Quantized Hall resistance and the measurement of the fine-structure constant

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An elementary, exact calculation of two-dimensional electrons in crossed electric and magnetic fields with a δ-function impurity is carried out in the quantum limit. A state localized on the impurity exists and carries no current. However, the remaining mobile electrons passing near the impurity carry an extra dissipationless Hall current exactly compensating the loss of current by the localized electron. The Hall resistance should thus be precisely h/e^2, as found experimentally by Klitzing *et al.* Other possible sources of deviation from this result are briefly examined.

In a recent Letter, v. Klitzing *et al.*[1] have reported a high-accuracy measurement of e^2/h (to one part in 10^5) which after improvements and together with the known value of the speed of light, c, potentially[2] would provide a measurement of the fine-structure constant of precision greater than that currently available (one part in 10^7). Their result is based on a measurement of the quantized Hall resistance in a two-dimensional electron gas, as realized in the inversion layer of a metal-oxide-semiconductor field-effect transistor. We here provide an elementary calculation which has a bearing on their result, and is a step in the direction of estimating theoretically the accuracy to which e^2/h is determined by the experiment.

Since free electrons which fill an integral number of Landau levels give a Hall resistance precisely an integral fraction of h/e^2, the problem is one of treating the imperfections, which might give rise to ordinary resistance, and/or to localized states which could cause the Hall resistance to deviate from its ideal value. (We do not treat the electrons as interacting, an omission which future work must remedy.) We here work out the instructive, elementary, and essentially exactly solvable case of two-dimensional electrons with a single δ-function impurity.

The main result is that (a) *a localized state exists,* which (b) carries *no current,* but (c) *the remaining nonlocalized states* carry an *extra Hall current* which *exactly compensates for that not carried* by the localized state. Thus, provided *all the nonlocalized states* of the appropriate Landau level are filled, the *total Hall current carried by the level is precisely the same as in the absence of impurities and localized states.*

We consider then free electrons of mass m ($=0.2m_e$ as appropriate for silicon) in the xy plane, subjected to an electric field E in the negative x direction, and a magnetic field B in the z direction. (The geometry is given in Fig. 1.) Spin and valley degeneracies are ignored, and attention is confined to the case in which only the ground Landau level is occupied. We choose m as the unit of mass, mc/eB as the unit of time, and $(hc/2\pi eB)^{1/2} \equiv l_B$ as the unit of length. (The cyclotron radius, $k_f hc/2\pi eB$, does not enter the problem.) The drift velocity, $cE/B = -e\Phi_H/W$ is denoted by v, where Φ_H is the Hall voltage

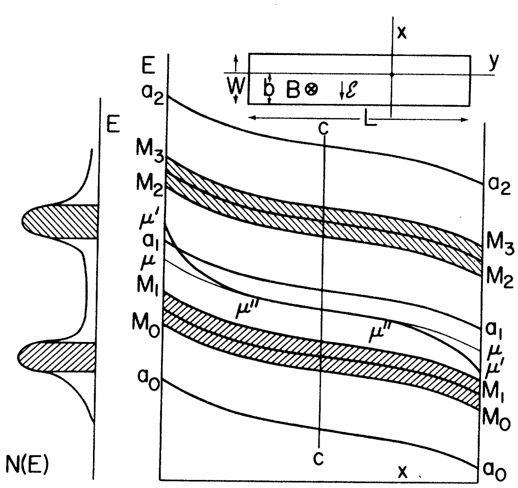

FIG. 1. The inset gives the geometry of the single-impurity problem. To the right below is the energy diagram of the lowest two Landau levels, as banded by impurities, and bent by the electric potential. The lines $a_i - a_i$ bound regions of energy states deriving from distinct levels. The lines $M_i - M_i$ are mobility edges separating localized and delocalized states. The hatched regions are those of delocalized states. The electrochemical potential is given by the line $\mu - \mu'' - \mu'' - \mu$. The position of this line depends on the total density of electrons, i.e., on the gate voltage. The states are occupied below the line $\mu' - \mu'' - \mu'' - \mu'$. The excess of electrons (holes) in the regions $\mu - \mu' - \mu''$ is the source of the Hall voltage. At the left is a schematic diagram of the density of states along the cut $c - c$.

tage, and W is the width of the device in the x direction. The Hamiltonian is $H = H_0 + 2\pi\lambda\delta^2(r)$, with $H_0 = \frac{1}{2}[-\partial^2/\partial x^2 + (p_y + x)^2] - vx$. The eigenstates of H_0 are

$$\psi_{np}(r) = \exp(ip_y y) H_n(x)\phi(x)/(2^n n! L)^{1/2} \ .$$

(The Landau gauge has been used.) The length of the sample in the y direction is L, H_n is the Hermite polynomial, $x = x + p$, $p = p_y - v$, and $\phi(x) = e^{-x^2/2}/\pi^{1/4}$. The eigenenergy corresponding to state ψ_{np} is $n + vp$. The values of p are $2\pi k/L$ with k integer and $-p$ ranges from $-b$ to $W - b$, where the impurity is at the origin of coordinates which is located a distance b above the lower edge of the sample. The total number of p values is thus $WL/2\pi$. This gives the degeneracy of the Landau level. For a sample as described in Ref. 1, we have in our units (with $B = 18$ T), $L = 7 \times 10^4$, $W = 8 \times 10^3$, and $v \sim 1/W \sim 10^{-4}$. We shall treat L and W as macroscopic and v as small, but it will be seen that $1/L$ is by no means the smallest number in the problem. The potential strength λ is of order unity, i.e., $\lambda >> 1/L$.

To find the full eigenstates of H, the state, denoted by ψ^α, is expanded in the "unperturbed" eigenstates of H_0 as $\psi^\alpha(r) = \sum c^\alpha_{np}\psi_{np}(r)$. It is easily seen that

$$c^\alpha_{np} = A^\alpha \psi_{np}(0)/(E_\alpha - n - pv) \quad (1)$$

with the eigenenergy E_α determined by

$$1 = 2\pi\lambda \sum_{np} \frac{|\psi_{np}(0)|^2}{E_\alpha - n - pv} \quad (2)$$

and with the amplitude determined by

$$(A^\alpha)^{-2} = \sum_{np} \frac{|\psi_{np}(0)|^2}{(E_\alpha - n - pv)^2} \ . \quad (3)$$

As is familiar, except for possible bound states breaking off above or below the bands of levels, the energies determined by (2) fall between the closely spaced levels of the successive p values. Thus, we may use as a label for the state α the nearest level of the system unperturbed by the impurity. For simplicity only levels belonging to the zeroth Landau level are considered.

We begin by making the approximation of retaining only the term $n = 0$ in the sums. This is suggested by the "strong magnetic field limit" which is usually taken in the literature.[3] The idea is that it is adequate to diagonalize the subspace consisting of the (nearly) degenerate states of one Landau level, if the levels are sufficiently separated. The current in this case may be obtained without the necessity of finding the explicit eigenfunctions and is completely independent of the form of the scattering potential. The (number) current operator in the x direction is $j_x = (1/i)(\partial/\partial x)$, and it is immediately seen that none of the states carries current along the electric field. In fact, for the exact solution to be found, the same is true. The current operator in the y direction is $j_y = p_y + x = p + x + v$. When this operator acts on a state ψ^α it gives $v\psi^\alpha$ plus a state orthogonal to ψ^α, because the operator $p + x$ changes the Landau level, and by assumption, there is only one Landau level in the sum defining ψ^α. Thus, all states belonging to the Landau level carry the same Hall current.

Since a localized state can certainly carry no current, we conclude that whenever the approximation is valid, there are no localized states. We shall see that for the case of the δ-function potential, there is a localized state and the approximation is not valid.

We therefore must solve the complete problem, keeping the admixture into the wave function of all Landau levels. An immediate difficulty arises because the sum on n in (2) does not converge if performed after the sum over p, at least if it is assumed that the latter sum may be approximated by an integral from $-\infty$ to ∞, using the largeness of L and W, and that, for large n, the term pv in the denominators may be neglected. However, the finiteness of the integration range will become a factor when the spatial extent of the wave functions ψ_{np} starts to become equal to the sample width, and there will be an effective cutoff at $n = M \sim W^2$. (The large magnitude of this cutoff is special to the δ-function potential. Finite-range potentials will have much smaller effective cutoffs.)

To evaluate (2) and (3), the sum over p is replaced by a principal value integral plus a contribution coming from the p values in the immediate neighborhood of the singular point. By introducing $k_\alpha - \delta_\alpha = LE_\alpha/2\pi v$, where k_α is an integer and $2|\delta_\alpha| < 1$, as well as $p_\alpha = 2\pi k_\alpha/L$ Eq. (2) may be rewritten as

$$1 = \lambda G(E_\alpha) + \lambda\phi(p_\alpha)^2\sigma_\alpha/v \ . \quad (2a)$$

Here $G(E)$ is the principal value integral, equal to $\sum_0^M (E-n)^{-1}$ for $|E-n| >> v$. (Since G is of order $\ln M$, the large but finite cutoff does not lead to an intolerably large G.) The discrete sum is convergent and is given by $\sigma_\alpha = -\pi\cot(\pi\delta_\alpha)$. In the same way the amplitudes are expressed as

$$(A^\alpha)^{-2} = -\left(\frac{\partial G}{\partial E_\alpha}\right)\bigg/2\pi + \phi(p_\alpha)^2 \frac{L(\sigma_\alpha^2 + \pi^2)}{(2\pi v)^2} \ . \quad (3a)$$

Consider first states for which $p_\alpha >> 1$, that is, states which hardly overlap the impurity and for which $\phi(p_\alpha)^2 << 1/L$ is very small indeed. Except for the case that $G(E_\alpha) = 1/\lambda$, σ_α will necessarily be enormous, the second term on the right of (3a) will dominate, and the state ψ^α will differ insignificantly from the corresponding unperturbed state.

For the special energy E_R satisfying $G(E_R) = 1/\lambda$, however, $\sigma_\alpha \sim 1$, and the first term on the right-hand side of (3a) dominates. Under the circumstances, there is exactly one level for which this is true. This also subsumes the case for impurities near the sample edges, which may have bound states lying outside the quasicontinuum. (One may also take the thermodynamic limit, $L \to \infty$, in which case the state in question becomes an exceedingly narrow resonance, without changing the result.) This is a localized state, whose wave function is $\psi^R = \exp[-(x^2 + y^2 - 2ixy)/4]/\sqrt{2\pi}$ (for $v=0$, and neglecting the interlevel mixing). It is a peculiarity of this system that the spatial extent of the localized state is controlled by the magnetic field when the potential fluctuation is short ranged and does not become large even when λ and E_R become small. (This may modify the theory of Anderson localization for weakly localized states.) The current of the localized state, j^R, of course vanishes as can be verified by direct computation which gives

$$j_y^R = v\left\{1 - \frac{(A^R)^2}{2\pi} \sum_l (n+1)[(E_R - n)^{-2} - (E_R - n - 1)^{-2}]\right\} = 0 \quad . \tag{4}$$

Next, the remaining class of states is considered, namely, those with energies E_α whose corresponding p_α is of order unity, that is, states made up of unperturbed eigenstates which have a significant overlap with the impurity. The eigenstates ψ^α do *not* overlap the impurity, of course, since they must be orthogonal to ψ^R which is localized at the site. The energies E_α are of order v, and thus $G(E_\alpha) \cong \mathrm{Re}G_0(p_\alpha)/v$ which is given by

$$G_0(p_\alpha) = \int dp \frac{\phi(p)^2}{p_\alpha - p + i\eta} \quad . \tag{5}$$

Then, $G \sim 1/v$, $\phi(p_\alpha) \sim 1$, $\sigma_\alpha \sim 1$, and the second term on the right-hand side of (3a) dominates because L is large. The Hall current carried by such a state may be evaluated to leading order in v as

$$j_y^\alpha = v + (A^\alpha)^2/\pi v \quad . \tag{6}$$

The sum over states α is smooth and may be replaced by an integral over p_α. It is found that

$$j^0 \equiv \sum_\alpha (j_y^\alpha - v) = v\left(\frac{2}{\pi}\right) \int dp \, \mathrm{Im}\left(\frac{1}{G_0(p)}\right) \quad . \tag{7}$$

The integral is evaluated as $\pi \int dp \, p^2 \phi(p)^2 = \pi/2$, by recognizing its analytic properties. Thus, j^0, which is the excess current carried by the electrons which pass near the impurity, is just $j^0 = v$, exactly enough to compensate for the failure of the localized state to carry a current.

The case of two δ-function potentials may be studied as well. In this case, if the potential sites are separated by a distance $l \gg 1$, they do not interact, and the single-impurity problem gives the answer. (This may be extended to many well separated impurities.) On the other hand, if the separation l is small compared with unity, the potential acts as a single site. Only when $l \sim 1$ do the impurity levels interact and start to form an impurity band.

The preceding results, as well as the lore of Anderson localization due to Mott and others,[1,4,5] support the conjecture[1,3] that the following picture holds. In the absence of an electric field, and in the presence of a considerable number of impurities, defects or other potential fluctuations, and in a sufficiently strong magnetic field, a given Landau level will be broadened into a band. The central region of this band will be delocalized states. Beyond a mobility edge there will be localized states, which can only conduct current by hopping processes. The delocalized states of different bands will thus be separated by a region of localized states. The delocalized states of each Landau "band," however, collectively carry a total Hall current I, in the presence of an electric field or potential gradient, which is

$$I = (-ev/L)LW/2\pi = e^2\Phi_H/2\pi (= e^2\Phi_H/h) \quad . \tag{8}$$

Our calculation thus tends to confirm the expectation that the quantum Hall resistance is h/e^2.

This result applies to a dilute system of δ-function impurities. Ando *et al.*[3] also obtained a quantized Hall current within the framework of their approximations, which are the assumption of "high magnetic field," the single-site approximation with the effects of scattering taken into account self-consistently, and the assumption that there are gaps between the resulting "impurity bands." The spirit of the "high-field" approximation seems to be the same as discussed earlier, but in detail it is different since the elegant formalism of Kubo *et al.*[6] is used. (This method breaks the current correlation expression for the conductivity into two parts, one of which is treated exactly, and the other approximately.) Although the conditions for the validity of the high-field approximation have not been spelled out, it seems likely that it will be valid when the field is so great that the potential hardly varies on the scale of l_B. The δ-function results on the other hand ought to be qualitatively valid when the potential is confined to a small region compared with l_B. Given the value of l_B appropriate for the experiment (70 Å), it is unlikely that either approximation is *a priori* very good. The experiment, however, is evidence that the results which have been obtained in these limiting cases must be valid under very general conditions.

Aside from these questions which somehow must receive a favorable answer if the experiment is to be explained at all, there are several other considerations which might lead to effects at the part in 10^6 level. In particular, W is not so large that corrections of order $1/W$ can be tolerated. Thus, the edge effects must be carefully investigated. It is known from the theory of the Landau diamagnetism, for example, that the surface Landau levels tend to carry current in a direction counter to that of the bulk. Perhaps this kind of correction can be avoided along the lines suggested by the foregoing, namely, that any surface states or other anomalous states will be compensated for by an increase in the Hall current of the delocalized states. Another small effect needing investigation is the nonparabolicity of the energy bands.

It is interesting to ask whether the quantum Hall current is a supercurrent in the sense of the theory of superconductivity, and whether a persistent current can be set up. There is no dissipation connected with the Hall current *per se,* since it is perpendicular to the field. It is a supercurrent in the sense that the wave function is locked into place by an energy gap, and it is because of the vector potential that the current exists.

Thus the more interesting question is whether there will be a small current parallel to the electric field, that is, whether in this direction, the system is a perfect insulator. In our model, such a current can come only by taking into account inelastic processes so far neglected, which give rise to a change in occupation of the states. When an entire level of current-carrying states are filled there is no possibility of changing their occupation without large energy cost. The localized states also are activated so they too are perfectly insulating at sufficiently low temperature. Thus we expect that the Hall potential can be maintained without dissipation and that a persistent current can be set up.

This raises the possibility of photoinducing a potential drop in the y direction. If light falls on the junction, it can excite electrons into the unfilled delocalized states of a higher Landau level, and these electrons will provide a current in the x direction which will in turn cause a shift in the direction of the Hall current and lead to a potential drop along the length of the sample. By controlling the frequency of the radiation, something about the position of the mobility edges may be inferred.

The approximation of constant electric field must also be examined. It is not known where the charge which gives rise to this field resides, at least in the case that the diagonal component of the conductivity, σ_{xx}, vanishes, and the actual electric field configuration may depend on how the Hall current is set up. If the charge is localized toward the edges of the sample, it will attract an equal and opposite image charge in the facing metal a few hundred nm away. The potential of such a line of dipoles will vary most pronouncedly in the first few hundred nm away from the edge. If this is the case, most of the Hall current will flow along the edges of the sample, and the inner part will carry practically no current. This will increase the effective value of the local v and corrections of order v^2 could start to play a role. There is, fortunately, no evidence thus far that the electric field is far from constant.[7] The situation might arise, however, in a Corbino (disk with center hole) geometry where a Hall voltage could be applied by moving up external charges.

In the actual experimental configuration,[1] the primary charge giving rise to the Hall potential presumably resides in the localized states. We thus envisage a situation which is schematically shown in the figure. The charges in these localized states will be unable to relax toward equilibrium at low temperature since they require thermally activated inelastic processes to change their state. The Landau band will then bend to follow the potential, and a picture as in the figure will result.

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