

## New method for scaling theory of localization. II. Multichannel theory of a "wire" and possible extension to higher dimensionality

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In this paper we use the techniques of the scattering formalism for localization to discuss a real wire of finite cross section. The paper derives the phenomenon of localization in such a wire and suggests a heuristic extension to higher dimensionality.

### I. INTRODUCTION

In a previous paper<sup>1</sup> I have discussed and evaluated the Landauer formula for quantum conductance of a one-dimensional chain of random scatterers, and in another<sup>2</sup> discussed how the Landauer formula in terms of a scattering formalism may be related to the Thouless formula in terms of boundary-condition sensitivity.

In the first paper we indicated that a corresponding general formula for the conductance of a real wire of finite cross section could be derived similarly, and that formula is repeated here. Az'bel has given similar results.<sup>3</sup> These results are actually virtually identical to formulas for the tunneling conductance of a barrier which are well known in the literature.<sup>4</sup>

In this paper I will try to show how to make a scaling theory whereby the conductance of a long wire may be derived in terms of the statistics of its component parts and the phenomenon of localization thereby demonstrated for this more realistic model. In a final section of the paper, I will heuristically extend the results to higher dimensionality, confirming the conjectures of previous work.

### II. CONDUCTANCE FORMULA IN THE MULTICHANNEL CASE

The preceding paper in this series gave an expression for the conductance of a multichannel system with elastic scattering only. (This is a valid model of a conducting wire at zero temperature.)

$$G = \frac{e^2}{2\pi\hbar} \sum_{\alpha} \frac{\sum_{\beta} |t_{\alpha\beta}|^2}{1 - \sum_{\beta} |t_{\alpha\beta}|^2}. \quad (1)$$

Here  $t$  and  $r$  are the transmission and reflection matrices on the energy shell in channel variables. As stated in Ref. 1, in the limit of a very large number of channels  $n$ ,  $t_{\alpha\beta} \sim 1/n$  and therefore

$$\sum_{\beta} (r_{\alpha\beta})^2 = R_{\alpha} = 1 - \sum_{\beta} |t_{\alpha\beta}|^2 \approx 1 - O\left(\frac{1}{n}\right) \approx 1.$$

In this case we can forget the denominators and write

$$G = \frac{e^2}{2\pi\hbar} \text{Tr}(tt^{\dagger}).$$

Let us define  $g$  as we did in the previous paper by

$$G = \frac{e^2}{2\pi\hbar} g, \quad g \approx \text{Tr}(tt^{\dagger}). \quad (2)$$

The definition of a channel, of course, varies depending on the physical model we envision, and is never very definitive but also never a serious problem. For a good metal, we define a channel as a  $\vec{k}_{\text{transverse}}$  which belongs to any  $\vec{k}$  near the Fermi surface. Those channels with no Fermi-level density in them will simply have no transmission  $t$ . This definition, and the question of what kind of medium is "outside" the scatterer, is almost physically irrelevant. We think for definiteness of a piece of impure metal sandwiched into a perfect crystalline wire, and recognize that any scattering due to mismatch will, of course, be a part of the contact resistance and legitimately included. In essence, the channels are defined as allowed states at the Fermi level in the highly conducting "contacts" through which we supply the current to our resistive wire. As for the concept of "reservoir" in statistical mechanics, the details are unimportant.

One is at first tempted to diagonalize  $r_{\alpha\beta}$  or  $t_{\alpha\beta}$  in channel variables  $\alpha$  and  $\beta$ , which can in fact be done in the case where  $H$  fields are absent and the scattering matrix is symmetric:  $\underline{S} = \tilde{\underline{S}}$ , so also  $\underline{r}$  is. We note that

$$\underline{S} = \begin{pmatrix} r & t \\ t & r' \end{pmatrix}.$$

$\underline{r}$  may be diagonalized by a pseudounitary trans-

formation  $\underline{r}_d = \underline{U}\underline{r}\underline{U}$ , with  $\underline{U}$  unitary, and then  $\underline{t}$  by a transformation on the ongoing channels. This temptation must be avoided. The correct channels to use are the natural ones for the perfect reservoir system attached to the resistive sample, i.e., channels where the incoming velocity  $v_x^\alpha = \delta E_\alpha / \delta p_x$  is a diagonal operator. This is because the cancellation of density of states and velocity factors which is responsible for the simple form of (1) does not work in any other representation. That is, Eq.(1) is to be thought of as multiplied by a factor

$$\langle (v_x)_\alpha \rangle_{av} \left\langle \left( \frac{dn}{dE} \right)_\alpha \right\rangle_{av},$$

and there is no reason at all why the averages of these factors over a wide variety of channels should be each others' inverse if they are not diagonal. That it is not appropriate to diagonalize can also be seen by a second way of deriving (1), which is by assuming a given flow of current  $J$  and calculating the momentum lost to the sample which must be proportional to  $V$ . Again, the calculation must be done in eigenstates of  $p_x$  to make sense.

The formula as given is clearly valid for a set of conducting wires of different resistance in parallel, which is not true of the one proposed by Az'bel.<sup>3,4</sup> But, in the only case either of us consider very seriously, where  $R_\alpha \approx 1$ , there is no real difference. Our later work can be used to show that the differences among the various formulas come in only in order  $n^{-3/2}$ .

We have shown in another paper<sup>2</sup> that for a truly one-dimensional chain our definition of conductance (which clearly can be extended in itself to any number of dimensions) can be related at least to that of Thouless, in terms of the effect of boundary conditions on energy levels, which itself was related by Thouless to the Kubo formula.<sup>5</sup> But our definition is more fundamental than that of Kubo, does not involve eigenstates of an artificial Hamiltonian or boundary conditions, and above all applies naturally to finite systems where Kubo's formula is ambiguous and Thouless' is not precise. Since finite systems such as tunnel junctions and small bits of wire are real and do exhibit finite resistance, it can be physically incorrect to use a formula depending on a large volume limit like Kubo's, or that of conventional transport theory for that matter. Among other properties, to specify densities and spacings of eigenstates one needs the scattering matrix as a function of energy  $S(E)$ , while our formula is on energy shell only.

### III. SCALING

As we did for a single channel, we will start from a microscopic scale at which we assume  $\underline{t}$

to be a member of some given probability distribution  $P(\underline{t})$ ; and to add "black boxes" in series (see Fig. 1) with the  $\alpha$  channels of box  $\underline{t}_1$  feeding into the  $\beta$  channels of  $\underline{t}_2$ . The probability distribution of the resultant conductance  $G$  is determined by the composition law

$$\underline{t} = \underline{t}_1 \frac{1}{1 - \underline{r}'_1 \underline{r}_2} \underline{t}_2. \quad (3)$$

The key assumption (very easily satisfied in the multichannel case) is complete phase randomness between matrix elements  $(t_1)_{\alpha\beta}$  and  $(t_2)_{\alpha\beta}$ , etc. This is surely correct for lengths past the mean free path  $l$  which is in this case always very much shorter than the localization length  $L_1$  (in general,  $L_1 = l n$  since  $|t|^2$  is of order  $1/e$  at  $l$ ).

We also assume a second kind of randomness: a random mixing of channels involving no correlation between the eigenchannels of  $\underline{t}_1$  and  $\underline{t}_2$ , so that any channel  $\alpha$  couples randomly to each channel  $\beta$ . This is the assumption of essential one-dimensionality. It states that there is no "transverse" localization, i.e., transverse dimensions are small compared to localization lengths.

Although we are free to redefine by unitary transformations four sets of channels (incoming and outgoing for scatterers 1 and 2), use of this freedom to diagonalize  $\underline{S}$ ,  $\underline{t}$ , or  $\underline{r}$  is not very physically meaningful. In any case the various factors of (3) cannot be simultaneously diagonalized as far as we can see; and, if they could be, the eigenvalues would have an obscure relationship to conductance because of the requirement of transforming to natural channels for the input and output problems.

It is meaningful, however, to discuss the eigenvalues of one factor of (3), namely

$$\underline{M} = \frac{1}{1 - \underline{r}'_1 \underline{r}_2}. \quad (4)$$

This matrix describes the multiple backscattering problem between scatterers 1 and 2, internally to the compound sample. Because in the limit of large  $n$ ,  $|\underline{r}_1|$  and  $|\underline{r}_2|$  are both of order  $1 - 1/n$ , this scattering is repeated  $\sim n$  times before the particle gets transmitted or reflected. Thus this is the sensitive, crucial part of the problem, and without treating it properly, it is not even possible to understand classical Ohm's law behavior. The other thing about  $\underline{M}$  is that, unlike  $\underline{t}$  or  $\underline{r}$ , it is

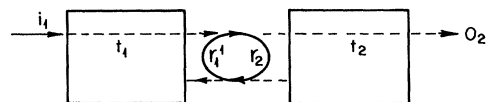


FIG. 1. Schematic representation of composition of two scatterers to increase scale by a factor 2.

a matrix both of whose indices refer to the same set of channels: ingoing from the left into scatterer 2. Thus it is appropriate to try to diagonalize it by a *unitary* transformation.

In order to understand the properties of  $M$ , we will have to study the large- $n$  limit. Unitarity of the  $S$  matrix gives us

$$1 = rr^\dagger + tt^\dagger$$

or

$$(rr^\dagger)_{\alpha\alpha} = 1 - (tt^\dagger)_{\alpha\alpha} = 1 - \sum_{\beta} |t_{\alpha\beta}|^2, \quad (5a)$$

$$(rr^\dagger)_{\alpha\beta} = - \sum_{\gamma} (t_{\alpha\gamma} t_{\beta\gamma}^*). \quad (5b)$$

Now in the large- $n$  limit, in order for the conductance to be finite on the scale where localization is relevant, we must have, by (2),

$$\sum_{\beta} |t_{\alpha\beta}|^2 \simeq g/n$$

or

$$t_{\alpha\beta} \sim 1/n.$$

Thus, since the phases in (5b) are surely random, (5b) is  $\sim 1/n^{3/2}$ , and so by (5a)  $rr^\dagger \simeq 1$ :  $r$  is unitary to lowest order in  $1/n$ . We may make  $r$  unitary to high accuracy by multiplying it by  $1/\sqrt{1-g/n}$  since the fluctuations in  $\sum_{\beta} |t_{\alpha\beta}|^2$  will also be of relative order  $1/\sqrt{n}$ .

In order to study  $\underline{M}$  consider the matrix

$$\underline{N} = r_1^\dagger r_2 \quad (6)$$

of which it is a function. Multiplying by the factors just mentioned,

$$\frac{N}{\left[ \left(1 - \frac{g_1}{n}\right) \left(1 - \frac{g_2}{n}\right) \right]^{1/2}}$$

is, as we have just shown, the product of unitary matrices and approximately unitary to order  $1/n^{3/2}$ . Thus we can diagonalize it to this order or better, and its eigenvalues are

$$\frac{N_\lambda}{\left[ \left[1 - (g_1/n)\right] \left[1 - (g_2/n)\right] \right]^{1/2}} \simeq \frac{N_\lambda}{\left[1 - (g_1 + g_2)/n\right]^{1/2}} = e^{i\theta_\lambda}. \quad (7a)$$

Thus the amplitudes of the  $N_\lambda$  are fixed and close to unity; the phases, however, are arbitrary, since the phases of  $r_2$  and  $r_1$  are completely uncorrelated. Thus, the eigenvalues  $N_\lambda$  are randomly arrayed with density  $n/2\pi$  in angle around a circle of radius  $(\langle N_\lambda^2 \rangle_{\text{av}})^{1/2} \simeq 1 - (g_1 + g_2)/2n$ . (It is not necessarily true that the eigenvalue *spacing* is random, but this does not appear to cause much effect: we return to this in the concluding sec-

tion.) In fact, we may achieve a slightly greater accuracy by defining  $g_1$  and  $g_2$  appropriate to each eigenvalue  $\lambda$  by

$$\frac{g_\lambda^{(1),(2)}}{n} = \sum_{\beta} |t_{\lambda\beta}^{(1),(2)}|^2$$

and then

$$N_\lambda \simeq e^{i\theta_\lambda} \left(1 - \frac{g_\lambda^{(1)} + g_\lambda^{(2)}}{n}\right)^{1/2}. \quad (7b)$$

Now that we understand the eigenvalues of  $N$  and thus of  $M$ , we may compute  $g$  from Eqs. (2) and (3):

$$g = \text{Tr} tt^\dagger,$$

$$g = \sum_{\alpha\beta\lambda\lambda'} t_{\alpha\lambda}^{(1)} M_\lambda t_{\lambda\beta}^{(2)} t_{\beta\lambda'}^{(2)*} M_\lambda^* t_{\lambda'\alpha}^{(1)}.$$

Now, again, the phases of  $M_\lambda$  and  $M_\lambda^*$  and of the various  $t$ 's are totally incoherent, so that to an order more accurate than  $n^{-1/2}$ , we can assume

$$g = \sum_{\alpha\lambda\beta} |t_{\alpha\lambda}^{(1)}|^2 |t_{\lambda\beta}^{(2)}|^2 |M_\lambda|^2, \quad (8)$$

and then this is by (7) equal to

$$\begin{aligned} g &= \frac{1}{n^2} \sum_{\lambda} g_\lambda^{(1)} g_\lambda^{(2)} \left| \frac{1}{1 - N_\lambda} \right|^2 \\ &= \frac{1}{n^2} \sum_{\lambda} g_\lambda^{(1)} g_\lambda^{(2)} \frac{1}{1 + |N_\lambda|^2 - 2|N_\lambda| \cos \theta_\lambda} \\ &= \frac{1}{n^2} \sum_{\lambda} \frac{g_\lambda^{(1)} g_\lambda^{(2)}}{(1 - |N_\lambda|)^2 + 2|N_\lambda|(1 - \cos \theta_\lambda)}, \end{aligned}$$

and we will recognize that aside from small fluctuations, this is by (7) equal to

$$g \simeq \sum_{\lambda} \frac{g_\lambda^{(1)} g_\lambda^{(2)}}{\left[ (g_\lambda^{(1)} + g_\lambda^{(2)})/2 \right]^2 + 2n^2(1 - \cos \theta_\lambda)}. \quad (9)$$

We recognize that this is a highly convergent sum, to which only values of  $\theta_\lambda \sim 1/n$  contribute appreciably, and for  $n$  large it is precise to replace the cosine by its expansion:

$$g = \sum_{\lambda} \frac{g_\lambda^{(1)} g_\lambda^{(2)}}{\left[ (g_\lambda^{(1)} + g_\lambda^{(2)})/2 \right]^2 + n^2 \theta_\lambda^2}. \quad (10)$$

The phase angles  $\theta_\lambda$  of the  $n$  eigenvalues are distributed evenly from  $-\pi$  to  $\pi$ . It is instructive to take a simple phase average of  $g$ , which gives us the ensemble average:

$$\begin{aligned} \langle g_{\text{av}} \rangle &= \left\langle \frac{n}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{g_1 g_2}{\left[ (g_1 + g_2)/2 \right]^2 + n^2 \theta^2} \right\rangle_{\text{av}} \\ &= \left\langle \frac{g_1 g_2}{g_1 + g_2} \right\rangle_{\text{av}} = \left\langle \frac{1}{\rho_1 + \rho_2} \right\rangle_{\text{av}}. \end{aligned}$$

This is then the same result as one gets for the average *transmission* in the pure one-dimensional,

one-channel case (see Paper I): this adds classically in that

$$\frac{1}{\langle T \rangle_{av}} = \frac{1}{\langle g \rangle_{av}} = [ \langle (\rho_1 + \rho_2)^{-1} \rangle_{av} ]^{-1}, \tag{11}$$

which gives us Ohm's law in the limit in which the distribution in  $g$  is sufficiently narrow to obey the law of large numbers. This will clearly be the case when the  $\theta_\lambda$  are closely spaced relative to the distance ( $| \langle N_\lambda \rangle_{av} | - 1$ ) between the circle of eigenvalues and 1, i.e., when  $g_1, g_2 \gg 1$ ; the large conductance limit. In this case many eigenvalues contribute but in the opposite case only the small-

lest few are relevant.

Again, as in the pure 1D case, all of the important action is in the statistics: it is essential to calculate probability distributions rather than means. Let us then calculate from (10) the probability distribution  $P(g)dg$ :

$$P(g) = \int P(g_\lambda^{(1)} g_\lambda^{(2)} \theta_\lambda) dg_1 dg_2 d\theta_\lambda \delta \times \left( g - \sum_\lambda \frac{g_1 g_2}{[(g_1 + g_2)/2]^2 + n^2 \theta_\lambda^2} \right).$$

We evaluate this by the well known technique of Markov and Kolmogorov<sup>6</sup>

$$P(g) = \frac{1}{2\pi} \int d\mu e^{-i\mu g} e^{-\psi(\mu)}, \tag{12a}$$

$$\psi(\mu) = n \int P(g_1, g_2, \theta) \left[ 1 - \exp \left( i \frac{\mu g_1 g_2}{[(g_1 + g_2)/2]^2 + n^2 \theta^2} \right) \right] d\theta dg_1 dg_2 \\ = \frac{1}{2\pi} \int P(g_1) P(g_2) dg_1 dg_2 \int_{-\infty}^{\infty} dx \left[ 1 - \exp \left( \frac{i\mu g_1 g_2}{[(g_1 + g_2)/2]^2 + x^2} \right) \right]. \tag{12b}$$

It does not seem to be easy to get  $P(g)$  explicitly. However, it seems adequate to study it in each of two limits:

$$g_1 + g_2 \gg 1 \text{ and } \ll 1.$$

First, consider  $g_1, g_2 \gg 1$ . In all cases, it is clear from (12b) that exceptionally large or small values of  $g_1$  and  $g_2$  are not very important (if they were, we should have had to deal more carefully with  $|N_\lambda|$ , for one thing). This is because the combination

$$\frac{g_1 g_2}{[(g_1 + g_2)/2]^2 + x^2}$$

is small if  $g_1$  or  $g_2 \rightarrow \infty$  or 0, so that exceptional values don't contribute strongly. It will be convenient to treat as secondary the fluctuations of  $g_1$  and  $g_2$  to begin with, therefore, and concentrate on those caused by the phase. Let us take  $g_1$  and  $g_2$  fixed and calculate the conditional probability

$$P(g | g_1, g_2) = \frac{1}{2\pi} \int d\mu \exp[-i\mu g - \psi(\mu | g_1, g_2)]$$

where

$$\psi(\mu | g_1, g_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left[ 1 - \exp \left( i \frac{\mu g_1 g_2}{\frac{1}{4}(g_1 + g_2)^2 + x^2} \right) \right] \\ \cong i\mu \frac{g_1 g_2}{g_1 + g_2} + \mu^2 \frac{g_1^2 g_2^2}{(g_1 + g_2)^3} + \dots \tag{13}$$

It is a well known theorem (Ref. 7) of this method that the power-series coefficients in  $\mu$  give the first, second, etc., moments of the prob-

ability distribution. Thus the first two moments are always given by [as we already saw in (11) for the first one]

$$\langle g \rangle_{av} = \frac{g_1 g_2}{g_1 + g_2}, \tag{14}$$

$$\langle \Delta g^2 \rangle_{av} = \frac{2g_1^2 g_2^2}{(g_1 + g_2)^3}. \tag{15}$$

This is irrespective of the size of  $g_1, g_2$ , although we will see that (14) and (15) become nearly irrelevant for small  $g$ . The behavior of  $P(g)$  depends crucially on the size of (15) since this is the first real term of  $\psi(\mu)$ . If  $g_1 g_2 \gg 1$ ,  $\langle \Delta g^2 \rangle_{av}$  [in (15)] is  $\gg 1$  and  $e^{-\psi(\mu)}$  falls off to small values for  $\mu \sim 1$ . Then the power series (13) is a good representation of  $\psi$ , the next terms in the power series become successively very much smaller in the relevant region, and a Gaussian representation of  $P(g)$  is accurate:

$$P(g | g_1, g_2) \cong \exp[-(g - \langle g \rangle_{av})^2 / 2 \langle \Delta g^2 \rangle_{av}]. \tag{16}$$

It is perhaps worthwhile to make this more explicit with a simplification: set  $g_1 = g_2 = g_0$ . (The same manipulations, slightly more complicated, work for  $g_1 \neq g_2$ .) Then

$$\psi(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left[ 1 - \exp \left( \frac{i\mu g_0^2}{g_0^2 + x^2} \right) \right] \\ = \frac{g_0}{2\pi} \int_{-\infty}^{\infty} dy \left[ 1 - \exp \left( \frac{i\mu}{1 + y^2} \right) \right] \\ = \frac{g_0}{2\pi} \phi(\mu).$$

Here  $\phi(\mu)$  is a universal function which has a good convergent power series well represented by its first terms {for  $\mu \gg 1$ }, but is very badly expressed by this power series for  $\mu \geq 1$ . (We will see that in this limit, it is  $\propto \mu^{1/2}$ .) Thus even though  $(\langle \Delta g^2 \rangle_{\text{av}})^{1/2}$  is not extremely small compared to  $\langle g \rangle_{\text{av}}$  in this region, the power-series argument tells us that the Gaussian is a good approximation throughout this region, since the expansion is in  $\mu \sim 1/g$ .

We now wish to transform (16) into a distribution in an appropriate variable  $f(g)$  which has the all-important property of *additive mean*, as explained in Paper I:

$$\langle f(g) \rangle_{\text{av}} = \langle f(g_1) \rangle_{\text{av}} + \langle f(g_2) \rangle_{\text{av}}.$$

To a zeroth approximation, this is the case for the resistance  $\rho = 1/g$  in the limit  $g \rightarrow \infty$ . That is, when

$$\frac{\langle \Delta g^2 \rangle_{\text{av}}}{(\langle g \rangle_{\text{av}})^2} = \frac{2}{g_1 + g_2} \ll 1,$$

we may write

$$\begin{aligned} \langle f \rangle_{\text{av}} &= \langle \rho \rangle_{\text{av}} - \alpha \langle \rho^2 \rangle_{\text{av}} = \langle (\rho_1 - \alpha \rho_1^2) \rangle_{\text{av}} + \langle (\rho_2 - \alpha \rho_2^2) \rangle_{\text{av}} \\ &= \rho_1 + \rho_2 + 2\rho_1\rho_2 - \alpha \left( \frac{1}{\langle g \rangle_{\text{av}}^2} + \frac{3\langle \Delta g^2 \rangle_{\text{av}}}{(\langle g \rangle_{\text{av}})^4} \right) + \dots \\ &= \rho_1 + \rho_2 + 2\rho_1\rho_2 - \alpha [(\rho_1 + \rho_2)^2 + 3\rho_1\rho_2(\rho_1 + \rho_2)] \\ &= \rho_1 - \alpha \rho_1^2 + \rho_2 - \alpha \rho_2^2 + (2 - 2\alpha)\rho_1\rho_2 + (\text{higher order}) + \dots \end{aligned}$$

If  $\alpha = 1$ , (19) is additive to lowest order. This result holds throughout the region of the Gaussian distribution.

Borrowing from the one-channel case, we guess that the form of the additive variable  $f(g)$  should be a version of that appropriate to that case, possibly involving a new conductance scale:

$$\begin{aligned} f(g) &= \rho_0 \ln \left( 1 + \frac{\rho}{\rho_0} \right) \\ &= \frac{1}{2} \ln(1 + 2\rho) = \frac{1}{2} \ln \left( 1 + \frac{2}{g} \right). \end{aligned} \quad (20)$$

(20) has the power-series expansion (19). We now confirm that a similar form (20) holds in the opposite limit  $\rho \gg 1$ ,  $g \ll 1$ .

In this opposite limit, we may neglect  $(g_1 + g_2)^2$  relative to  $x^2$  in the exponent and

$$\begin{aligned} \langle \rho \rangle_{\text{av}} &= \left\langle \frac{1}{g} \right\rangle = \frac{1}{g} \left\langle \left( 1 - \frac{\Delta g}{\langle g \rangle_{\text{av}}} + \frac{\Delta g^2}{(\langle g \rangle_{\text{av}})^2} \dots \right) \right\rangle \\ &\simeq \frac{1}{\langle g \rangle_{\text{av}}} + \frac{\langle \Delta g^2 \rangle_{\text{av}}}{(\langle g \rangle_{\text{av}})^3} + \dots \\ \langle \rho \rangle_{\text{av}} &= \frac{1}{g_1} + \frac{1}{g_2} + \frac{2}{g_1 g_2} + \dots \end{aligned} \quad (17)$$

This is the result if  $g_1$  and  $g_2$  are fixed: averaging (17) gives

$$\langle \rho \rangle_{\text{av}} = \langle \rho_1 \rangle_{\text{av}} + \langle \rho_2 \rangle_{\text{av}} + 2\langle \rho_1 \rangle_{\text{av}} \langle \rho_2 \rangle_{\text{av}} + \dots, \quad (18)$$

which is additive to lowest order but exhibits localization in higher, and in fact the localizing term is identical to that in the one-channel case.

In the one-channel case the appropriate linearizing variable was

$$f(g) = \ln(1 + \rho) = \rho - \frac{\rho^2}{2} + \frac{\rho^3}{3} - \dots$$

In the present case there seems no easy way to determine the full functional form of  $f(g)$  but we can approach it from its limiting values: first as a power series in  $\rho$ . That is, we write

$$f(g) \cong \rho - \alpha \rho^2 + \dots \quad (19)$$

We determine  $\alpha$  by demanding additive means:

$$\begin{aligned} \psi(\mu) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left[ 1 - \exp \left( \frac{i\mu g_1 g_2}{[(g_1 + g_2)/2]^2 + x^2} \right) \right] \\ &\simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left[ 1 - \exp \left( i \frac{\mu g_1 g_2}{x^2} \right) \right] \\ &= \left( \frac{\mu g_1 g_2}{i\pi} \right)^{1/2}. \end{aligned} \quad (21)$$

The corresponding probability distribution

$$P(g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mu \exp \left[ -i\mu g - \left( \frac{\mu g_1 g_2}{i\pi} \right)^{1/2} \right]$$

is calculated by an exact saddle-point integration about the saddle point

$$\mu_0 = \frac{g_1 g_2}{4i g^2 \pi},$$

and one obtains

$$P(g) \cong \frac{e^{-\epsilon_1 \epsilon_2 / 4g\pi} \left( \frac{g_1 g_2}{g^3} \right)^{1/2}}{2\pi} \quad (22)$$

Note that (22) has no finite mean or variance, since at large  $g$  it falls off only as  $g^{-3/2}$ . A more exact calculation would confirm the values (14) and (15), since when the correction

$$x^2 \rightarrow \frac{(g_1 + g_2)^2}{4} + x^2$$

is made, a cutoff for large  $g$  results which must converge the means. In fact, the cutoff occurs approximately at

$$g = \text{const} \frac{(g_1 g_2)}{(g_1 + g_2)^2},$$

which is consistent with the two means we have quoted in Eqs. (14) and (15)

When  $\rho \gg 1$ , the logarithmic form (19) reduces to  $\rho_0 \ln \rho / \rho_0 = \rho_0 \ln(1/\rho_0 g)$ . We are therefore interested in  $\langle \ln \rho \rangle_{\text{av}}$  which may be calculated from (22) as follows:

$$\langle \ln g \rangle_{\text{av}} = \frac{1}{2\pi} \int_0^\infty e^{-\epsilon_1 \epsilon_2 / 4g\pi} \ln g dg \left( \frac{g_1 g_2}{g^3} \right)^{1/2}$$

Let  $1/g = x^2$ :

$$\langle \ln g \rangle_{\text{av}} = \langle -2 \ln x \rangle_{\text{av}} = -\frac{\sqrt{g_1 g_2}}{2\pi} \int_0^\infty e^{-\epsilon_1 \epsilon_2 / 4\pi x^2} (\ln x) dx,$$

which is a known integral:

$$\frac{2}{\pi} \int_0^\infty e^{-x^2/4\pi} \ln x dx = \ln \pi - \gamma \cong 0.568$$

so that

$$\langle \ln g \rangle_{\text{av}} = \ln g_1 + \ln g_2 - (\ln \pi - \gamma), \quad (23)$$

where  $\gamma$  is Euler's constant, 0.57722. This means that the combination which is additive in this limit is (choosing the sign to obtain a positive quantity, and noting that multiplicative factors are irrelevant)

$$\begin{aligned} f(g) &= -\ln \left( \frac{e^{-\gamma}}{g} \right) \\ &= \ln \left( \frac{\rho}{\pi e^{-\gamma}} \right) = \ln \frac{\rho}{1.764}, \end{aligned} \quad (24)$$

i.e.,

$$\left\langle \ln \left( \frac{\rho}{1.764} \right) \right\rangle_{\text{av}} = \ln \left( \frac{\rho_1}{1.764} \right) + \ln \left( \frac{\rho_2}{1.764} \right).$$

This limiting result fits on to the result in the extended limit only if we allow the resistivity scale to vary slowly between the two limits. We may write

$$f(g) = \rho_s \ln \left( 1 + \frac{\rho}{\rho_s} \right), \quad (25)$$

where the scale resistivity  $\rho_s$  is a slowly varying function of  $\rho$  with the two limits

$$\rho_s = 2, \quad \rho \rightarrow 0$$

$$\rho_s = \pi e^{-\gamma} = 1.764, \quad \rho \rightarrow \infty.$$

The scaling, then, is not exactly calculable except in these limits, but it is probably quite accurate to treat  $\rho_s$  as though it were constant and then we would have, starting at a length  $L_0$  with starting resistivity  $\rho_0$ , the Landauer form

$$\begin{aligned} \frac{\rho}{\rho_s} &= \left\{ 1 + \frac{\rho_0}{\rho_s} \left[ \exp \left( \frac{\rho_c - \rho_0}{\rho_s} \right) - 1 \right] \right\} \\ &\cong e^{\rho_c / \rho_s} - 1, \end{aligned}$$

where  $\rho_c$  is, as in the previous paper, the classically calculated additive resistivity, and the second version holds if we start at weak scattering with  $\rho_0$  very small.

It is now essential to carry out the second stage of the prescription of the earlier paper<sup>1</sup>: to make sure that the variable with additive mean has a variance which scales at least according to a weak law of large numbers, so that in the limit of large scale lengths, the quantity  $f(g)$  will have a Gaussian distribution.

This may be checked in the two limits. At very small  $\rho$ , it is unnecessary to include the small correction term in the power series in  $\rho$ , and we have

$$\begin{aligned} \langle \Delta^2 [f(g)] \rangle_{\text{av}} &= \langle \Delta \rho^2 \rangle_{\text{av}} = \langle \rho \rangle_{\text{av}}^4 \langle \Delta g^2 \rangle_{\text{av}} \\ &= 2\rho_1 \rho_2 (\rho_1 + \rho_2), \end{aligned} \quad (26)$$

which goes adequately to zero as  $\rho \rightarrow 0$ . It is fairly clear, since the distribution is most singular as  $\rho \rightarrow \infty$ , that  $\langle \Delta f^2 \rangle_{\text{av}}$  rises monotonically from this to its value in the limit  $\rho \rightarrow \infty$ . The integral for  $\ln^2 g$  can be done only numerically as far as was ascertained: the relevant integral is

$$\frac{1}{\pi} \int_0^\infty e^{-x^2/4\pi} (\ln x)^2 dx \cong 1.33.$$

From this we obtain that

$$\lim_{g \rightarrow 0} \langle [\Delta f(g)]^2 \rangle_{\text{av}} = \langle (f - f^2) \rangle_{\text{av}} \cong 5.01. \quad (27)$$

It is interesting that this is about the same ratio to the one-channel result as the scale factor on the resistance.

To repeat the argument of the previous paper,<sup>1</sup> we may majorize the variance in the distribution  $P(f)$  after scaling by noting that the added variance at each scaling step is less than (27). Thus the total variance satisfies

$$\Delta^2(L) < \Delta_m^2(L)$$

with

$$\Delta_m^2(L) = \Delta_m^2(L_1) + \Delta_m^2(L_2) + \Delta_0^2,$$

which in turn is satisfied by

$$\Delta_m^2(L) = -\Delta_0^2 + \text{const} \times L. \quad (28)$$

We plotted this result, as well as a plausible actual course for  $\Delta^2(L)$ , in Fig. 3 of that paper which is reproduced here as Fig. 2. This also compares our later results [see Eq. (29)] with the one-channel case. We see that the rms breadth asymptotically approaches  $\sqrt{L}$  in this variable, which is suitably small.

As, again, for the simple chain, it is possible to study  $\Delta^2(L)$  a little more precisely. In the first place, it is possible to solve the scaling equation implied by the "extended" limit (26). We must add the variances of the starting distributions  $P(f_1)$  and  $P(f_2)$  to that caused by the composition process, namely (26):

$$\Delta^2(L) = \Delta^2(L_1) + \Delta^2(L_2) + 2\rho_1\rho_2(\rho_1 + \rho_2).$$

This is satisfied by

$$\Delta^2(L) = \frac{2}{3}\rho^3(L), \quad (29)$$

because  $\rho$  is to first approximation linear in  $L$  and

$$L^3 = L_1^3 + L_2^3 + 3L_1L_2(L_1 + L_2).$$

Thus the variance in the extended limit rises much more slowly, initially, than for a single chain, where it rises as  $L^2$ .

The ultimate slope of the linear portions of  $\Delta^2(L)$  for large  $\rho$  may, again, be estimated by using the exact mean value of  $g$ , (13). If  $\langle g \rangle_{\text{av}}$  is not to vary exponentially in this limit, we must have

$$\int df e^{-(f-\langle f \rangle_{\text{av}})^2 / 2\Delta^2} e^{-f/\rho_s} \rightarrow \text{const},$$

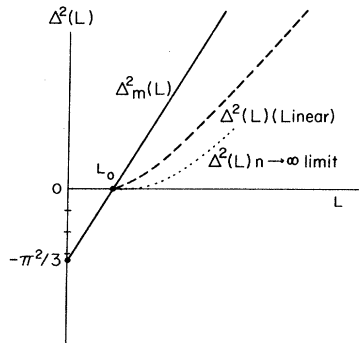


FIG. 2. Schematic representation of the variance of  $f(L)$ . The solid line gives the rigorous upper limit, the dashed line gives the estimate for the linear chain from the previous paper, and the dotted line gives our best estimate from the present work.

which is to say that

$$\exp\left(-\frac{\langle f \rangle_{\text{av}}}{\rho_s} + \frac{1}{2}\Delta^2/\rho_s^2\right) \rightarrow \text{const}$$

or

$$\begin{aligned} \Delta^2 &\rightarrow 2\langle f \rangle_{\text{av}}\rho_s = 2\rho_c\rho_s = 3.33\rho_c \\ &= 3.33\rho_0L. \end{aligned}$$

#### IV. DISCUSSION AND IMPLICATIONS FOR HIGHER DIMENSIONALITIES

Unlike the results for truly one-dimensional chains of one or a few channels discussed in the previous paper, these results are not rigorously derivable nor entirely precise quantitatively. The deep assumption is that the different channels or eigenstates  $\lambda$  of the back reflection matrix  $N$  can be discussed independently. There is the problem already mentioned, that in fact they may not be quite independent in that transmission between two of them can interfere after taking into account the  $t$  matrices of the separate boxes 1 and 2, but I am sure this is not serious because of the smallness and random phasing of these terms.

All of the effects of such interactions of the different channels would surely be to enhance localization. In the first place, one might expect that insofar as the  $N_\lambda$ 's are eigenvalues of a random matrix problem, they may to some extent repel like the eigenvalues of the energy in a random potential. I do not, however, believe that this is physically the case, although I have not produced a rigorous argument. (Unpublished calculations by Lee show, however, that individual eigenvalues of  $r$  can be very different in magnitude, but we see no reason for such a correlation in  $N$ .) In any case this simply enhances their relative spacing, which plays no special role in the problem. In the second place, transverse localization will act to anomalously decrease the  $g$ 's which determine the magnitudes of the  $N_\lambda$  and space the eigenvalues further apart, because they will lead to a correlation between the  $N_\lambda$ 's and the  $g_1$  and  $g_2$  matrix elements which couple to them. That is, once the sample width is wider than the localization length, it should be necessary to treat it roughly as  $W/L_{\text{loc}}$  separate one-dimensional systems (where  $W$  is the width). Interpolation between this concept and the considerations to be given shortly should give quite a satisfactory heuristic theory of the  $d$ -dimensional system.

In the opposite case, that of a  $d$ -dimensional sample not too wide relative to the localization length, we may derive the scaling for  $d$  dimensions directly from that for one dimension by the following transformation: Let us consider a sample with one-dimensional topology, but which does

not have a constant cross section. In our terms this means that it does not have equal numbers of ingoing channels from the left and outgoing channels to the right (see Fig. 3). In this case the matrices  $r'_1$  and  $r_2$  are no longer equal in size: one is  $N \times N$  and the other  $M \times M$ , while  $t$  is  $N \times M$ . Unitarity conditions still require that

$$r r^\dagger + t t^\dagger = r'^\dagger r' + t t^\dagger = 1,$$

so that the conductance is still symmetric and for practical purposes given by

$$\sum_{\alpha=1}^N \sum_{\beta=1}^M |t_{\alpha\beta}|^2.$$

Two such unsymmetrical objects may still be compounded by the law

$$t = t_1 \frac{1}{1 - r'_1 r_2} t_2$$

and the properties of the matrix  $N$  are the same. Thus, as far as we can see, the manipulations of the previous section are perfectly valid so long as transverse localization does not interfere.

We may make such a nonuniform sample in the shape of a sector of an annulus of opening angle  $\theta$  (see figure). In scaling up such a sample, we note that we get equal increments of resistance

not by adding equal lengths but by adding pieces with equal ratios of radii

$$\ln \frac{R_3}{R_2} = \ln \frac{R_2}{R_1}.$$

In this case the fundamental scaling law (18)

$$\langle \rho \rangle_{av} = \langle \rho_1 \rangle_{av} + \langle \rho_2 \rangle_{av} + 2 \langle \rho_1 \rangle_{av} \langle \rho_2 \rangle_{av}$$

becomes

$$\begin{aligned} \langle \rho \rangle_{av} (R_3, R_1) &= \langle \rho_1 \rangle_{av} (R_2, R_1) \\ &+ \frac{1}{\sigma_{cl} \theta} \ln \frac{R_3}{R_2} (1 + 2 \langle \rho_1 \rangle_{av}). \end{aligned}$$

Here we have set  $R_3 - R_2 \ll R_2 - R_1$  and so small, in fact, as to make the resistance of the sector  $R_3 - R_2$  perfectly classical. This may be converted into a differential equation:

$$\frac{d\rho}{1 + 2\rho} = \frac{1}{\sigma_{cl} \theta} d \ln R$$

$$\frac{1}{2} \ln(1 + 2\rho) \Big|_{R_1}^R = \left( \ln \frac{R_1}{R} \right) \frac{1}{\sigma_{cl} \theta}.$$

Now there is nothing in this argument which prevents us from allowing  $\theta$  to approach  $2\pi$ , at which point the sample becomes strictly two dimensional. Thus this expression in the limit  $\theta = 2\pi$  should give us  $g(L)$ , the effective two dimensional conductance as a function of scale length  $L$ . This may be simply done by comparing

$$\frac{\rho(L)}{\frac{1}{2\pi\sigma_{cl}} \ln \frac{L}{R_1}} = \frac{g_0}{g(L)}$$

which leads to

$$g(L) = g_0 - \frac{1}{2\pi} \ln L/L_0 + \dots$$

This happens to be *exactly* the result obtained by perturbation theory in Ref. 8. We do *not* believe this to be a coincidence but think the reasoning correct in the large- $g$  limit. Incidentally, the result [Eq. (29)] shows that fluctuations are much less severe in the higher dimensionalities than in the 1D chain.

The same trick may be extended to higher dimensionality, where indeed the numerical coefficient of the first term agrees perfectly with perturbation theory in three dimensions as well. Thus this method seems to obtain a valid first-order correction in powers of  $1/g$ . Combined with the idea of transverse localization with Az'bel's methods, one could undoubtedly interpolate a quite satisfactory estimate of the  $\beta(g)$  curve of Ref. 8

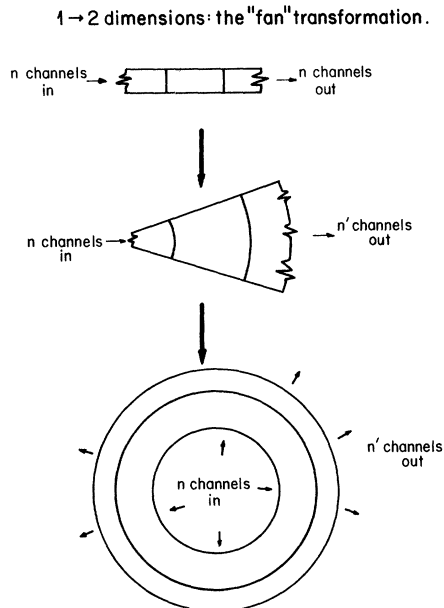


FIG. 3. Sketch of the fan transformation which carries the one-dimensional, many-channel case into 2 or higher dimensions. Instead of symmetric scatterers, we use random scatterers with  $n$  channels on the left and  $n' > n$  on the right. Doing this repetitively gives us a sample which is topologically 2 or  $n$  dimensional.



at physical dimensionalities of 2 or 3.

The next and more difficult problem is to apply this method to the essentially two- or three-dimensional questions of Hall effect and magnetoresistance, and possibly to generalize to interacting systems. We feel that as it is it places the quantum transport theory on a much firmer foundation than heretofore.

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<sup>1</sup>P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, *Phys. Rev. B* 22, 3519 (1980).

<sup>2</sup>P. W. Anderson and P. A. Lee *Prog. Theor. Phys.* (in press).

<sup>3</sup>M. Az'bel, unpublished.

<sup>4</sup>M. Az'bel, unpublished.

<sup>5</sup>D. J. Thouless, in *III-Condensed Matter*, edited by R. Balian, R. Maynard, and G. Toulouse (North-Hol-

land, Amsterdam, 1979), p. 1.

<sup>6</sup>See, for example, S. Chandrasekhar, *Rev. Mod. Phys.* 15, 1 (1943); the method was used in P. W. Anderson, *Phys. Rev.* 109, 1492 (1958).

<sup>7</sup>See, for an early use, P. W. Anderson, *J. Phys. Soc. Jpn.* 9, 316 (1954).

<sup>8</sup>E. Abrahams *et al.*, *Phys. Rev. Lett.* 42, 673 (1979).