have

$$L(\{\sigma\}, E) = L(\{\sigma\}, -E) \ . \tag{3.5}$$

This last relation will turn out to be an important symmetry of the problem, especially near the band center, $E=0$. In the practical calculation $\rho(\epsilon)$ will be chosen to be a simple distribution, parametrized by its width σ_2, with σ_1 fixed at $\sigma_1=0$. From our discussions so far we can anticipate the possible phase diagram, shown in Fig. 1. The broken line corresponds to the band edges which, for a rectangular distribution of width σ, is given by[3,30] $E=\pm(Z+\frac{1}{2}\sigma)$. The localization length is expected to diverge as the solid line is crossed from the region of localized states. In what follows we develop a scaling theory for the localization length L. Since L is a function of the moments $\{\sigma\}$ and the energy E, these will be identified as the natural scaling variables.

Let us suppose that we can construct a RG transformation which maps Eq. (3.1) onto

$$H'|\psi\rangle' = E'|\psi\rangle' \tag{3.6}$$

such that in the new problem described by Eq. (3.6) all lengths are rescaled by a factor $b>1$, i.e., $\vec{r} \rightarrow \vec{r}' = \vec{r}/b$. The new Hamiltonian H' is assumed to be similar to the old Hamiltonian H in that it can be expressed in terms of a set of matrix elements between renormalized orbitals $|r\rangle'$ whose spatial density has been reduced by a factor of b^d

$$H' = \sum_{\vec{r}} \epsilon'_{\vec{r}} |\vec{r}\rangle' \langle \vec{r}|' + \frac{1}{2} \sum_{\vec{r}\,\vec{r}'} (V'_{\vec{r}\,\vec{r}'} |r\rangle' \langle \vec{r}'|' + \text{H.c.}) \ . \tag{3.7}$$

We note that even though in the original problem the

off-diagonal matrix elements $V_{\vec{r},\vec{r}'}$, were uniform, and connected only nearest-neighbor sites, the RG transformation may generate further neighbor interactions which would, in general, be random. Thus, in addition to random-site energies and the energy E, we should also include the probability distributions (e.g., moments) of the off-diagonal matrix elements as scaling variables. We suppose that the moments parametrizing these probability distributions are included in the set $\{\sigma\}$.

The new problem is thus characterized by an energy E' and a set of moments $\{\sigma'\}$ which are, in general, functions of E and $\{\sigma\}$ and the rescaling factor b. We further assume that the probability distributions are simple enough so that only a finite number of moments need be considered. Let us label these moments by the index i, where i runs from 1 to n. We then have the RG equations

$$\sigma_{i'} = f_i(\sigma_1, \sigma_2, \ldots, \sigma_n, E) \ , \tag{3.8}$$

$$E' = g(\sigma_1, \sigma_2, \ldots, \sigma_n, E) \ , \tag{3.9}$$

where we have suppressed the b dependence of the functions f_i and g.

Furthermore, if the state $|\psi\rangle$ is localized over a length L then the state $|\psi\rangle'$ is localized over a length $L' = b^{-1}L$ since all lengths are rescaled by a factor b. This yields the following scaling equation for L:

$$L(\sigma_{1'}, \sigma_{2'}, \ldots, \sigma_{n'}; E') = b^{-1} L(\sigma_1, \sigma_2, \ldots, \sigma_n; E) \ . \tag{3.10}$$

Let us suppose that for symmetric distributions the RG transformation treats the states with energy E and $-E$ on an equal footing. This means that these two states will give rise to the same randomness $\{\sigma'\}$. Moreover, if the energy E maps onto E' then the energy $-E$ maps onto $-E'$. In particular, the band center ($E=0$) maps onto itself ($E'=0$). As we will show later, for arbitrary E the new randomness $\{\sigma'\}$ may be nonsymmetric. However, for the band center, the new distribution is symmetric. Therefore the band center maps onto itself under repeated iteration; in other words, the line $E=0$ is a trajectory of the RG transformation.

We now look for the fixed-point solutions of the RG equations

$$\sigma_i^* = f_i(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*, E^*) \ , \tag{3.11}$$

$$E^* = g(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*, E^*) \ . \tag{3.12}$$

The general fixed-point structure can be established from the following considerations. For zero randomness, the transformation cannot produce any randomness. Therefore, the line $\sigma_i = 0$ for all i is invariant under the RG transformation. Similarly, for infinite randomness, the states are localized with $L=0$, so that $L' = b^{-1}L = 0$. Therefore the line $\sigma_i = \infty$ is also invariant. It is obvious that the intersection of any two trajectories must be a fixed point of the RG equations. Therefore, the points $(\sigma_i = 0, E=0)$ and $(\sigma_i = \infty, E=0)$ are two fixed points.

For the moment let us confine our attention to the line $E=0$. If the localization length L is finite for a given randomness than by definition $L' = b^{-1}L < L$ so that the scaling is toward larger randomness. Under repeated iterations L' decreases until the fixed point at $\sigma_i = \infty$ is reached, which is therefore "stable" or attractive. Suppose that, in addition, the fixed point at zero randomness ($\sigma_i = 0$) is also stable, that is, for small randomness the scaling is toward this point. It is obvious that in this case L cannot be finite, so that the state is extended. The situation is depicted schematically in Fig. 1 where for simplicity we consider only one moment, namely, the width σ. The scaling is indicated by the arrows. It is clear from the figure that there must be an "unstable" or repulsive fixed point for some $\sigma = \sigma_c > 0$ which signals the transition from an extended to a localized state.

Consider now $E \neq 0$. The solid curve separating the two regions of Fig. 1 is called a "critical surface." In the localized regime the scaling is again toward larger randomness since $L' < L$. We also assume that in the extended regime the scaling is toward smaller randomness. The critical surface is a "trajectory" of the RG equations and is unstable since the flows are away from this surface. In general there are one or more points on the surface which are stable for flows along the surface. These points are called critical points. If a fixed point on the critical surface is unstable then it is called a multicritical point. It is clear from the $(E, -E)$ symmetry that if there is only one critical fixed point then it must be located at the band center. If, on the other hand, the fixed point at the band center $(\sigma_c, E=0)$ is doubly unstable, we expect to find a pair of critical points which are located symmetrically with respect to the band center.

Any RG procedure (with a few exceptions) produces a large (sometimes infinite) number of couplings, or parameters in the new Hamiltonian H'. For example, in spin systems, even though one starts with a model that contains, say, nearest-neighbor interactions only, second-neighbor, third-neighbor, four-spin, etc., interactions are generated. In practical calculations one truncates this parameter space and retains the minimal number of "dominant" ones needed to represent the important physical variables. In similar spirit, we will use a procedure that keeps only single-site energies and nearest-neighbor hopping terms. Furthermore, since we deal with a random system, the Hamiltonian is characterized by *distributions* of these terms. Here again we proceed in the spirit of position-space RG studies of random

spin systems, working with distributions of some specific fixed form, parametrized by a few low moments. Obviously, if different fixed-point distributions exist, one needs a larger space of allowed functions in order not to lose any of these distributions. For the sake of simplicity we now assume that it is sufficient to work with one simple distribution and follow the way its width or second moment $\sigma=\sigma_2$ changes under the RG iterations. Under this assumption, Eqs. (3.8)–(3.10) reduce to

$$\sigma'=f(\sigma,E) \ , \tag{3.13}$$

$$E'=g(\sigma,E) \ , \tag{3.14}$$

$$L'=L(\sigma',E')=b^{-1}L(\sigma,E) \ . \tag{3.15}$$

Consider the case of a fixed point at the band center at $(\sigma^*,0)$. As usual, $f(\sigma,E)$ and $g(\sigma,E)$ are assumed to be smooth analytic functions of their arguments (since as will be shown later, they are derived from finite cells). The stability properties of the fixed points can be established by expanding f and g in powers of the deviations $\Delta\sigma=\sigma-\sigma^*$ and E as follows

$$\Delta\sigma'=b^{\lambda_1}\Delta\sigma+\alpha(b)E^2+\cdots \ , \tag{3.16}$$

$$E'=b^{\lambda_2}E+\cdots \ , \tag{3.17}$$

where the ellipses represent higher-order terms and we have used the $E\rightarrow -E$ symmetry discussed above and the fact that there cannot be any term independent of E in Eq. (3.17) since $E'=0$ when $E=0$.

Since the fixed point is unstable along the band center, $\lambda_1>0$. If it is stable along the energy direction then $\lambda_2<0$, i.e., $|E'|<|E|$. The corresponding flow patterns are shown in Fig. 1. Along the band center the localization length diverges as

$$L=L_0(\sigma-\sigma^*)^{-\nu} \ , \tag{3.18}$$

where $\nu=1/\lambda_1$. To find the exponent along any other direction we define the scaling fields[31] h_1 and h_2 by

$$h_1\simeq\Delta\sigma+\beta E^2 \ , \tag{3.19}$$

$$h_2\simeq E \ , \tag{3.20}$$

with the corresponding RG equations

$$h_i'=b^{\lambda_1}h_i, \quad i=1,2 \tag{3.21}$$

so that $\beta=\alpha(b)/(b^{\lambda_1}-b^{2\lambda_2})$. The line $h_1=0$ is the equation for the critical curve separating the localized from the extended states (Fig. 1). As this line is approached from the localized regime, L diverges as

$$L\sim h_1^{-\nu}=(\sigma-\sigma_c+\beta E^2)^{-\nu} \ . \tag{3.22}$$

Usually the existence of a single relevant scaling field implies that there is a single exponent characterizing the divergence of L. Approaching the critical line at *fixed* σ, we have

$$L\sim(E^2-E_c^2)^{-\nu} \ . \tag{3.23}$$

Denoting $\Delta E=E-E_c$, this reads

$$L\sim\Delta E^{-\nu}(2E_c+\Delta E)^{-\nu} \ . \tag{3.24}$$

As long as $2E_c>>\Delta E$, one obtains $L\sim(E-E_c)^{-\nu}$. In the regime $\Delta E\approx 2E_c$, however, one may (numerically) find an effective exponent $\nu<\nu'<2\nu$. Thus to obtain the correct exponent ν' for finite E_c in the neighborhood of the band center one has to consider a "critical region" ΔE much smaller than E_c. This, in our opinion, is the reason why different numerical estimates of ν' show large variations (see Table I).

Finally, we consider a situation which may be important for the two-dimensional Anderson model. Assuming that the critical fixed point is located on the band center we restrict our attention to the recursion relation

$$\sigma'=f(\sigma) \ . \tag{3.25}$$

If there is no unstable fixed point at finite σ then the fixed point at $\sigma=0$ must be unstable since at large σ the states are localized, whereas at $\sigma=0$ they are definitely extended. Expanding in powers of σ we obtain after the lth iteration

$$\sigma_{l+1}=b^{\lambda}\sigma_l+a_2\sigma_l^2+\cdots \ . \tag{3.26}$$

For $\lambda>0$, σ is relevant and the analysis is as before. For $\lambda=0$ the field is marginal in the linear region and we have to retain the higher-order terms to study the RG flow. In this case we can rewrite (3.26) in differential form

$$\frac{\partial\sigma}{\partial l}=a_2\sigma^2 \ . \tag{3.27}$$

We note that $a_2>0$ since the fixed point is unstable. Integrating, we obtain

$$L(\sigma)\sim\exp\frac{1}{a_2\sigma}, \quad \sigma\rightarrow 0 \ . \tag{3.28}$$

IV. EXPLICIT CONSTRUCTION OF RENORMALIZATION-GROUP TRANSFORMATION

We now consider the explicit construction of the RG mapping and the determination of the functions $g(\sigma,E,b)$ and $f(\sigma,E,b)$. Our method is an application of position-space RG ideas of Niemeijer and van Leeuwen[22] and is expected to be valid when σ is not too small.

To establish the mapping we shall use nearly degenerate perturbation theory.[32] The Hamiltonian

$$H=\sum_{\vec{r}}\epsilon_{\vec{r}}|\vec{r}\rangle\langle\vec{r}|+\frac{1}{2}\sum_{\langle\vec{r}\,\vec{r}'\rangle}(|\vec{r}\rangle\langle\vec{r}'|+|\vec{r}'\rangle\langle\vec{r}|) \ , \tag{4.1}$$

$$\langle\epsilon_{\vec{r}}\rangle=0; \quad \langle\epsilon_{\vec{r}}\epsilon_{\vec{r}'}\rangle=\sigma^2\delta_{\vec{r}\,\vec{r}'} \ , \tag{4.2}$$

is written as a sum of two parts

$$H = H_0 + H_1 \ . \tag{4.3}$$

The eigenvalues E_n and the normalized eigenvectors of H_0 are assumed to be known

$$H_0|\phi_n\rangle = E_n|\phi_n\rangle \ , \tag{4.4}$$
$$\langle\phi_n|\phi_m\rangle = \delta_{nm} \ .$$

The Hilbert space Ω in which H operates can be spanned by the vectors $|\phi_n\rangle$. For the Anderson model the number of such states is equal to the number of sites N of the lattice. The RG mapping we are seeking to establish reduces N to $N' = N/b^d$. To achieve this reduction we divide the Hilbert space Ω into two subspaces D and $\bar{D}$, containing N' and $N - N'$ states, respectively.

If $|\psi\rangle$ is an eigenstate of H,

$$H|\psi\rangle = E|\psi\rangle \tag{4.5}$$

denote by $|\psi_D\rangle$ the projection of $|\psi\rangle$ onto D, e.g.,

$$|\psi\rangle = P_D|\psi\rangle \ . \tag{4.6}$$

Then $|\psi_D\rangle$ satisfies the equation

$$H_D|\psi_D\rangle = E|\psi_D\rangle \ , \tag{4.7}$$

where

$$H_D = H_0 + V \tag{4.8}$$

and V solves the operator equation

$$V = H_1 + H_1\frac{1-P_D}{E-H_0}V \ . \tag{4.9}$$

We have thus reduced the problem from that of diagonalizing an $N \times N$ matrix to that of an $N' \times N'$ one, provided we can calculate the matrix elements of V in the subspace D. In general this is nontrivial since (i) V depends explicitly on the exact energy E, which can only be obtained by solving the original secular equation and (ii) one has to solve an operator Eq. (4.9). The first difficulty can be overcome by noting that for a very large lattice the spectrum of H forms a continuum, so that for an E within the band there is a solution. Therefore the energy E can be used as a parameter in Eq. (4.9).

So far the analysis has been exact. To overcome the second difficulty, i.e., the calculation of V, we resort to perturbation theory, and expand V in powers of H_1

$$V = H_1 + H_1GH_1 + H_1GH_1GH_1 + \cdots \ , \tag{4.10}$$

where

$$G = \frac{1-P_D}{E-H_0} \ . \tag{4.11}$$

Whether the series (4.10) converges rapidly, so that keeping only the first few terms yields a reliable estimate for H_D, depends on our choice of H_0 and the model space D. The eigenvalues and eigenvectors of H_0 should approximate those of H fairly closely so that H_1 can be considered a small effect. For small randomness the eigenfunctions of H will spread over many lattice spacings; hence those of H_0 should also be fairly extended. A natural choice in this regime is the plane-wave states for $\sigma = 0$, labeled by the momentum q:

$$\Phi_{\vec{q}} = \langle\Phi|\Phi_{\vec{q}}\rangle = \frac{1}{\sqrt{N}}e^{i\vec{q}\cdot\vec{r}} \ . \tag{4.12}$$

The appropriate method in this regime could probably be some version of the momentum-space renormalization-group methods of Wilson.[24]

In the other regime, when σ is sufficiently large, the eigenstates of H are expected to be localized. Hence in this regime the position-space renormalization-group ideas of Niemeyer and van Leeuwen[22] are applicable. The amplitude of $|\psi\rangle$ on a given lattice site will strongly depend on the amplitudes on nearby lattice sites and only weakly on those far away. Accordingly, we break up the lattice into small cells of volume b^d, and include in H_0 all the elements of (4.1) which connect sites in the same cell. Hopping elements that connect sites of neighboring cells are included in H_1. H_0 is thus a sum of uncoupled cell Hamiltonians which we label by R:

$$H_0 = \sum_{\vec{R}} h_{\vec{R}} \ , \tag{4.13}$$

with

$$h_R = \sum_{\vec{r}\in\vec{R}} \epsilon_{\vec{r}}|\vec{r}\rangle\langle\vec{r}| + \frac{1}{2}\sum_{\vec{r},\vec{r}',\in\vec{R}} V_{\vec{r}\vec{r}'}(|\vec{r}\rangle\langle\vec{r}'| + |\vec{r}'\rangle\langle\vec{r}|) \ , \tag{4.14}$$

and

$$H_1 = \frac{1}{2}\sum_{\vec{r}\in\vec{R},\vec{r}'\in\vec{R}'} V_{\vec{r}\vec{r}'}(|\vec{r}\rangle\langle\vec{r}'| + |\vec{r}'\rangle\langle\vec{r}|) \ . \tag{4.15}$$

The eigenstates of the unperturbed Hamiltonian H_0 are obtained by diagonalizing each h_R separately. Each cell will have b^d states. Let $|\vec{R}i\rangle$ be the eigenvectors and $e_{\vec{R}i}$ the corresponding eigenvalues of the cell Hamiltonian $h_{\vec{R}}$ (i runs from 1 to b^d). The Hamiltonian H can be expressed in this basis

$$H = \sum_{\vec{R}i} e_{\vec{R}i}|\vec{R}i\rangle\langle\vec{R}i| + \frac{1}{2}\sum_{\vec{R}i}\sum_{\vec{R}'j}(H_1{}_{\vec{R}i\vec{R}'j}|\vec{R}i\rangle\langle\vec{R}'j| + \text{H.c.}) \ , \tag{4.16}$$

where H.c. means Hermitian conjugate and

$$H_1{}_{\vec{R}i\vec{R}'j} = \langle\vec{R}i|H_1|\vec{R}'j\rangle$$

and is nonzero only when $\vec{R}, \vec{R}'$ are nearest-neighbor cells (see Fig. 2).

Now we construct the model subspace D by keeping only a fraction of b^d states in each cell. Since we want our RG procedure to produce a new Hamiltonian which is similar to the old one, we keep only one state in each cell. Note that the operator G connects only states which are in the excluded space $\bar{D}$; thus if $\bar{D}$ contains states with eigenvalues close to E the perturbation theory will converge slowly because of small energy denominators. We therefore choose that state from the cell $\vec{R}$, for which $|E - e_{\vec{R}i}|$ is smallest. Denoting this state by $|\vec{R}\rangle$, the Hamiltonian H_D takes the form

$$H_D = \sum_{\vec{R}} (e_{\vec{R}} + V_{\vec{R},\vec{R}}) |\vec{R}\rangle \langle \vec{R}| \tag{4.17}$$

$$+ \tfrac{1}{2} \sum_{\vec{R} \neq \vec{R}'} (V_{\vec{R},\vec{R}'} |\vec{R}\rangle \langle \vec{R}'| + \text{H.c.}) \ . \tag{4.18}$$

Note that so far the formalism is exact. However, since the operator equation (4.9) cannot be solved exactly, we must resort to some approximation. As a first approximation, we keep the first term only, i.e., $V \simeq H_1$. Keeping higher-order terms will generate higher-order interactions (e.g., next neighbor); these could, in principle, be also included in a more elaborate RG calculation (see Sec. VI). To first order in H_1 we can write

$$H_D^{\text{approx}} = \sum_{\vec{R}} e_{\vec{R}} |\vec{R}\rangle \langle \vec{R}|$$

$$+ \tfrac{1}{2} \sum_{\vec{R}\vec{R}'} V'_{\vec{R}\vec{R}'} (|\vec{R}\rangle \langle \vec{R}'| + \text{H.c.}) \tag{4.19}$$

where $V'_{\vec{R}\vec{R}'} = \langle \vec{R}|H_1|\vec{R}'\rangle$. The Hamiltonian H_D describes a problem on a lattice with $N' = N/b^d$ sites; the new distance between two such sites is reduced by a factor b with respect to the "old" distance.

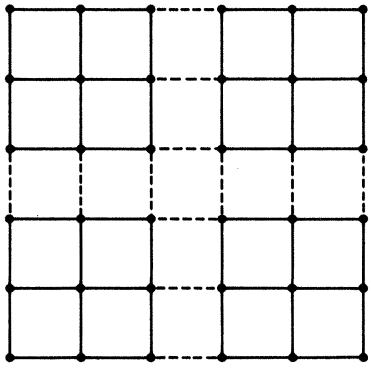

FIG. 2. Section of an infinite lattice: each dot represents a diagonal element of the Hamiltonian (1); solid and dashed lines represent nearest-neighbor hopping elements. The hopping elements that connect different cells are indicated by the dashed lines. H_0 contains all intracell operators (dots and solid lines); H_1 contains the intercell hopping elements.

The Hamiltonian H_D of Eq. (4.19) looks very much the same as the Anderson Hamiltonian (1.1). The "site" energies $e_{\vec{R}}$, being functions of different sets of random variables, are themselves random and independent of each other. However, the hopping elements $V_{\vec{R}\vec{R}'}$ are no longer uniform. They vary both in sign and in magnitude. As we shall see from actual calculations, the variation in magnitude $|V'|$ is small compared to that of the site energies.

We proceed, following the ideas outlined in Sec. III, by replacing the actual $V'_{\vec{R}\vec{R}'}$ distribution that is generated by some fixed distribution. First we remove all randomness of $V_{RR'}$ by a replacement

$$V'_{RR'} \to V_{\text{eff}} \ . \tag{4.20}$$

Two measures of V_{eff} were considered; the average absolute value of $V'_{RR'}$, e.g.,

$$V_{\text{eff}} = \langle |V'_{\vec{R}\vec{R}'}| \rangle \tag{4.21}$$

and the root mean square

$$V_{\text{eff}} = [\langle (V'_{\vec{R}\vec{R}'})^2 \rangle]^{1/2} \ . \tag{4.22}$$

Most of our calculations were done with (4.21); however we did check the effect of using (4.22) and found no qualitative change in our results. By the replacement (4.20) we have eliminated both amplitude and sign randomness of the $V'_{\vec{R}\vec{R}'}$. Alternatively, one can keep track at least of the sign randomness, by using the replacement

$$V'_{\vec{R}\vec{R}'} \to V_{\text{eff}} \eta_{RR'} \ , \tag{4.23}$$

where $\eta_{RR'}$ are independent random variables taken from the distribution

$$P(\eta_{\vec{R}\vec{R}'}) = p\,\delta(\eta_{\vec{R}\vec{R}'} - 1)$$

$$+ (1-p)\,\delta(\eta_{\vec{R}\vec{R}'} + 1) \ . \tag{4.24}$$

As we will show below, when (4.20) is used, H' can easily be put in a form similar to (4.1) and (4.2). When (4.23) and (4.24) are used, the new Hamiltonian is more general than (4.1); however, further iterations will leave H in this larger space. In this case the recursion relations for σ and E must be supplemented by one for p. Note that the simpler choice (4.20) is contained in the more general case by forcing $p = 1$ (i.e., $\eta_{\vec{R}\vec{R}'} = 1$).

To bring (4.19) to the form of (4.1) and (4.2) we substitute

$$e_{\vec{R}} = \langle e_{\vec{R}} \rangle + \epsilon_{\vec{R}} V_{\text{eff}} \ . \tag{4.25}$$

Within the approximations taken, we now have

$$H_D^{\text{approx}} \cong \langle e_{\vec{R}} \rangle I + V_{\text{eff}} H' \ , \tag{4.26}$$

where $\langle e_{\vec{R}} \rangle$ is the mean of the $e_{\vec{R}}$'s; I is the unit matrix, and

$$H' = \sum_{\vec{R}} |\vec{R}\rangle \langle \vec{R}| \epsilon_{\vec{R}} + \frac{1}{2} \sum_{\vec{R}\vec{R}'} (\eta_{\vec{R}\vec{R}'} |\vec{R}\rangle \langle \vec{R}'| + \text{H.c.}) \quad . \tag{4.27}$$

The state $|\psi_D\rangle$ [see Eq. (4.7)] satisfies now

$$H'|\psi_D\rangle = E'|\psi_D\rangle \quad , \tag{4.28}$$

with

$$E' = (E - \langle e_{\vec{R}} \rangle)/V_{\text{eff}} \quad . \tag{4.29}$$

By subtracting the average value of the $e_{\vec{R}}$ from the diagonal elements of H_D, we obtained a distribution of new site energies with zero mean, i.e., $\langle \epsilon_R \rangle = 0$. Thus, the first moment of the new distribution is adjusted to be the same as that of the old one [see Eq. (4.2)].

Equations (4.20) supplemented by one of (4.21)–(4.24), together with (4.29) constitute the recursion relation for the energy E, Eq. (3.14).

When (4.21) or (4.22) is used, H' is of the same form as H. Since the new site energies ϵ_R are completely independent and random, they can be characterized by a set of moments σ_n'. The shape of the new distribution will depend on the old distribution, and also on the energy E. For example, if we start out with a symmetric distribution, the new distribution is expected to be asymmetric for nonzero E. This can be seen by considering a state near the band edge with E large and positive. Since the average density of cell states is a decreasing function of E, an asymmetric distribution of $\epsilon_{\vec{R}}$ will be generated. Whereas for a symmetric distribution the width $\sigma' = (\langle \epsilon_{\vec{R}}^2 \rangle)^{1/2}$ is a suitable measure of randomness, for the asymmetric case one should include at least one odd moment in the parameter space (e.g., the third moment). Near the band center, however, we expect the distribution to be symmetric. In the present work we study the neighborhood of the band center ($E = 0$) and we choose for our scaling variable the width σ', giving the second RG equation (3.14), by

$$\sigma' = (\langle \epsilon_{\vec{R}}^2 \rangle)^{1/2} \quad . \tag{4.30}$$

When the replacement (4.23) is used, Eqs. (4.29) and (4.30) must be supplemented by a third equation for p, the parameter characterizing the distribution of the sign variable [see Eq. (4.24)].

Note that by an appropriate gauge transformation the sign of any hopping element can be changed. Thus a Hamiltonian H with all $\eta_{RR'} = +1$ can be transformed into one where some $\eta_{RR'}$ are negative. Such gauge transformations will affect neither the spectrum nor the localization properties of the eigenstates of H. Therefore, in order to characterize the hopping sign randomness in a Hamiltonian, one has to specify a quantity which is invariant under such gauge transformations. Such a quantity is the frustration[33] ratio of the lattice, q, defined as

$$q = N_f/N \quad , \tag{4.31}$$

where N_f is the number of frustrated plaquettes (elementary squares) and N is the total number of plaquettes. A plaquette is frustrated if the product of $\eta_{RR'}$ around it is negative. Thus, if a given set $\eta_{RR'}$ is generated by our procedure, we determine q using (4.31) and determine the distribution of the $\eta_{RR'}$ variables by the relation

$$q = 1 - \tfrac{1}{2}(2p-1)^4 \quad . \tag{4.32}$$

Equations (4.31) and (4.32) together with (4.29) and (4.30) constitute the set of RG recursion relations used when the replacement (4.23) is made.

To conclude this section, note that we have constructed RG recursion relations that are approximations to an underlying exact procedure. At each step of renormalization b^d lattice sites are replaced by a single site, all lengths are scaled by b, and a mapping in the space of Hamiltonians is obtained. Thus all basic ingredients of a valid RG procedure are contained in our method.

V. NUMERICAL METHOD, AND RESULTS OF FIRST-ORDER CALCULATION

In this section we describe our numerical procedure and analyze the results obtained therefrom. The RG equations (4.29)–(4.32) were studied extensively for a square ($d = 2$) and simple cubic ($d = 3$) lattice. The cell sizes were chosen to be b^d with $b = 3$.

The numerical procedure consisted of the following steps. (i) A sample of matrices corresponding to cell Hamiltonians $h_{\vec{R}}$ was created. This was done by selecting the diagonal elements from a Gaussian distribution with zero mean and width σ. For the procedure that neglects the sign randomness of the hopping terms, all off-diagonal elements were set to be $+1$. When the sign randomness was also studied, the off-diagonal elements were chosen from the distribution (4.24), for various values of p. The cell matrices were numerically diagonalized; the b^d eigenvectors and the corresponding eigenvalues were stored. This meant diagonalizing 9×9 matrices in $d = 2$ and 27×27 matrices for $d = 3$. Altogether, a sample of 900 cells were created for the square and 512 for the simple cubic lattice for a sequence of values of σ and p. (ii) The 900 cells in two dimensions were arranged to form a 30×30 supperlattice. To generate the recursion relations for any value of E, the state whose eigenvalue was closest to E was picked from

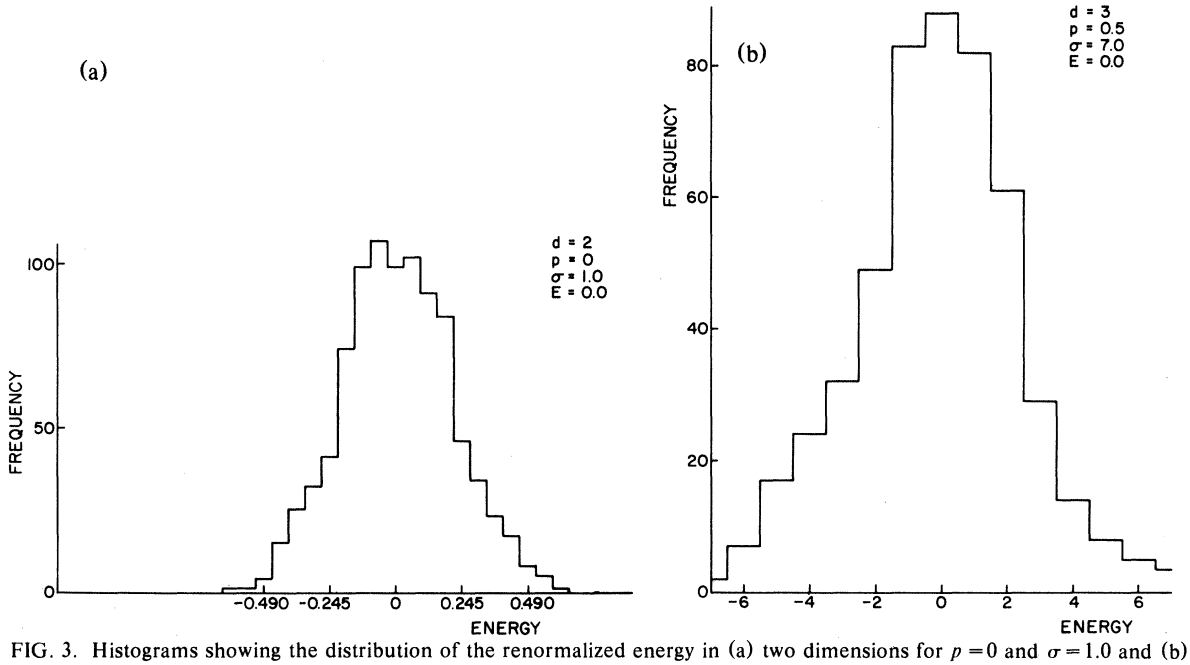

FIG. 3. Histograms showing the distribution of the renormalized energy in (a) two dimensions for $p=0$ and $\sigma=1.0$ and (b) in three dimensions for $p=0.5$ and $\sigma=7.0$.

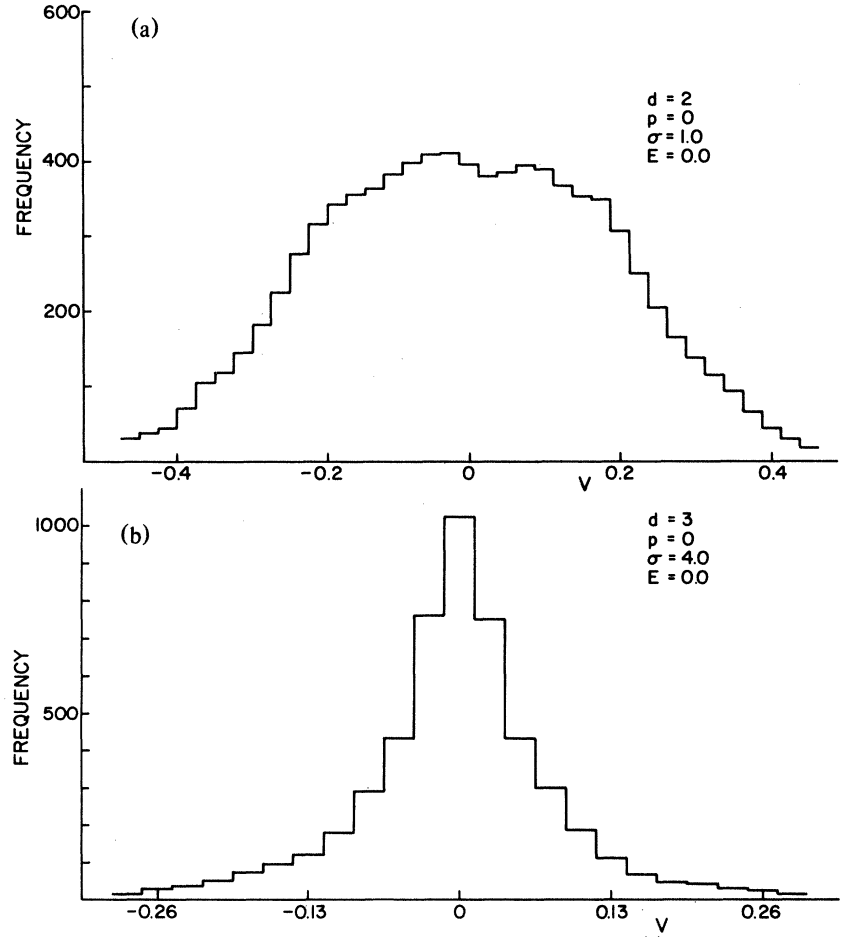

FIG. 4. Histograms showing the distribution of the new hopping elements $V_{\vec{R}\vec{R}'}$ in (a) two dimensions and (b) three dimensions for the values of the parameters shown.

each cell, and the distribution of the corresponding eigenvalues was studied. In three dimensions a similar procedure was carried out by forming an $8\times8\times8$ superlattice. (iii) The new hopping elements $V_{\vec{r}\,\vec{r}'}$ were calculated from the equation

$$V_{\vec{r}\,\vec{r}'}=\sum_i a_{\vec{R}i}a_{\vec{R}'i}\eta_i \ , \tag{5.1}$$

where $a_{\vec{R}i}$ is the amplitude of the chosen state at the site i on a surface (or an edge in $d=2$) of the cell R, $a_{\vec{R}'i}$ is the amplitude at the corresponding point i on the adjacent surface of a neighboring cell $\vec{R}'$. For the procedure that neglects sign randomness all $\eta_i=1$; when sign randomness was studied, η_i were also generated from the distribution (4.24). In this latter case, the number of frustrated squares (plaquettes) was then easily calculated by multiplying together the $V_{\vec{R}\vec{R}'}$, around an elementary square and counting only those squares for which the product was negative. The frustration probability q' was then given by Eq. (4.31); and p' was determined using (4.32). σ' and E' were determined from Eqs. (4.30) and (4.29).

This numerical program was carried out for several values of σ in both two ($0.1\leq\sigma\leq20$) and three ($0.75\leq\sigma\leq10$) dimensions for $0\leq p\leq0.5$. For the study of the energy scaling, four energies were chosen near the band center in each case. Since the band center should map onto itself, the calculated values of E' for $E=0$ were taken to be a measure of statistical error. The superlattices were constructed in five different ways in two dimensions and four ways in three dimensions, in order to improve accuracy and provide an estimate of the statistical error.

Histograms were compiled for the distribution of the eigenvalues $e_{\vec{R}}$ as well as the hopping elements $V_{\vec{R}\vec{R}'}$ and their absolute values $|V_{\vec{R}\vec{R}'}|$. Typical histograms are shown in Figs. 3–5. The distribution of $V_{\vec{R}\vec{R}'}$ was always found to be symmetric, with a maximum at the center accompanied by a sharp drop and a long tail. For $V_{\rm eff}$ we chose the arithmetic mean of the absolute values of $V_{\vec{R}\vec{R}'}$. We checked and found that using the root mean square did not qualitatively change our results.

Examining the histograms for the site energy distributions (Fig. 3) we see that the new distribution is symmetric about the origin for $E=0$. This remains valid for other values of E near the band center. The values of E', σ', g' were calculated from Eqs. (4.29)–(4.31), respectively, and are shown in Tables II and III for $E=0$ along with the corresponding values of σ, q, $V_{\rm eff}$ and the standard deviation of the absolute values of $V_{\vec{R}\,\vec{R}'}$.

The first striking feature that emerges from an examination of the tables is the rapid flow of the frustration probability q to its fixed-point value $q^*=0.5$. The line $q=0.5$ is therefore a trajectory of the RG

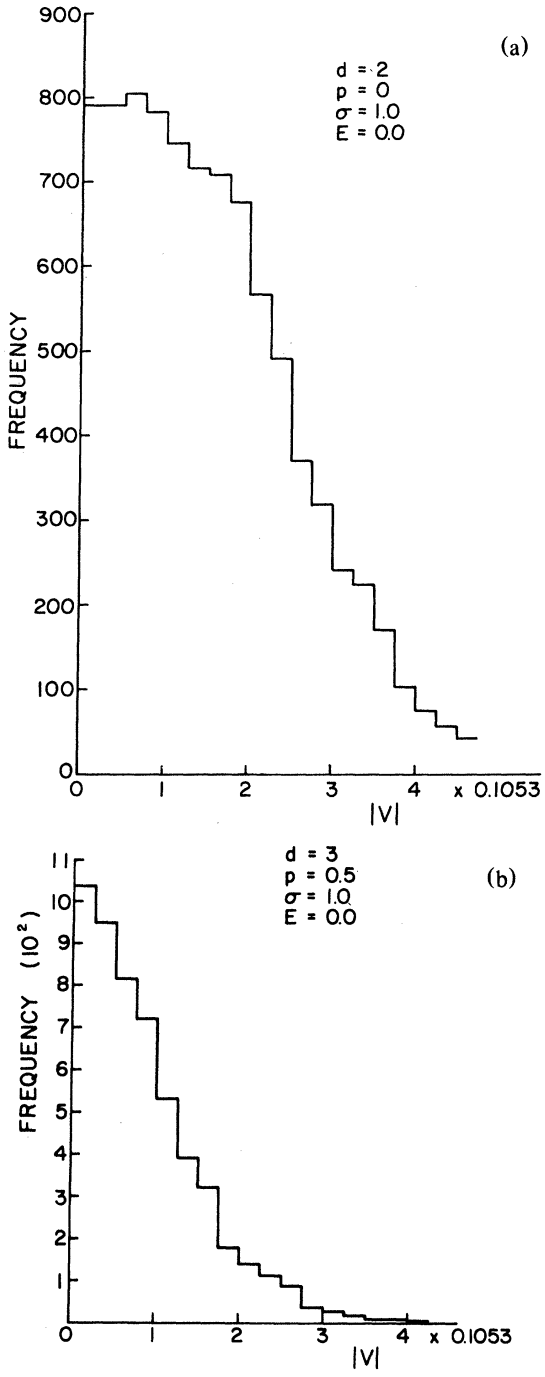

FIG. 5. Distribution of the absolute values of the hopping element in (a) two dimensions and (b) three dimensions for the values of the parameters shown.

equation, and is stable against perturbations along the q direction. According to the discussions in Sec. III, this implies that the localization properties will be determined by the critical value $p^*=q^*=0.5$. Moreover, the values of other quantities (σ', $V_{\rm eff}$, etc.) are found to be quite insensitive to the variation of q.

TABLE II. Calculated values of σ', q', and V_{eff} for given values of σ, q, and for $E=0$ in two dimensions.

σ	q	V_{eff}	σ'	q'
1.0	0.0	0.164 ± 0.003	1.256 ± 0.023	0.448 ± 0.010
	0.5	0.155 ± 0.002	1.757 ± 0.023	0.506 ± 0.005
2.0	0.0	0.148 ± 0.001	2.168 ± 0.025	0.485 ± 0.015
	0.5	0.159 ± 0.002	2.350 ± 0.038	0.494 ± 0.014
3.0	0.0	0.129 ± 0.002	3.295 ± 0.068	0.490 ± 0.018
	0.5	0.128 ± 0.004	3.584 ± 0.112	0.497 ± 0.015
4.0	0.0	0.116 ± 0.002	5.433 ± 0.082	0.471 ± 0.013
	0.5	0.117 ± 0.003	4.985 ± 0.129	0.499 ± 0.023
5.0	0.0	0.104 ± 0.005	7.313 ± 0.347	0.481 ± 0.008
	0.5	0.102 ± 0.002	7.336 ± 0.303	0.05 ± 0.021
10.0	0.0	0.071 ± 0.005	21.91 ± 1.75	0.498 ± 0.020
	0.5	0.073 ± 0.002	22.61 ± 0.79	0.502 ± 0.015
20.0	0.0	0.058 ± 0.003	58.86 ± 3.22	0.494 ± 0.014
	0.5	0.053 ± 0.004	64.10 ± 4.94	0.500 ± 0.010

This is in accord with the results obtained by Theodorou and Cohen and Antoniou and Economou[34] who found that the neighborhood of the band center is insensitive to off-diagonal randomness as far as localization properties are concerned.[35]

To establish the localization properties we need to study the function $\sigma'=f(\sigma,E)$. Tables IV and V contain the values of σ' for different values of σ and E for $q=0$. Figures 6 and 7 are the plots of σ' vs σ for $q=0$ and 0.5 as evaluated for $E=0$. Note that the function shown in Fig. 6 that was derived with $q=0$ constitutes the recursion relation based on the procedure (4.21) (i.e., neglecting sign randomness of $V_{RR'}$). Figure 7 describes the recursion relation for σ along the invariant subspace $E=0$, $q=p=0.5$ for the procedure defined by (4.23), (4.24), and (4.31). The fixed point is determined by the point where the curve for σ' vs σ crosses the straight line $\sigma'=\sigma$. In two dimensions we have not found a fixed point, whereas in three dimensions there is a fixed point at

TABLE III. Calculated values of σ', q', and V_{eff} for given values of σ and q for $E=0$ in three dimensions. Asterisk represents values of q' calculated for $E=0.5$.

σ	q	V_{eff}	σ'	q'
0.75	0.0	0.110 ± 0.002	0.551 ± 0.011	
1.90	0.0	0.095 ± 0.002	1.220 ± 0.021	0.497 ± 0.016*
2.0	0.5	0.087 ± 0.003	1.487 ± 0.057	0.489 ± 0.007*
3.0	0.0	0.080 ± 0.001	1.952 ± 0.028	0.499 ± 0.012*
	0.5	0.079 ± 0.001	2.08 ± 0.041	0.473 ± 0.024*
4.0	0.0	0.070 ± 0.002	2.808 ± 0.095	0.497 ± 0.005
	0.5	0.072 ± 0.001	2.784 ± 0.045	0.517 ± 0.015
5.0	0.0	0.063 ± 0.002	4.027 ± 0.118	0.508 ± 0.022*
	0.5	0.065 ± 0.002	4.244 ± 0.097	0.490 ± 0.010*
6.0	0.0	0.057 ± 0.002	5.779 ± 0.231	0.487 ± 0.014
	0.5	0.058 ± 0.002	5.320 ± 0.154	0.499 ± 0.010
7.0	0.0	0.054 ± 0.004	6.796 ± 0.367	0.494 ± 0.013
	0.5	0.053 ± 0.002	6.767 ± 0.522	0.494 ± 0.016
8.0	0.0	0.046 ± 0.002	8.969 ± 0.309	
	0.5	0.048 ± 0.003	8.256 ± 0.446	
10.0	0.0	0.040 ± 0.002	14.56 ± 0.87	0.495 ± 0.017
	0.0	0.041 ± 0.002	12.97 ± 0.75	0.504 ± 0.017

TABLE IV. Calculated values of σ' and E' for various σ and E and for $q=0$ in two dimensions.

σ	E	E'	σ'
1.0	0.0	−0.045	1.256 ± 0.023
	0.4	0.154 ± 0.001	1.419 ± 0.163
	0.8	0.006 ± 0.000	1.565 ± 0.023
	1.2	−0.052 ± 0.004	1.345 ± 0.010
2.0	0.0	−0.004	2.168 ± 0.025
	0.4	0.071 ± 0.001	2.122 ± 0.020
	0.8	−0.072 ± 0.001	2.158 ± 0.022
	1.2	−0.144 ± 0.002	2.165 ± 0.017
3.0	0.0	0.036	3.295 ± 0.068
	0.4	0.217 ± 0.003	3.372 ± 0.059
	0.8	0.219 ± 0.003	3.571 ± 0.051
	1.2	0.134 ± 0.002	3.736 ± 0.054
5.0	0.0	0.176	7.313 ± 0.347
	0.4	−0.208 ± 0.019	7.372 ± 0.666
	0.8	0.404 ± 0.030	7.466 ± 0.565
	1.2	0.320 ± 0.017	7.661 ± 0.417

TABLE V. Calculated values of σ' and E' for various σ and E and for $q=0$ in three dimensions.

σ	E	E'	σ'
1.9	0.0	0.097	1.220 ± 0.021
	0.5	−0.094 ± 0.001	1.279 ± 0.023
	1.0	−0.017 ± 0.000	1.395 ± 0.013
	1.5	0.023 ± 0.000	1.437 ± 0.044
3.0	0.0	0.081	1.952 ± 0.028
	0.5	0.056 ± 0.001	1.966 ± 0.060
	1.0	0.057 ± 0.001	1.913 ± 0.039
	1.5	0.126 ± 0.004	2.289 ± 0.076
4.0	0.0	0.072	2.808 ± 0.095
	0.5	0.057 ± 0.001	2.724 ± 0.076
	1.0	0.313 ± 0.003	2.758 ± 0.026
	1.5	0.184 ± 0.007	3.021 ± 0.020
5.0	0.0	0.065	4.027 ± 0.118
	0.5	0.031 ± 0.001	3.993 ± 0.113
	1.0	0.027 ± 0.000	3.897 ± 0.062
	1.5	0.504 ± 0.005	4.241 ± 0.047
6.0	0.0	0.059	5.779 ± 0.231
	0.5	0.235 ± 0.010	5.468 ± 0.247
	1.0	−0.034 ± 0.000	5.732 ± 0.068
	1.5	−0.054 ± 0.001	5.782 ± 0.195
7.0	0.0	0.058	6.796 ± 0.367
	0.5	0.122 ± 0.003	6.903 ± 0.209
	1.0	−0.026 ± 0.001	7.291 ± 1.178
	1.5	0.236 ± 0.005	7.067 ± 0.156
8.0	0.0	0.048	8.969 ± 0.309
	0.5	−0.186 ± 0.009	9.739 ± 0.506
	1.0	0.640 ± 0.046	9.168 ± 0.673
	1.5	0.553 ± 0.020	9.355 ± 0.345
10.0	0.0	0.042	14.56 ± 0.87
	0.5	−0.186 ± 0.009	14.705 ± 0.719
	1.0	−0.164 ± 0.002	15.107 ± 0.177
	1.5	−0.232 ± 0.012	14.745 ± 0.775

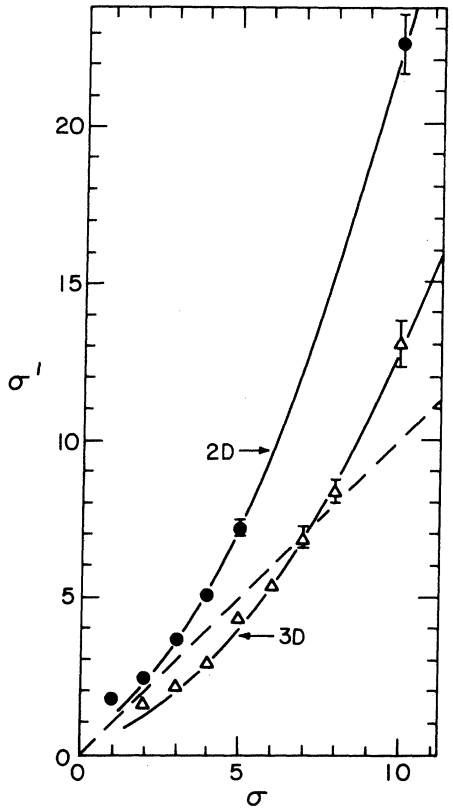

FIG. 6. σ' vs σ in two and three dimensions for $E=0$. Results based on representing the hopping elements by uniform V_{eff}, i.e., for $p=0$. In two dimensions no fixed point is found, while in three dimensions there is a fixed point at $\sigma_c \approx 7.0$.

$\sigma_c \approx 7.0$. Similar results hold for other values of the energy $E=0.5$, 1.0, 1.5 for $d=3$ and $E=0.4$, 0.8, 1.2 for $d=2$.

From Tables IV and V it is apparent that the energy scales toward the band center in both two and three dimensions for all values of σ considered. For an estimate of the error involved, we have also listed the largest value of the renormalized energy E' corresponding to the band center $E=0$. In Figs. 8 and 9 we show the plot of E' vs E for various values of σ in two and three dimensions, respectively. The solid lines correspond to $E'=E$. The points at the origin are rough measures of the statistical error. From these graphs it is clear that $E' < E$ in all cases, i.e., the fixed point at the band center is stable against perturbations in the energy direction. This in turn tells us that the fixed point is a "critical" fixed point and that there is only one independent exponent characterizing the divergence of the localization length.

To calculate the exponent we have to obtain the slope of the curve σ' vs σ at $\sigma_C \approx 7.0$. By Eq. (3.16)

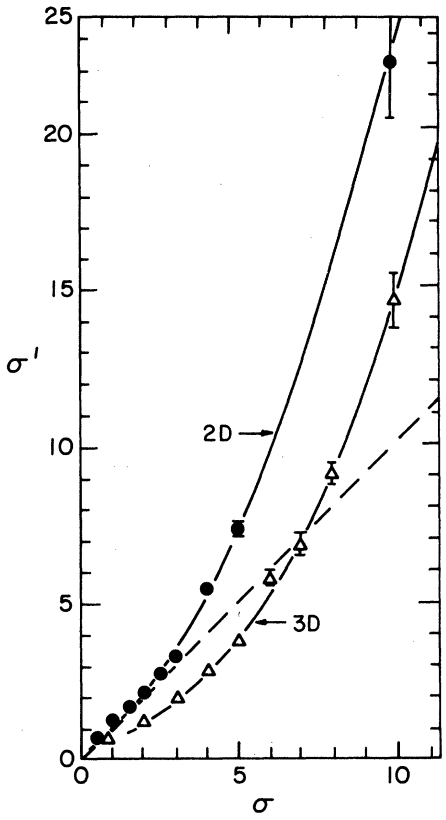

FIG. 7. σ' vs σ in two and three dimensions. Results based on hopping elements with random signs [Eqs. (56c) and (56d) with $p=0.5$]. No fixed point in two dimensions; in three dimensions $\sigma_c \approx 7.0$.

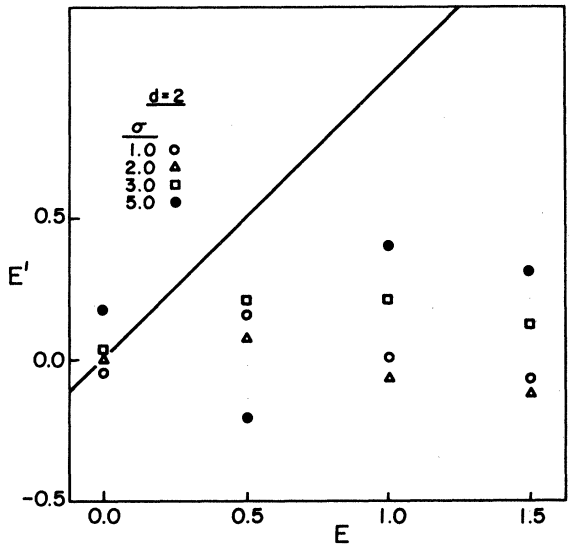

FIG. 8. E' vs E in two dimensions for various values of σ and $p=0$. The points at the origin can be taken as a statistical error. The straight line corresponds to $E'=E$.

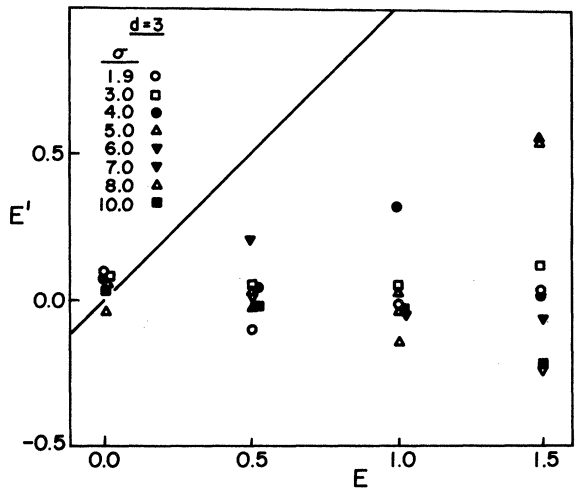

FIG. 9. E' vs E in three dimensions for various values of σ and $p=0.0$. The points at the origin can be taken as a measure of statistical error. The straight line corresponds to $E'=E$.

this slope s is equal to b^{λ}. Since $\nu=1/\lambda$, we have

$$\nu=\frac{\ln b}{\ln s} \quad . \tag{5.2}$$

Because of the statistical uncertainty a precise determination of ν was not possible; values of $1.25<\nu<1.75$ are consistent with the data.

VI. FINITE-LATTICE APPROXIMATION

In Sec. IV we presented a general framework for a renormalization-group treatment of a quantum-mechanical problem. The actual calculation summarized in Sec. V is based on a first-order approximation to the exact procedure. This is the quantum equivalent to the first-order cumulant approximation used in position space RG treatments of classical models. The quantum-mechanical equivalent has also been used by various authors, especially to study d-dimensional quantum-spin problems that are related to $d+1$-dimensional classical ones.

A different approximation method, which was used extensively and with remarkable success for classical spin systems, is the finite-lattice approximation.[22] In this approach the effective couplings between a group of neighboring cells are calculated exactly; however, only for a finite lattice that contains the group of cells being studied. The basic physical idea behind this approach is the belief that couplings between neighboring cells will be only slightly affected by the presence or absence of more distant cells.

To our knowledge this finite-lattice approximation has not been applied previously to quantum systems.

In the language of a perturbative approach to the exact equation for the effective-cell Hamiltonian, e.g., Eqs. (4.9) and (4.10), this approximation is equivalent to summation to infinite order of a selected subset of the operators that constitute the perturbation H_1. To be more specific, in order to calculate the effective hopping between two neighboring cells, $V_{RR'}$, the part of H_1 that connects these two cells is summed to infinite order. As an example, consider the four cells of Fig. 2. After each cell Hamiltonian is diagonalized, one state, $|R\rangle$, from each cell, is assigned to the model space D. All matrix elements $V_{RR'}=\langle R|V|R'\rangle$ in the space of these four states, are then calculated by solving the operator equation (4.9) *exactly*. In order to see how this is done, note that (4.9) can be written as an algebraic equation for the matrix elements

$$V_{\vec{R}\,\vec{R}'}=H_{1\,\vec{R}\,\vec{R}'}+\sum_{\vec{R}''\alpha''}\frac{H_{1\,\vec{R},\vec{R}''\alpha''}V_{\vec{R}''\alpha'',\vec{R}'}}{E-e_{\vec{R}''\alpha''}}\ , \tag{6.1}$$

where $|R\alpha\rangle$ is an eigenstate of cell $\vec{R}$ with eigenvalue $e_{\vec{R},\alpha}$, that was not assigned to D (i.e., $|\vec{R}\alpha\rangle\in\bar{D}$) and $H_{1\,R,R'\alpha'}=\langle\vec{R}|H_1|\vec{R}'\alpha'\rangle$. Thus, the evaluation of $V_{RR'}$, an element in D, requires the knowledge of all matrix elements of V between any state in D and any state in $\bar{D}$. The equation for such elements, as obtained from (4.9), is given by

$$V_{\vec{R}\alpha,\vec{R}'}=H_{1\,\vec{R}\alpha,\vec{R}'}+\sum_{\vec{R}''\alpha''}\frac{H_{1\,\vec{R}\alpha,\vec{R}''\alpha''}V_{\vec{R}''\alpha'',\vec{R}'}}{E-e_{\vec{R}''\alpha''}}\ . \tag{6.2}$$

Note that matrix elements $V_{\vec{R}\alpha,\vec{R}'}$ of the same kind (i.e. that connect $\bar{D}$ to D) appear on both sides of this equation, and therefore, $V_{\vec{R}\alpha,\vec{R}'}$ can be calculated by solving the set of algebraic equations

$$\sum_{\vec{R}''\alpha''}M_{\vec{R}\alpha,\vec{R}''\alpha''}V_{\vec{R}''\alpha'',\vec{R}'}=H_{1\,\vec{R}\alpha,\vec{R}'}\ , \tag{6.3}$$

where

$$M_{\vec{R}\alpha,\vec{R}''\alpha''}=\delta_{\vec{R}\,\vec{R}'',\alpha\alpha''}-\frac{H_{1\,\vec{R}\alpha,\vec{R}''\alpha''}}{E-e_{\vec{R}''\alpha''}}\ . \tag{6.4}$$

Once $V_{\vec{R}''\alpha'',\vec{R}}$ are known, substitution into (6.1) immediately yields $V_{R,R'}$. Thus *diagonal and off-diagonal* elements of the effective-cell Hamiltonian are obtained to infinite order in the unrenormalized hopping elements that connect the cells in our finite cluster.

This procedure was carried out in $d=2$, using finite clusters that contain two cells only. These two cells were chosen at random from a population of 400 cells, and the corresponding matrix elements $V_{\vec{R}\,\vec{R}'}$ were calculated. We created 300 such pairs to generate a distribution of $V_{\vec{R}\,\vec{R}'}$. Again, the off-diagonal elements were treated as in Sec. IV, to yield V_{eff}. The diagonal elements form a distribution whose second moment determines σ'. The resulting σ' vs σ curve for $E=0$ and $q=0$ is shown in Fig. 10. For the sake of comparison we have also included the first-order result in the same figure. It is clear that the two results are identical for $\sigma\gtrsim 3.0$, whereas for small σ, although there is a small quantitative discrepancy, the two results are in qualitative agreement with each other. In particular the infinite-order calculation does not produce a localization edge in two dimensions. We have also checked the energy scaling in the neighborhood of the band center and have found that the scaling is always toward $E=0$.

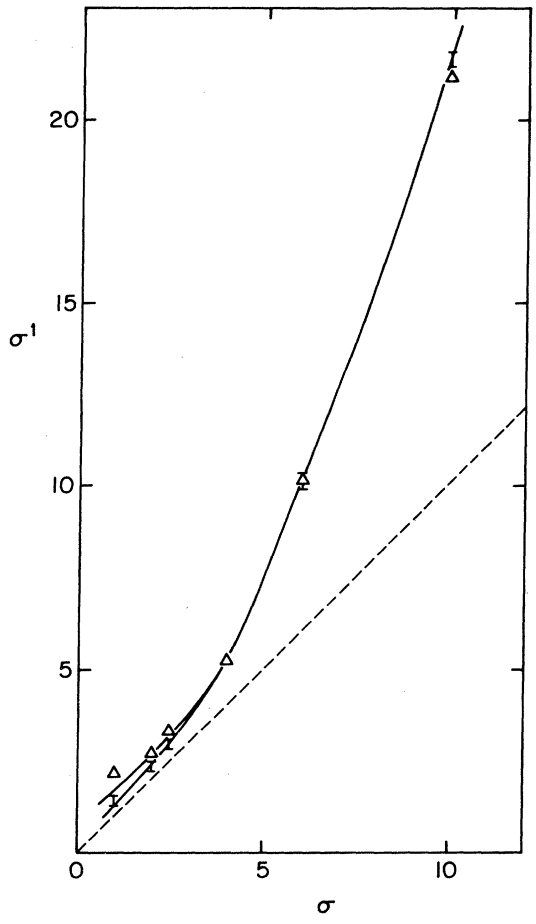

FIG. 10. σ' vs σ in two dimensions for $p=0.0$. Results based on the finite-lattice approximation (triangles). For comparison we include the results of first-order calculation (solid bars). No fixed point is found.

VII. SUMMARY AND CONCLUSIONS

In this paper we have developed a scaling theory of Anderson localization based on the ideas of position-space renormalization group. Following the methods used in the studies of critical phenomena, we have constructed an *exact* RG transformation which maps the Hamiltonian onto a new one. The *probability distribution* of the Hamiltonian (i.e., of the matrix elements) and the energy E of an electron eigenstate emerge as the natural scaling variables. Our method is closely related to the one proposed by Wegner[23] and differs sharply from that of Abrahams *et al.*[16]

The diagonal matrix elements were taken from a Gaussian distribution of width σ, whereas the off-diagonal elements were taken to be $+1$ or -1 with probability p and $1-p$, respectively. A sequence of approximate calculations based on an exact perturbation series were performed to obtain recursion relations for E, σ, and p. To lowest order we found a fixed point at $E=0$, $p=0.5$, and $\sigma_c \simeq 7.0$ in three dimensions. Along the line $E=0$, $p=0.5$ the localization length L diverges as $L \sim (\sigma-\sigma_c)^{-\nu}$ with $1.25 < \nu < 1.75$. The fixed point is stable with respect to perturbation in the E and p directions. By using the symmetry properties of the Hamiltonian and simple scaling arguments we were able to predict a parabolic phase boundary $E_c^2(\sigma) \propto (\sigma_c - \sigma)$. To the same order we have found no fixed point in two dimensions for finite σ implying that all states are localized.

It is difficult to make direct comparison with the results derived by other methods, since most workers use a rectangular distribution of width W for the diagonal matrix elements. If, however, we associate with σ a value of W that yields the second moment, i.e., $W_c \approx \sigma_c (12)^{1/2}$, our result could be interpreted as $W_c \approx 24$. In Table I we have listed values of W_c obtained by other workers. In particular, Licciardello and Economou[36] quote an upper bound of 24, attributed to the numerical work of Schönhammer and Brenig.[37]

As to the exponent ν, the values reported in the literature correspond to a definition $L \sim (E-E_c)^{-\nu'}$ with the values ν' varying over a wide range. The existence of a parabolic phase boundary implies that $L \sim (\sigma - \sigma_c + \beta E^2)^{-\nu}$ near the critical point, so that for E_c close to zero (band center) $\nu' = 2\nu$. Far away from the band center one obtains $\nu' = \nu$. This apparent discrepancy in the values of the critical exponent ν' is due to the use of the wrong scaling variable (i.e., E instead of E^2) and not because of any multicriticality as suggested by Stein and Krey.[21]

Mott's notion of minimum metallic conductivity[2] which predicts a sharp discontinuity of the dc conductivity at the mobility edge for zero temperature contradicts the RG theories[16,17,23] which assume the physical quantities scale with a relevant length, in this case the localization length. This scaling view is also supported by the percolation picture of Cohen[38] and Cohen and Jortner.[39] To reconcile this difference Mott[40] has recently argued that if $\nu' d > 2$ the conductivity has a discontinuity, whereas for $\nu' d > 2$ the conductivity vanishes continuously as the mobility edge is approached from the extended region as $\sim |E-E_c|^s$ where the exponent s is related to ν'. Because of the ambiguities in the definition of the exponent ν' it is difficult to test the Mott hypothesis. It would be more appropriate to replace ν' by ν in the Mott condition. Since the present work is based on a scaling theory, our value of ν does not agree with the existence of such a criterion.

The absence of a fixed point in two dimensions implies that there is no mobility edge for any finite σ. This is in agreement with the results of Abrahams *et al.*[16] The approach taken by these authors is very different from ours and is based on a numerical calculation of Licciardello and Thouless which anticipates the absence of a mobility edge. Since the first-order perturbation theory breaks down for low σ it was necessary to study the effects of higher-order terms. To this end we have used a finite-lattice approximation which allowed us to sum the perturbation series to all orders. Within this approximation no qualitive change was found in $d=2$; in particular, the inclusion of the higher-order terms does not delocalize the states and induce a transition for finite σ in two dimensions.

The experimental situation is somewhat clouded. Earlier experiments[41–44] indicated the existence of a mobility edge and of minimum metallic conductivity. Recent experiments on dirty metal films by Dolan and Osheroff[6] and on inversion layers by Bishop *et al.*[5] seem to be in support of the scaling and localization picture.

A very important question that remains unanswered concerns the role of electron-electron interactions particularly in view of the suggestion of Altshuler, Aronov, and Lee[45] that the observed behavior in recent experiments[5,6] can be explained within the framework of an interacting system without resorting to localization.

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