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Strong-coupling expansions for truncated Hamiltonian $O(n)$-spin systems

Herbert W. Hamber* and John L. Richardson
Institute for Theoretical Physics and Department of Physics, University of California, Santa Barbara, California 93106
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The Hamiltonian formulation of lattice spin systems is used to study the critical properties of a truncated quantum $O(n)$-spin model in one and d spatial dimensions for arbitrary n. Strong-coupling expansions for the mass gap, ground-state energy density, and susceptibility have been computed up to 14th order and used to search for phase transitions. For $n=0$ we obtain the exponents for the self-avoiding random-walk problem. In the $O(2)$ case we find that the correlation length diverges with an essential singularity at the critical point in $1+1$ dimensions. No phase transition is found for the $O(3)$ and $O(4)$ models in the same dimension. Exponents for other values of n are presented and higher dimensions are considered.

I. INTRODUCTION

Recently there has been renewed interest in understanding the phase structure of $O(n)$-spin systems in two dimensions.[1] Hamiltonian strong-coupling expansions have proven particularly successful in analyzing their critical properties and in computing exponents.[2] In this paper we introduce and study a new class of $O(n)$-symmetric Hamiltonian spin models that are obtained from the quantum $O(n)$ Heisenberg model by a truncation of the space of states. Since the truncation preserves the global $O(n)$ symmetry, one might expect that the two classes of models share the same long-distance behavior. The reduction of the space of states we perform consists in retaining only $n+1$ states at every lattice site, and this has the effect of significantly reducing the effort in computing strong-coupling series. Since our series explicitly depend on the parameter n, they may be analytically continued to nonintegral and nonpositive values, in particular $n=0$ (corresponding to the self-avoiding random-walk problem[3]) and negative n.[4] In this paper we will present evidence that truncated and untruncated models do share the same infrared properties and use the first to compute critical exponents.

The structure of the paper is as follows. We shall first introduce the truncated $O(n)$ Hamiltonian as a simplified version of the quantum $O(n)$ Heisenberg model. We will then present strong-coupling expansions for the mass gap, ground-state energy density, and susceptibility for arbitrary n in $1+1$ dimensions, and for the mass gap and ground-state energy density in $d+1$ dimensions. We then use the ratio test and Padé approximant techniques to search for critical points and compute exponents. The case $n=1$ corresponds to the Ising model in a transverse field for which an exact solution is available,[5] and we use it therefore as a test of our methods. We then study the $n=0$ case which is expected to describe the self-avoiding random walk on an anisotropic lattice, and find good agreement with previous calculations of the exponents. The $O(2)$ model is identical to the model studied by Luther and Scalapino.[6] We find that the mass gap vanishes with an essential singularity at the critical point with an exponent $\sigma=0.85 \pm 0.4$ in rough agreement with the Kosterlitz renormalization-group prediction for the infinite spin model.[7] We also find $\eta=0.27\pm0.1$. We have found the series in this case quite irregular due to the presence of strong competing singularities in the complex plane, and as a consequence of this the error bars are quite large. For $n>2$ we find no indication of a phase transition at finite couplings. We have also computed the thermal exponent for several other values of n $(-2\leq n<2)$ and compared it with a recent conjecture.[8] The agreement is rather satisfactory, particularly in the neighborhood of $n=2$. Finally, we present some results concerning the quantum $O(n)$ models in higher dimensions.

II. TRUNCATED σ MODEL

We shall consider the classical $O(n)$ Heisenberg model in two dimensions described by the action

$$I=\frac{1}{2g}\int d^2x\,(\partial_\mu \vec{S})^2 \ , \tag{2.1}$$

where $\vec{S}$ is an n-component scalar field of unit length, $\vec{S}^2=1$. By relabeling one space direction as imaginary time, one can then construct the quantum Hamiltonian on a one-dimensional spatial lattice

$$H=\frac{g}{2a}\sum_{l=1}^{N}\left[\vec{J}_l^2-\frac{2}{g^2}\vec{S}_l\cdot\vec{S}_{l+1}\right] \ . \tag{2.2}$$

Here $\vec{J}^2$ is the quadratic Casimir operator for the group $O(n)$ whose eigenvalues are given by

$$\vec{J}^2|j,\{m\}\rangle = j(j+n-2)|j,\{m\}\rangle , \quad j=0,1,2,\ldots . \tag{2.3}$$

H is invariant under a global $O(n)$ rotation $\vec{S}_l \to R\,\vec{S}_l$ where R is an $n \times n$ orthogonal matrix.

In order to simplify the above Hamiltonian further, we follow the suggestion of Luther and Scalapino for the $O(2)$ model,[6] and truncate the space of states at every site to a finite number of states.[9] In the strong-coupling limit ($x \to 0$) the Hamiltonian (2.2) reduces to the sum of the quadratic Casimir operators. In this limit the minimal set of states at each lattice site that are lowest in energy and maintain the full $O(n)$ global symmetry of the Hamiltonian are the $j=0$ and the n-fold-degenerate $j=1$ states. Using the fact that

$$\begin{aligned} &\langle j=0|\vec{S}|j=0\rangle = 0 , \\ &\langle j=1,\ m|\vec{S}|j=1,\ m'\rangle = 0 , \\ &\langle j=0|\vec{S}\cdot\hat{e}_k|j=1,\ m\rangle = \frac{1}{\sqrt{n}}\delta_{km} , \end{aligned} \tag{2.4}$$

we arrive at the following form for H_T:

$$H_T = \frac{g}{2a}(n-1)\sum_{l=1}^{N}\left[\vec{L}_l^2 - \frac{2}{n(n-1)}\frac{1}{g^2}\vec{\lambda}_l\cdot\vec{\lambda}_{l+1}\right] , \tag{2.5}$$

where $\vec{L}^2$ and $\lambda^{(\alpha)}$ are $(n+1)\times(n+1)$ matrices

$$\vec{L}^2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix} , \quad \lambda^{(1)} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & & \vdots \\ 0 & 0 & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{bmatrix} ,$$

$$\lambda^{(2)} = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & & \vdots \\ 1 & 0 & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{bmatrix} , \ldots , \quad \lambda^{(n)} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & & 0 \end{bmatrix} \tag{2.6}$$

It is convenient to rescale the Hamiltonian (2.5) and define

$$W = \frac{2a}{g(n-1)}H_T = \sum_{l=1}^{N}(\vec{L}_l^2 - x\,\vec{\lambda}_l\cdot\vec{\lambda}_{l+1}) , \tag{2.7}$$

with

$$x = \frac{2}{n(n-1)g^2} . \tag{2.8}$$

The truncation we have performed has preserved the global $O(n)$ symmetry of the original Hamiltonian and at the same time has significantly reduced the degrees of freedom.

For $n=1$ the Hamiltonian (2.5) describes the Ising model in a transverse field, for which an exact solution is available.[5] For $n=2$ we recover the model studied by Luther and Scalapino.[6] The case $n=0$ is also of interest since it describes the self-avoiding random-walk problem,[3] which is related to the study of polymers in solutions.[10] Finally for $n=-2$ we recover the Gaussian model.[4]

It is of interest to find out what the Euclidean counterpart of the truncated Hamiltonian (2.5) is.[11] One can derive easily that the partition function

$$z = \mathrm{Tr}\prod_{\langle ij\rangle}(1+\omega\vec{S}_i\cdot\vec{S}_j) \tag{2.9}$$

gives rise to an infinitesimal transfer matrix of the form (2.5) when the lattice spacing in the time direction is taken to zero. The sum is over nearest-neighbor pairs on a two-dimensional square lattice, and the $\vec{S}$'s are again n-component unit vectors. The $O(n)$ symmetry is explicit in this formulation, and we notice that for $n=1$ the truncated and untruncated model are of course the same. From this we see that the truncation we have adopted is not restricted to the particular model we have chosen to study, but can be generalized to a wide class of theories, including those with a local symmetry.

III. STRONG-COUPLING EXPANSION

We now turn to the problem of computing the spectrum of the Hamiltonian (2.7) using strong-coupling expansion techniques. We have chosen this method because it has proven to be quite successful in determining the phase structure of other spin models.[2,12] The expansion in $x=2/[n(n-1)g^2]$ is closely related to the high-temperature expansion in statistical mechanics, since small x corresponds to large g or high temperatures. Due to the different cutoff procedures in the Euclidean and Hamiltonian version of the theory, establishing a relationship between the two couplings turns out to be a nontrivial task. Space and time are both treated symmetrically in the Euclidean version, whereas in the quantum model the lattice spacing in the time direction is taken to zero from the start. But, the infrared properties of the two models should of course be the same and independent of the lattice structure.

To develop a strong-coupling expansion for the spectrum of the Hamiltonian (2.7), we separate W into two parts

$$W = H_0 - xV , \tag{3.1}$$

where

$$H_0 = \sum_{l=1}^{N}\vec{L}_l^2 , \tag{3.2}$$

$$V = \sum_{l=1}^{N}\vec{\lambda}_l\cdot\vec{\lambda}_{l+1} . \tag{3.3}$$

TABLE I. Strong-coupling expansion coefficients for the mass gap for the $O(2)$, $O(0)$, and $O(n)$ models in $1+1$ dimensions. The expansion variable is x. Mass gap $\mu_0=1$.

N	$O(2)$	$O(0)$	$O(n)$
1	-2	-2	-2
2	1	-1	$n-1$
3	$\frac{1}{2}$	$-\frac{1}{2}$	$(n-1)/2$
4	$-\frac{3}{4}$	$-\frac{5}{4}$	$(n-1)(5-4n)/4$
5	$\frac{1}{8}$	$-\frac{13}{8}$	$(n-1)(13-6n)/8$
6	$\frac{11}{16}$	$-\frac{65}{16}$	$(n-1)(30n^2-87n+65)/16$
7	$-\frac{17}{32}$	$-\frac{209}{32}$	$(n-1)(48n^2-209n+209)/32$
8	$-\frac{77}{128}$	$-\frac{2265}{128}$	$(n-1)(-550n^3+2574n^2-4119n+2265)/128$
9	$\frac{55}{128}$	$-\frac{489}{16}$	$(n-1)(-1775n^3+11493n^2-23600n+15648)/512$
10	$\frac{903}{1024}$	$-\frac{89515}{1024}$	
11	$-\frac{977}{8192}$		

H_0 is already diagonal and we denote its ground state as $|0\rangle$

$$\vec{L}_l^2|0\rangle=0,\quad \forall l\ . \tag{3.4}$$

The lowest energy state above the vacuum is

$$|n\rangle=\frac{1}{\sqrt{N}}\sum_{l=1}^{N}\lambda_l^{(n)}|0\rangle\ . \tag{3.5}$$

We have computed the ground-state energy density ω_0 and the mass gap

$$\mu=\frac{g}{2a}(\omega_1-\omega_0)=\frac{g}{2a}F(x) \tag{3.6}$$

as a power series in x up to 11th order using Raleigh-Schrödinger perturbation theory. Here $F(x)$ is a dimensionless function of the parameter x and corresponds to the inverse correlation length in statistical mechanics. The expansion coefficients for these quantities are shown in Tables I and II for $1+1$

TABLE II. Ground-state energy density for the Ising, $O(2)$ and $O(n)$ models in $1+1$ dimensions.

N	$O(1)$	$O(2)$	$O(n)$
0	0	0	0
2	$-\frac{1}{2}$	-1	$-n/2$
4	$-\frac{1}{8}$	$\frac{1}{2}$	$-n(4-3n)/8$
6	$-\frac{1}{8}$	$-\frac{5}{16}$	$-n(19n^2-56n+41)/32$
8	$-\frac{25}{128}$	$\frac{9}{64}$	$-n(-309n^3+1448n^2-2249n+1160)/256$
10	$-\frac{49}{128}$	$\frac{1}{64}$	$-n(5719n^4-37119n^3+90003n^2-96761n+38942)/2048$
12	$-\frac{441}{512}$	$-\frac{233}{2048}$	
14	$-\frac{1089}{512}$	$\frac{34105}{393216}$	
N	$-2^{N-1}\left[\frac{(N-3)!!}{N!!}\right]^2$		

TABLE III. Mass gap and ground-state energy density for the $O(n)$ model in $d+1$ dimensions.

N	Mass gap (arbitrary d) μ_N
0	1
1	$-2d$
2	$-d(2d-n-1)$
3	$-\frac{1}{2}d(2d-1)(4d-3-n)$
4	$-\frac{1}{4}d[40d^3-48d^2-8d+21-n(16d^2+14d-21)+n^2(6d-2)]$
N	Ground-state energy (arbitrary d) ω_{0_N}
0	0
1	0
2	$-\frac{1}{2}n$
3	0
4	$-\frac{1}{8}n[2(9d-7)-(4d-1)n]$

dimensions and in Table III for $1+d$ dimensions. From the ground-state energy one can compute the specific heat, which is given by

$$\frac{1}{g}C_h=-\left(\frac{\partial^2\omega_0}{\partial g^2}\right)=-n(n-1)x^2(3\omega_0'+2x\omega_0'') \quad , \quad (3.7)$$

although for our purposes it will be sufficient to study $\partial^2\omega_0/\partial x^2$.

It is also useful to study the Hamiltonian (2.7) in an external field in the (1) direction

$$W=\sum_{l=1}^{N}(\vec{L}_l^2-x\vec{\lambda}_l\cdot\vec{\lambda}_{l+1}-h\lambda_l^{(1)}) \quad . \qquad (3.8)$$

TABLE IV. Susceptibility series in $1+1$ dimensions.

N	Susceptibility $\chi_0=1$ $O(1)$	$O(2)$	$O(n)$
1	4	4	4
2	11	$\frac{19}{2}$	$(25-3n)/2$
3	28	$\frac{35}{2}$	$(77-21n)/2$
4	$\frac{799}{12}$	$\frac{329}{12}$	$(87n^2-1201n+2112)/24$
5	154	$\frac{113}{3}$	$(729n^2-4979n+7946)/24$
6	$\frac{37469}{108}$	$\frac{40661}{864}$	$(-8631n^3+156705n^2-668789n+820467)/864$
7	$\frac{62387}{81}$	$\frac{141293}{2592}$	
8	$\frac{8741521}{5184}$	$\frac{599521}{10368}$	
9	$\frac{7113365}{1944}$		
10	$\frac{1101794737}{139968}$		
11	$\frac{294520877}{17496}$		

The zero-field susceptibility is then defined as

$$\chi = -\left(\frac{\partial^2 \omega_0}{\partial h^2}\right)_{h=0} \tag{3.9}$$

We have computed the ground-state energy as a power series in both x and h, and after isolating the second-order terms in the expansion in h we obtain the coefficients listed in Table IV.

IV. SERIES ANALYSIS AND PHASE STRUCTURE

We shall now briefly digress on the various methods we have used for analyzing the above series.[13] Let us assume an algebraic singularity for the mass gap (as an example) of the form

$$F(x) \underset{x\to x_c}{\sim} A(x_c-x)^{\nu} \tag{4.1}$$

for the x close to x_c. The β function $\beta(g)$ defined by

$$\frac{\beta(g)}{g} = \frac{F(x)}{F(x)-2xF'(x)} \tag{4.2}$$

will then have a simple zero at x_c

$$\frac{\beta(g)}{g} \underset{x\to x_c}{\sim} \frac{1}{2\nu x_c}(x_c-x) \ . \tag{4.3}$$

The simplest test consists in taking ratios of successive coefficients in the series of $F^{-1}(x)$, which should then behave as

$$R_l = \frac{a_l}{a_{l-1}} \underset{l\to\infty}{\sim} x_c^{-1}\left[1+\frac{\nu-1}{l}+O\left(\frac{1}{l^2}\right)\right] \ . \tag{4.4}$$

From these one then computes the linear extrapolant

$$E_l = lR_l-(l-1)R_{l-1} \underset{l\to\infty}{\sim} x_c^{-1} \tag{4.5}$$

and the slopes

$$S_l = (R_l-R_{l-1})/[1/l-1/(l-1)] \underset{l\to\infty}{\sim} x_c^{-1}(\nu-1) \ . \tag{4.6}$$

We have found that the convergence can sometimes (in the presence of confluent singularities) be improved by using quadratic extrapolants

$$\tfrac{1}{2}[l^2R_l-2(l-1)^2R_{l-1}+(l-2)^2R_{l-2}] \underset{l\to\infty}{\sim} x_c^{-1} \ , \tag{4.7}$$

$$3-2l+x_c^{(l)}[l^2R_l-(l-1)^2R_{l-1}] \underset{l\to\infty}{\sim} \nu \ , \tag{4.8}$$

where $x_c^{(l)}$ in Eq. (4.8) stands for the inverse of the left-hand side (lhs) of Eq. (4.7).

In the presence of strong competing singularities in the complex plane we have found the Padé approximant method more useful than the ratio test. Assuming an algebraic singularity of the form (4.1) we can construct the logarithmic derivative, which has a simple pole at x_c

$$\frac{d}{dx}\ln F^{-1}(x) \underset{x\to x_c}{\sim} \frac{\nu}{x_c-x} \ . \tag{4.9}$$

Padé approximants to this function give then estimates of the pole and residue. If a value of ν is assumed, Padé approximants to the function

$$[F^{-1}(x)]^{1/\nu} \underset{x\to x_c}{\sim} \frac{A^{-1/\nu}}{x_c-x} \tag{4.10}$$

give a biased estimate of x_c, and vice versa. Once a value for x_c is assumed, Padé approximants to

$$(x_c-x)\frac{d}{dx}\ln F^{-1}(x) \underset{x\to x_c}{\sim} \nu \tag{4.11}$$

yield a biased estimate for ν.

In certain cases, like the $O(2)$ model in $1+1$ dimensions, the assumption of an algebraic singularity in the mass gap might not be the correct one. If we assume

$$F^{-1}(x) \underset{x\to x_c}{\sim} A\exp\left[\frac{a}{(x_c-x)^{\sigma}}\right] \ , \tag{4.12}$$

then the double logarithmic derivative should have a simple pole[14]

$$\frac{d}{dx}\ln\left[\frac{d}{dx}\ln F^{-1}(x)\right] \underset{x\to x_c}{\sim} \frac{1+\sigma}{x_c-x} \ , \tag{4.13}$$

and the same should be true for

$$\left[\frac{d}{dx}\ln F^{-1}(x)\right]^{1/1+\sigma} \underset{x\to x_c}{\sim} \frac{(\sigma a)^{1/1+\sigma}}{x_c-x} \ , \tag{4.14}$$

and for the logarithmic derivative of the β function,

$$\frac{d}{dx}\ln\frac{\beta(g)}{g} \underset{x\to x_c}{\sim} -\frac{1+\sigma}{x_c-x} \ . \tag{4.15}$$

We have found some series [like the $O(2)$ mass gap and susceptibility] quite irregular because of the presence of competing singularities in the complex plane, besides the physical one on the positive real axis. To reduce the effects of these singularities we have frequently mapped the series conformally using the Euler transformation in the general form

$$x = \frac{a+bz}{c+dz} \ . \tag{4.16}$$

The transformation

$$x = \frac{1}{2}\frac{z}{1-z} \tag{4.17}$$

has proven particularly beneficial in analyzing the $O(2)$ mass-gap series, but improved results were obtained also in other cases that we have studied. We now turn to the results we have obtained from the series in $1+1$ dimensions.

A. Ising model

The Ising model in a transverse field in one space dimension is solvable and provides a good test for the methods we have summarized in the preceding section. We recover the series for the Ising model when we set $n=1$. The mass-gap series then truncates and gives the exact result

$$F(x)=1-2x \ , \tag{4.18}$$

and therefore we find $x_c=\frac{1}{2}$ and $\nu=1$. We have also computed the ground-state energy density to 20th order in x and checked it against the exact solution. Using the ratio test on the specific heat we find $x_c=0.5001\pm0.0001$ and $\alpha=0.01\pm0.01$. From the susceptibility series we obtain using Padé approximants $x_c=0.49999\pm0.00001$ and $\gamma=1.7495\pm0.0005$. Using the scaling argument

$$\chi \underset{x\to x_c}{\sim} \xi^{2-\eta} \tag{4.19}$$

and therefore

$$\frac{\ln\chi}{\ln\xi} \underset{x\to x_c}{\sim} 2-\eta \ , \tag{4.20}$$

we have computed η and found $\eta=0.2501\pm0.0001$. The methods seem therefore to be quite reliable for this model, and we can turn now with some confidence to less trivial cases.

B. $n=0$ case

We have already mentioned the fact that the $n=0$ limit corresponds to the self-avoiding random walk and is equivalent to the long-polymer problem.[3–10] Since each closed loop has a factor of n in front of it in the strong-coupling series expansion, for $n=0$ there are no closed loops. This leads to a vanishing ground-state energy density ω_0, although the limit ω_0/n remains finite as $n\to0$ and can be analyzed for singularities.

Padé approximants to the logarithmic derivative of the mass-gap[15] series give $x_c=0.366\pm0.002$ and $\nu=0.75\pm0.01$. The rate of convergence in the Padé table (see Table V) is rather slow due to the presence of a competing singularity at $x_0=-0.392\pm0.005$. The transformation $x=z/(1-z)$ moves it away from the origin, and the ratio test on the conformally mapped series now gives (see Table VI)

$$x_c=0.3665\pm0.001 \tag{4.21}$$

and

$$\nu=0.750\pm0.005 \ . \tag{4.22}$$

The series for $(\mu^{-1})^{4/3}$ gives, using Padé approxi-

TABLE V. Padé method analysis of the series for $D\ln\mu^{-1}$ in the $O(0)$ model. Listed are the positions and (in brackets) residues of the first pole on the positive real axis, for Padé approximants of various order. Our final estimates are $x_c=0.366\pm0.002$, $\nu=0.75\pm0.01$.

		$n=0,\ D\ln\mu^{-1}$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1	⋯	0.3871 (0.929)	0.3690 (0.791)
2	0.3605 (0.726)	0.3692 (0.792)	0.3690 (0.792)
3	0.3680 (0.782)	0.3681 (0.783)	0.3676 (0.775)
4	0.3681 (0.782)	0.3674 (0.771)	0.3674 (0.772)
5	0.3674 (0.772)		

TABLE VI. Ratio test applied to the mass-gap series for the $O(0)$ model, in the variable $z=x/(1+x)$. Successive columns list the order N, the ratios of coefficients R_N, the estimate for z_c given by the inverse linear extrapolant, z_N, and the corresponding estimate for the index, μ_N. The final estimates for x_c and μ are: $z_c=0.2682\pm0.001$, $x_c=0.3665\pm0.001$, $\nu=0.750\pm0.005$.

		$n=0,\ \mu^{-1}$ ratios in Z	
N	R_N	z_N	μ_N
1	2	0.5	1
2	3.5	0.2	0.4
3	3.5	0.2857	1
4	3.5408	0.2730	0.8663
5	3.5692	0.2716	0.8460
6	3.5912	0.2702	0.8211
7	3.6077	0.2698	0.8135
8	3.6207	0.2694	0.8038
9	3.6311	0.2692	0.7988
10	3.6396	0.2691	0.7939

TABLE VII. Padé approximants to the series $(\mu^{-1})^{4/3}$ in the $O(0)$ model. The estimate for x_c is 0.3667 ± 0.001.

		$n=0,\ (\mu^{-1})^{4/3}$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1	⋯	0.3529	0.3606
2	0.3647	0.3634	0.3671
3	0.3641	0.3656	0.3661
4	0.3670	0.3664	0.3669
5	0.3665	0.3667	

TABLE VIII. Ratio method analysis of the series for $\partial^2\omega_0/\partial x^2$ in the $O(0)$ model. Format as in Table VI. ($x_c=0.359\pm0.01$, $\alpha=0.55\pm0.1$.)

		$n=0$, ω_0''	
N	R_N	x_N^2	α_N
1	6	0.1666	1
2	6.4063	0.1468	0.8807
3	6.6016	0.1430	0.8324
4	6.7441	0.1394	0.7616

mants, the estimate (Table VII) $x_c=0.3667\pm0.001$, in agreement with the previous one, and an amplitude $A=1.042\pm0.01$ which leads to the approximate result

$$\mu \underset{x\to x_c}{\sim} 1.042(1-x/0.3667)^{0.750} \quad . \tag{4.23}$$

The ratio test applied to the second derivative of the ground-state energy density gives $x_c=0.359\pm0.01$ and (Table VIII)

$$\alpha=0.55\pm0.1 \quad . \tag{4.24}$$

The large uncertainty in this case is due to the shortness of the series.

Padé approximants for the logarithmic derivative of the susceptibility series give (Table IX)

$$x_c=0.3667\pm0.0002 \quad , \tag{4.25}$$

in good agreement with the estimate from the mass-gap series, and

$$\gamma=1.333\pm0.003 \quad . \tag{4.26}$$

From the series for $D\ln\chi/D\ln\mu^{-1}$ we obtain

$$\eta=0.23\pm0.02 \quad . \tag{4.27}$$

These results therefore suggest $\nu=\frac{3}{4}$, $\alpha=\frac{1}{2}$, $\gamma=\frac{4}{3}$, $\eta=\frac{2}{9}$ and therefore $\beta=\frac{1}{12}$ and $\delta=17$ from the scaling relations $\beta=\nu\eta/2$ and $\delta=1+\gamma/\beta$. We can compare our exponents with similar estimates from exact enumeration techniques which yield $\nu=0.755\pm0.005$.[16,17] The agreement is quite good and provides further evidence for universality of critical behavior.

TABLE IX. Padé approximants to the series $D\ln\chi$ in the $O(0)$ model. Format as in Table V. ($x_c=0.3667\pm0.0002$, $\gamma=1.333\pm0.003$.)

		$n=0$, $D\ln\chi$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1	...	0.3051 (0.837)	0.3668 (1.336)
2	0.4370 (2.463)	0.3666 (1.333)	0.3667 (1.335)
3	0.3669 (1.338)		

C. $O(2)$ model

For $n=2$ the Hamiltonian (2.5) describes the model studied by Luther and Scalapino.[6] Their calculations indicate the presence of a phase transition at some finite coupling, as in the infinite-spin (or classical) planar model for which some presumably exact results have been derived by Kosterlitz and Thouless.[7] Here we will try to address the question of whether the two models exhibit universal behavior at the critical point using the strong-coupling expansion.

Ratio test and Padé approximant techniques applied to the mass gap and susceptibility series reveal that the series are quite irregular due to the presence of competing singularities in the complex plane. In the case of the mass-gap series there are two singularities at $x\simeq0.25\pm0.73i$ and $\simeq-0.89$, whereas the physical singularity is around $x\simeq1.67$. The mapping $x=z/2(1-z)$ reduces the effect of the unphysical singularities and moves the physical one closer to the origin. The series becomes now more regular and the results of the Padé analysis are shown in the Tables X–XVI.

When we analyze the mass-gap series assuming an algebraic singularity, we find that Padé approximants to the logarithmic derivative suggest a phase transition at $z_c=0.75\pm0.02$ or

$$x_c=1.52\pm0.3 \quad , \tag{4.28}$$

with a large exponent $\nu=2.6\pm0.8$. The fact that the values for x_c and ν are not stable suggests that an algebraic singularity might not be the right ansatz. The large value for ν indeed suggests that the mass

TABLE X. Padé method analysis of the series $D\ln\mu^{-1}$ in the $O(2)$ model, in the variable $z=2x/(1+2x)$. I denotes a small imaginary part. ($z_c=0.75\pm0.02$, $\nu=2.6\pm0.8$.)

		$n=2$, $D\ln\mu^{-1}(z)$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1	...	0.5480 (0.75)	...
2	0.6239 (1.11)	0.7490 (3.44)	0.4401 (1.22)
3	0.7270 (2.57)	0.7314 (2.73)	0.7155(I)
4	0.7252 (2.52)	0.7723(I)	0.7231 (1.73)
5	0.7723(I)	0.7685 (3.39)	

TABLE XI. Padé method analysis of the series for $D\ln^2\mu^{-1}$ in the $O(2)$ model, in the variable $z=2x/(1+2x)$. Dagger denotes the presence of a secondary pole close to z_c. ($z_c=0.82\pm0.05$, $1+\sigma=1.86\pm0.3$.)

		$n=2$, $D\ln^2\mu^{-1}(z)$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1		0.8598 (2.13)	0.8714 (2.18)
2	0.8030 (1.73)	0.6829 (0.69)	0.8114 (1.92)
3	0.7665 (1.45)	0.8430 (2.18)†	$1.0510\pm0.1753i$
4	0.8823 (2.54)†	0.9722 (4.21)	0.8964† (2.41)
5	0.9373 (3.29)†		

TABLE XIII. Padé method analysis of the series for $(D\ln\mu^{-1})^{2/3}$ in the $O(2)$ model, in the variable z. Dagger denotes the presence of a secondary pole close to z_c. ($z_c=0.771\pm0.01$, $x_c=1.67\pm0.1$.)

		$n=2$, $(D\ln\mu^{-1})^{2/3}(z)$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1		0.7101	0.7900
2	0.7543	0.7778	0.7507
3	0.7716	0.7712	0.7848
4	0.7716	0.8255†	0.8549†
5	0.8504†	0.8464†	

gap might go to zero exponentially as first suggested by Kosterlitz and Thouless.[7]

When we analyze the mass-gap series for an essential singularity of the type

$$\mu \underset{x\to x_c}{\sim} \exp\left[-\frac{a}{(x_c-x)^\sigma}\right] \tag{4.29}$$

we find using double logarithmic derivatives $z_c=0.82\pm0.05$ or

$$x_c=2.3\pm0.6 \tag{4.30}$$

and

$$1+\sigma=1.85\pm0.3\ . \tag{4.31}$$

Padé analysis of the logarithmic derivative of the β function leads to the estimate $z_c=0.78\pm0.05$ or

$$x_c=1.88\pm0.4 \tag{4.32}$$

and

$$1+\sigma=1.90\pm0.3\ . \tag{4.33}$$

The convergence in the estimates for σ appears slow, as is the case in the infinite spin model. An algebraic

TABLE XII. Padé method analysis of the series for $D\ln[\beta(g)/g]$ in the $O(2)$ model, in the variable z. ($z_c=0.78\pm0.05$, $1+\sigma=1.90\pm0.3$.)

		$n=2$, $D\ln[\beta(g)/g](z)$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
2	· · ·	· · ·	0.7107 (1.22)
3	· · ·	0.7928 (1.73)	0.7989 (1.78)
4	0.8091 (1.90)	0.7857 (1.67)	0.8337 (2.03)
5	0.6605 (2.53)	0.8739 (2.35)	

singularity would mean $\sigma=0$, which we can rule out with some confidence. If we assume $\sigma=\frac{1}{2}$, we can improve our estimate of x_c (see Table XIII), and we find $z_c=0.77\pm0.01$ and

$$x_c=1.67\pm0.1\ . \tag{4.34}$$

These results agree rather well with finite lattice approaches.[9]

We now turn to the ground-state energy density. The renormalization-group approach for the (infinite-spin) planar model predicts that the specific does not diverge at the critical point. Our series for the second derivative of ω_0 with respect to x does in fact not show a singularity on the real axis (the only singularities are at $x_c\simeq\pm0.68i$) in agreement with the previous results.

From the scaling hypothesis we know that the susceptibility should diverge as a power of the correlation length as x approaches x_c. We therefore expect an exponential divergence of the type (4.12) as in the case of the mass gap. The ratio test applied to the

TABLE XIV. Ratio method analysis of the susceptibility series in the $O(2)$ model, in the variable z. Format as in Table VI. ($z_c=0.753\pm0.02$, $\gamma=3.4\pm0.3$.)

		$n=2$, $\chi(z)$	
N	R_N	x_N	γ_N
1	2	0.5	1
2	2.375	0.4210	0.8421
3	1.7536	0.5703	1.4949
4	1.6593	0.6027	1.6935
5	1.5955	0.6268	1.8810
6	1.5498	0.6453	2.0547
7	1.5158	0.6597	2.2128
8	1.4894	0.6714	2.3583

TABLE XV. Padé method analysis of the series for $D\ln\chi$ in the $O(2)$ model, in the variable z. ($z_c=0.775\pm0.01$.) I denotes a small imaginary part.

		$n=2$, $D\ln\chi(z)$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1	0.6194	0.5547	0.7351
2	0.6194	0.7945(I)	0.7480(I)
3	0.7672(I)	0.7734(I)	0.7930(I)
4	0.7598(I)		

susceptibility series gives

$$x_c=1.64\pm0.04 \tag{4.35}$$

and

$$\gamma=7.1\pm0.5\ , \tag{4.36}$$

which suggests again that the assumption of an algebraic singularity is probably not correct. After transforming to the variable $z=2x/(1+2x)$ the ratio test gives $z_c=0.753\pm0.02$ and $\gamma=3.4\pm0.3$ (see Table XIV), and Padé approximants to the logarithmic derivative of χ give $z_c=0.775\pm0.01$, which corresponds to a critical point at

$$x_c=1.72\pm0.1 \tag{4.37}$$

in satisfactory agreement with the estimates from the mass-gap series. The estimates for σ from the susceptibility series converge rather slowly at this order, and although they seem to favor a nonzero value for σ, the series is still too short to provide a reliable estimate.

From the knowledge of the critical point we can extract the exponent η using the series for $\ln\chi/\ln\mu^{-1}$ Again the convergence is rather slow and we find (see Table XVI)

$$\eta=0.27\pm0.1\ , \tag{4.38}$$

which is consistent with $\eta=\frac{1}{4}$.[7] Although we favor this result, we cannot entirely rule out $\eta=1/\sqrt{8}$, for example.[6]

D. Other values of n

In two dimensions we have computed the mass gap, ground-state energy, and susceptibility series coefficients through a few orders for arbitrary n. This allows us to study, besides the previous cases, the case $n>2$, where no phase transition is expected,[1] and $n=-2$, which corresponds to the Gaussian model,[4] among others.

The ratio test and Padé analysis applied to the mass gap and susceptibility series (see Table XVII) for the $O(3)$ model give no evidence for a singularity at finite real x. After a conformal mapping to the variable $z=2x/(1+2x)$, we find that the only singularity is at $z\simeq1$, corresponding to $x=\infty$. The large index γ ($\gamma\sim3$) indicates that the assumption of an algebraic singularity for the susceptibility at $1/x=0$ is probably not correct. Similar results seem to hold for $n>3$. In these cases we again do not find evidence for a critical point at finite x.

In the large-n limit the correct expansion variable is n^2x, and Padé approximants give evidence of two singularities at $x_c\simeq\pm(n)^{-1/2}0.55i$ and no singularity on the real axis.

The case $n=-2$ is also of interest since it describes the Gaussian model. The free energy and the two-point correlation function take their noninteracting values, and the exponents are $\nu=\frac{1}{2}$, $\eta=0$, $\gamma=1$, and $\alpha=2-d/2$ ($d\leq4$). From the strong-coupling series for the mass gap we get

$$x_c=0.2666\pm0.001 \tag{4.39}$$

and

$$\nu=0.58\pm0.08\ . \tag{4.40}$$

The convergence is rather slow and is reflected in the

TABLE XVI. Padé method analysis of the series for $\ln\chi/\ln\mu^{-1}$ in the $O(2)$ model, in the variable z. Our final estimate for η is 0.27 ± 0.1.

		$n=2$, $\ln\chi/\ln\mu^{-1}(z)$	
N	$[N,N-1]$	$[N,N]$	$[N,N+1]$
1		0.1196	0.0984
2	0.1074	0.0904	0.0958
3	0.0980	0.4370	0.3545
4	0.3547	0.4296	

TABLE XVII. Ratio method analysis of the susceptibility series in the $O(2)$ model, in the variable $z=2x/(1+2x)$. Format as in Table VI. ($z_c=1.0\pm0.1$, $\gamma=3\pm1$.)

		$n=3$, $\chi(z)$	
N	R_N	x_N	γ_N
1	0.6667	1.5	1
2	1.3333	0.5	0.3333
3	1.2865	0.8383	1.2357
4	1.2480	0.8830	1.4080
5	1.2149	0.9235	1.6102
6	1.1855	0.9632	1.8511
7	1.1584	1.0043	2.1437

TABLE XVIII. Conjectured ν and mass-gap series for various values of n.

n	Conjectured ν	Series
−1	$\frac{5}{8}=0.625$	0.67 ± 0.05
0.5	0.8458	0.865 ± 0.02
1.5	1.3367	1.330 ± 0.02
1.75	1.8041	1.80 ± 0.3

large uncertainty in ν. The specific-heat series gives a similar value for x_c and

$$\alpha = 1.05 \pm 0.05 \quad , \tag{4.41}$$

which is consistent with $\alpha=1$. From the susceptibility series we have also computed $\gamma = 1.05 \pm 0.05$.

Recently one of us (H.H.) has conjectured an analytical form for the thermal exponent of the n-vector model as a function of n.[8] The conjecture is

$$\nu = \frac{1+y}{4y} \quad , \tag{4.42}$$

where

$$y = (2/\pi)\cos^{-1}[(2+n)^{1/2}/2]$$

for $-2 \leq n \leq 2$. We have used the mass-gap series to test the validity of the conjecture, and some of the results we have already presented. We have shown evidence in the preceding sections for ν being $\frac{3}{4}$ exactly for the $O(0)$ model and ∞ (i.e., an essential singularity) for the planar model. For other values of n we find, using Padé approximants for the logarithmic derivative of the mass gap, the data in Table XVIII. We see that the agreement is particularly good close to $n=2$ and quite consistent with a divergence of the type

$$\nu \underset{n\to 2}{\sim} (\pi/4)(2-n)^{-1/2} \quad . \tag{4.43}$$

V. HIGHER DIMENSIONS

Table III contains the coefficients of the mass gap and ground-state energy series for the truncated $O(n)$ model on a hypercubical spatial lattice in d dimensions. Padé approximants to the logarithmic derivative of the mass gap give the critical points and exponents shown in Tables XIX and XX. The agreement with previous calculations is rather good if one takes into account the fact that the series is quite short.[18] In $3+1$ dimensions we have analyzed the mass-gap series for the Ising model and found

TABLE XIX. Summary of extimates of the critical point and exponent ν for integer n, $-2 \leq n \leq 2$, from the analysis of mass-gap series in $1+1$ dimensions.

	$d=1+1$	
n	x_c	ν
2	1.67 ± 0.1	(∞)
1	0.5	1
0	0.3667 ± 0.0001	0.750 ± 0.005
−1	0.3050 ± 0.001	0.66 ± 0.03
−2	0.2666 ± 0.0002	0.58 ± 0.06

$$x_c = 0.09807 \pm 0.00002 \tag{5.1}$$

and

$$\nu = 0.577 \pm 0.05 \quad . \tag{5.2}$$

In this case we would expect $\nu=\frac{1}{2}$, but the convergence appears to be slow as in the case $n=-2$.

In the large-d limit we can sum the leading term in the series for the mass gap and we obtain

$$\mu \underset{d\to\infty}{=} 1-2dx-2(dx)^2-4(dx)^3-10(dx)^4-\cdots$$

$$=(1-4dx)^{1/2} \quad , \tag{5.3}$$

which gives correctly the mean-field exponent $\nu=\frac{1}{2}$, independent of n, and a critical point at $x_c = 1/4d$.

TABLE XX. Critical points and exponents ν for the $O(n)$ model in $2+1$ dimensions. The results were obtained using Padé approximants on $D\ln\mu^{-1}$.

	$d=2+1$	
n	x_c	ν
3	0.209 ± 0.01	0.75 ± 0.12
2	0.1835 ± 0.01	0.68 ± 0.06
1	0.1650 ± 0.002	0.643 ± 0.01
0	0.1513 ± 0.003	0.63 ± 0.04
−2	0.132 ± 0.01	0.55 ± 0.05

VI. CONCLUSIONS

We have shown in the preceding sections how Hamiltonian strong-coupling expansions can be used to study the critical properties of the truncated $O(n)$ models we have introduced. The analysis we have presented suggests that in spite of their simplicity these models share the same infrared properties with the infinite-spin σ model. They can therefore be regarded as an effective tool for studying the critical behavior of other models for which a truncation of the type we have discussed can be performed. Among further applications of the method we see SU(n) chiral models in two dimensions and gauge theories in four dimensions.

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*Present address: Department of Physics, Brookhaven National Lab., Upton, N.Y. 11793.

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