Anisotropic sine-Gordon model and infinite-order phase transitions in three dimensions

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A three-dimensional anisotropic sine-Gordon model, derived as the spin-wave approximation to the biaxial ($m = 2$) Lifshitz point problem in a uniform magnetic field, is shown to possess [in close analogy to the isotropic two-dimensional (2D) sine-Gordon theory which is well known to describe the critical behavior of the 2D XY model], a surface of infinite-order phase transitions. This critical surface separates a phase characterized by infinite correlation length ξ and power-law decay of correlations, and controlled by a stable fixed line, from one with finite ξ and exponential decay. As the critical surface is approached from the latter phase, ξ diverges as $\exp(\sigma t^{-\nu})$ where $\nu = 1$ is a universal number, t measures the distance from the critical surface, and σ is nonuniversal. On the critical surface correlations decay like $r^{-\eta}(\ln r)^{-\tilde{\eta}}$, where $\eta = 4$ and $\tilde{\eta} = 0.88 \cdots$. Speculations on the occurrence of an infinite-order transition in liquid-crystal mixtures exhibiting nematic, smectic-A, and smectic-C phases are advanced.

I. INTRODUCTION

The Lifshitz point[1] (LP) is a special critical point which, as illustrated schematically in Fig. 1, connects three distinct phases: (in magnetic language) paramagnetic, ferromagnetic, and helical or modulated. The phases are, respectively, characterized by an order parameter, $\vec{M}(\vec{x})$, which is zero, finite but spatially uniform, and spatially varying. The paramagnetic-ferromagnetic transition is typically second order, as is the ferromagnetic-helical[2]; fluctuations are believed to render the paramagnetic-helical transition first order.[3]

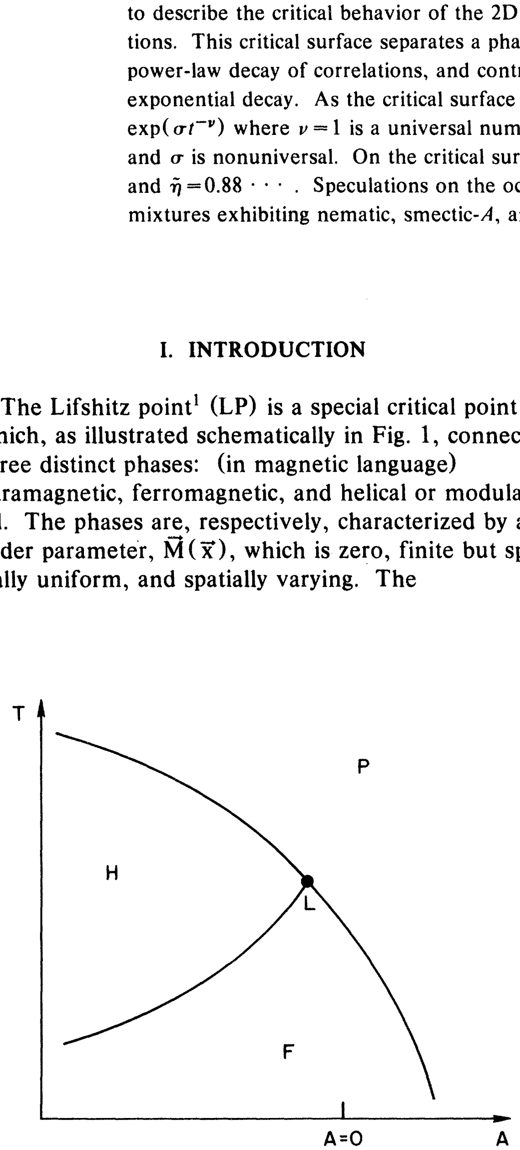

FIG. 1. Schematic Lifshitz-point phase diagram as a function of temperature (T) and the parameter A of Eq. (1). The paramagnetic (P), ferromagnetic (F), and helical (H) phases meet at the Lifshitz point (L).

Critical exponents at the LP itself have been computed in ϵ expansion about the appropriate upper[1,4] or lower[5] critical dimensionality for a variety of values of n, the number of components of $\vec{M}$, and m, the number of spatial dimensions in which $\vec{M}$ can vary. The $m = 2$ LP is particularly interesting in that a precise physical realization of the model exists in the form of bulk liquid-crystal mixtures which exhibit nematic, smectic-A, and smectic-C phases,[2,6,7] and in that its lower critical dimensionality is three[5,8,9]: in any dimension $d \leq 3$ a simple argument (reviewed in Sec. II) shows that for $n \geq 2$ fluctuations prohibit long-range order on the phase boundary between the ferromagnetic and helical phases. This suggests that for $n \geq m = d - 1 = 2$ no LP occurs at finite temperature (T) and the phase diagram is as shown in Fig. 2. Calculations in $3 + \epsilon$ dimensions[5] confirm this expectation for $n > 2$ but indicate that for $n = 2$ there is a finite-temperature LP as in Fig. 1. Explicit spin-wave[8,9] (SW) computations for $n = m = d - 1 = 2$ show that, at least at very low T where the SW approximation is valid, the ferromagnetic-helical boundary is characterized by $\vec{M} = 0$, an infinite correlation length, ξ, and *power-law* decay of correlations.[10] Presumably this power-law behavior on the phase boundary persists right up to the LP where a transition restores the paramagnetic phase and exponentially decaying correlations. As always[11] the SW approximation is too crude to provide any information about the transition itself (that is, about the LP).

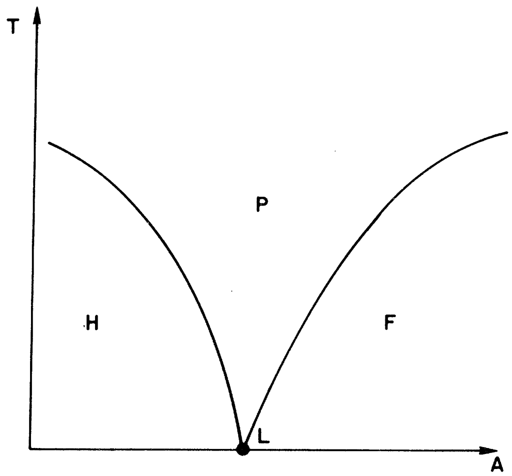

FIG. 2. Schematic Lifshitz-point phase diagram for $n > m = d - 1 = 2$. Fluctuations prevent the existence of a LP at finite T. The paramagnetic (P), ferromagnetic (F), and helical (H) phases meet at a $T = 0$ Lifshitz point (L).

The ferromagnetic-helical phase boundary in the $n = m = d - 1 = 2$ LP problem is a three-dimensional (3D) analog of the low-T phase, likewise characterized by $\vec{M} = \xi^{-1} = 0$ and algebraic decay, of the 2D XY model.[11] Of course the transition to the high-T phase in the 2D problem is well understood: it is mediated by vortex excitations omitted in the SW picture.[12] Bound in zero-vorticity *(neutral)* pairs in the low-T phase, the vortices unbind with increasing temperature, producing, at the critical temperature, T_c, a continuous transition to the high-T state characterized by unpaired vortices. While paired vortices are ineffective in screening vortex-vortex interactions—the screening length is infinite in the low-T phase—the unpaired vortices give rise to a finite screening length. This fact accounts for ξ being, respectively, infinite and finite at low and high T in the 2D XY model.[12]

The "SW plus vortices" approximation to the 2D XY model is mathematically represented by a 2D Coulomb gas[13] or, equivalently by the 2D sine-Gordon (SG) theory.[14] Renormalization-group (RG) analysis of either of these models shows[13,15] that ξ diverges exponentially as $T \to T_c$ from above. The free-energy density, f, has, correspondingly, an *essential* critical singularity[13]: f and all its derivatives are finite at T_c; the phase transition is therefore designated *infinite order.*[16]

Curiously enough, the pure SW approximation to the 2D XY model in a uniform magnetic field, h, is also described mathematically by the 2D SG theory[17] (see Appendix A); this approximation therefore predicts that in finite field the 2D XY model undergoes an infinite-order transition from a phase with $\xi = \infty$, finite M, and algebraic decay of correlations at high T[18] to one with finite ξ, finite M, and exponential decay at low T.[19] (See Fig. 3.) This is of course wrong physics: the 2D XY model *does not* undergo a transition in finite field. The error results from neglect of vortices. It is nonetheless intriguing that the mathematical model—the 2D SG theory—which correctly describes the infinite-order transition in the zero-field XY model emerges from as crude an approximation as SW theory when $h \neq 0$.

In this paper we study a 3D anisotropic SG theory [Eq. (3)] which is the SW approximation to the $n = m = d - 1 = 2$ LP problem in a uniform magnetic field, h. The field clearly favors the uniformly magnetized state over the helical one. For fixed h and sufficiently large and negative A (the parameter in Fig. 1) however, the system still undergoes a transition from the paramagnetic, uniformly magnetized phase to the helical one. The phase diagram of Fig. 4 results. Our concern here is with the paramagnetic-helical phase boundary. We show that in this boundary [which is a 2D surface in the 3D (T,A,h) space—Fig. 5] the 3D SG theory has (like its 2D isotropic counterpart) a line of infinite-order phase transitions separating a state with finite, uniform M, $\xi = \infty$, and power-law decay of correlations at high T from one with finite, uniform M, finite ξ, and exponential decay at low T. As this critical line, say $T = T_c(h)$ (Fig. 5), is approached from below along a line of constant h lying in the paramagnetic-helical boundary, ξ diverges as $\exp(\sigma t^{-\nu})$, where σ is a

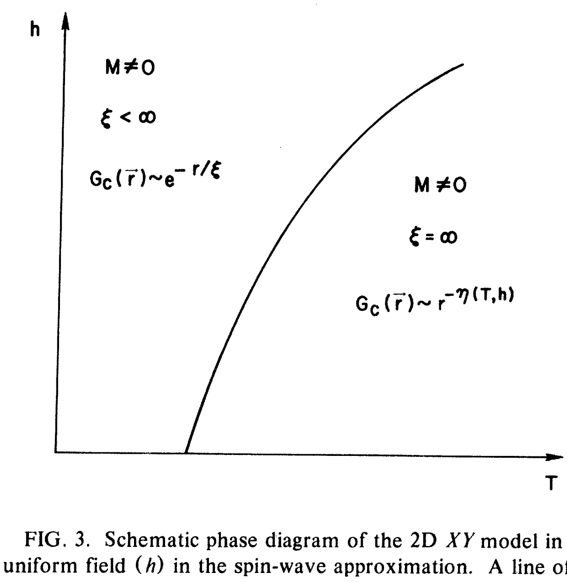

FIG. 3. Schematic phase diagram of the 2D XY model in uniform field (h) in the spin-wave approximation. A line of infinite-order transitions separates a high-T phase with finite magnetization and power-law decay of correlations from a low-T phase with finite magnetization and exponential decay of correlations.

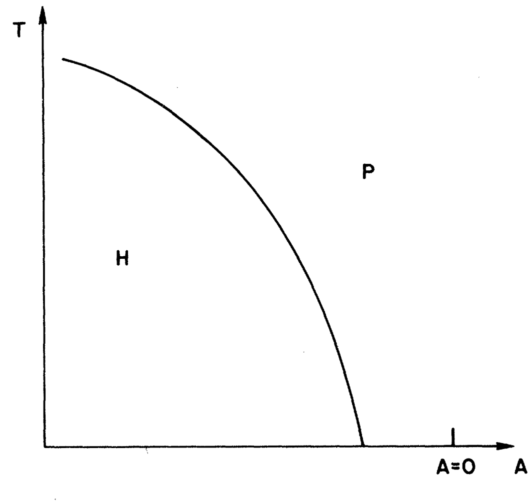

FIG. 4. Schematic phase diagram of the Lifshitz-point model in a finite uniform field, h. For large enough T the system is a uniformly magnetized paramagnet for all values of A, the parameter of Eq. (1). For smaller T the system undergoes a transition, for sufficiently negative A, into a helical state where the magnetization varies spatially.

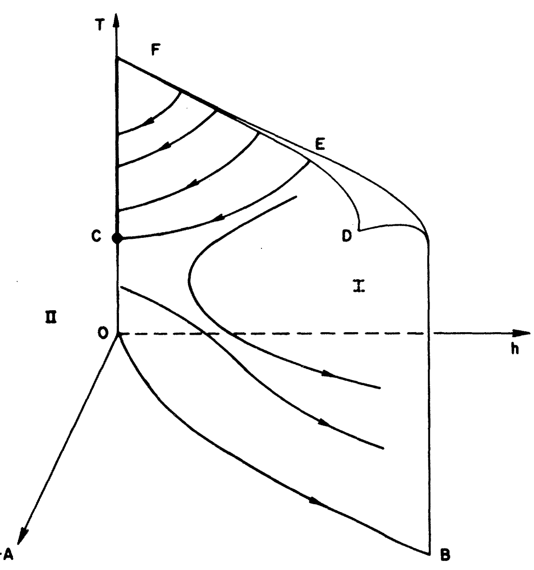

FIG. 5. Schematic phase diagram of the finite-field $n=m=d-1=2$ LP problem in (T,A,h) space. The surface *FOBD* separates a phase (I) (behind *FOBD*) of uniform magnetization from a phase (II) of nonuniform magnetization. In the surface *FOBD* RG flows, computed in the spin-wave approximation, are schematically sketched. The critical trajectory (or line) *CE* [described by $T=T_c(h)$] flows into the critical fixed point *C*. Points on *FOBD above CE* [i.e., $T>T_c(h)$] flow into the fixed line *CF*. Points below *CE*[$T<T_c(h)$] flow off to large values of h and A.

nonuniversal constant (it varies with h), $t\equiv T_c(h)-T$, and the exponent $\nu=1$, is constant (universal) along the critical line. As $h\to 0$ along a line of constant T lying in the boundary below the critical line, ξ diverges like $h^{1/4\delta^*}(\ln h)^{(1+x)/4\delta^*}$.[20] Here $x=-1.1\cdots$ is a universal number and δ^* is a negative function of T, nonuniversal in that it depends on the details of the ultraviolet cutoff prescription used in the calculation; $\delta^*\to 0$ as the critical line is approached from below.[21] The free-energy density behaves like ξ^{-d} near the critical line and so has the weak essential singularity familiar from the 2D problem.[13] At any point (T,A,h) lying in the boundary and above the critical line the connected spin-spin correlation function $G(\vec{r})$ decays at large distances like $r^{-\eta(h,T)}$ (apart from logarithmic corrections). The exponent η assumes the universal value four along the critical line.

The above results are derived through RG analysis of the anisotropic SG model. Though more complex in detail than the analogous 2D calculation, this analysis has many of the same features.[13] The portion of the paramagnetic-helical boundary above the critical line is controlled by a stable *fixed line* at $h=0$, $T_c(0)\leq T<\infty$ (Fig. 5). The critical fixed point at $h=0$, $T=T_c(0)$ controls the behavior along the entire critical line and is responsible for the universality of ν and η along that line. The fixed line at $h=0$ persists for $0\leq T<T_c(0)$, but is unstable with respect to the cosine operator of the SG theory. The major new element of the 3D RG analysis is the existence of *two* operators which are *marginal*[16] with respect to the critical fixed point: the cosine operator familiar from 2D and a *quartic* SW operator.[9] {In this respect the 3D problem is somewhat reminiscent of the 2D problem in the presence of a fourfold symmetry-breaking field, which is also marginal at the critical fixed point.[14(b)] The similarity is superficial, however, since the quartic spin-wave operator in 3D is marginal with respect to the *entire* fixed line whereas the fourfold symmetry-breaking operator in 2D is marginal only at the critical fixed point. This difference is of course manifest in the recursion relations for the two problems which [see Eq. (7a) and compare with Eq. (5.17c) of Ref. 14(b)] differ qualitatively.}

The SG theory is, we believe, the first model with purely short-ranged interactions to exhibit an infinite-order transition in 3D.[22] As an approximation to the LP problem in a field it is presumably as inadequate as is the 2D SG theory for describing the 2D XY model in a field. In both cases the predicted infinite-order transition is an artifact of the SW approximation, that is, of the neglect of topological defects. In the 2D case one knows[12] that the important omitted defects are pointlike objects—vortices—and that the 2D SG theory in fact gives a correct description of the 2D XY transition in *zero* field when they are properly treated. We can at this point make no such claim for the 3D LP problem, where not only the nature of the zero-field transition but the character of the relevant defects is unclear. The analysis presented here is at most suggestive that the transition at the LP in zero field is of infinite order.

The outline of this paper follows: Sec. II defines the general LP Hamiltonian, reviews the SW theory for the $n=m=d-1=2$ case in zero field, and shows the equivalence between the SW theory in finite field and the 3D SG model; Sec. III contains the RG analysis of the 3D SG system and the results; Sec. IV is devoted to discussion and conclusions; Appendix A briefly reviews, for comparison, the relation between the 2D SG theory and the 2D XY model with and without a magnetic field; Appendixes B–E contain technical details of the RG analysis.

II. $n=m=d-1=2$ LIFSHITZ POINT: SW THEORY

A. Definitions and zero-field SW theory

The general "m,n,d" LP problem is described[1] by the Hamiltonian

$$H_{\rm LP}=\frac{J}{2}\int d^dy\,[m_0^2(\nabla_\perp\vec{S})^2+A(\nabla_\parallel\vec{S})^2+(\nabla_\parallel^2\vec{S})^2]\ . \tag{1}$$

Here J is the exchange strength, the parameter A has dimension (mass)^2 and will assume both positive and negative values, the mass m_0 is inserted only to give all terms of (1) the same dimension, and the spin $\vec{S}$ has n components and fixed magnitude: $\vec{S}^2=m_0^{d-4}$. Of the d spatial dimensions m are arbitrarily denoted *parallel*, the remaining $d-m$ perpendicular:

$$(\nabla_\parallel\vec{S})^2\equiv\sum_{\alpha=1}^{n}\sum_{i=1}^{m}(\nabla_iS_\alpha)^2\ ,$$

$$(\nabla_\perp\vec{S})^2\equiv\sum_{\alpha=1}^{n}\sum_{i=m+1}^{d}(\nabla_iS_\alpha)^2\ ,$$

and

$$(\nabla_\parallel^2\vec{S})^2\equiv\sum_{\alpha=1}^{n}\left[\sum_{i=1}^{m}\nabla_i^2S_\alpha\right]^2\ .$$

Hamiltonian $H_{\rm LP}$ gives rise to the phase diagram of Fig. 1. The LP is the critical point connecting the paramagnetic ($\langle\vec{S}\rangle=0$), ferromagnetic [$\langle\vec{S}(\vec{q}=0)\rangle\neq0$], and helical [$\langle\vec{S}(q_\perp=0,q_\parallel\sim|A|^{1/2})\rangle\neq0$] phases.

Grest and Sak[5] pointed out that for $n\geqslant2$, $G_\perp(\vec{q})$, the transverse spin-spin correlation function on the phase boundary separating the helical and ferromagnetic phases behaves for small $\vec{q}$ like $[q_\perp^2+(q_\parallel^2)^2]^{-1}$ whenever long-range order exists on the boundary. Thus

$$G_\perp(\vec{x}=0)\sim\int d^mq_\parallel\,d^{d-m}q_\perp\,G_\perp(\vec{q})$$

diverges at small $\vec{q}$ for $d\leqslant2+m/2$; for $m=2$ there can therefore be no long-range order on the phase boundary in 3D.[23] The obvious inference that for $n\geqslant2$ the $m=d-1=2$ problem has no LP at finite T (rather a phase diagram like Fig. 2) is confirmed for $n\geqslant3$ by calculations[5] in $3+\epsilon$ dimensions which also indicate the existence of a LP at finite T for $n=2$ (i.e., a phase diagram like Fig. 1).

The SW theory for $n=m=d-1=2$ is constructed, as in the 2D XY case,[11,16] by the substitution

$$S_x+iS_y=m_0^{-1/2}\exp(i\beta_0m_0^{1/2}\tilde{\theta})\ ,$$

with $\beta_0^2\equiv k_BT/J$; the partition function $T_r\exp(-H_{\rm LP}/k_BT)$ becomes $T_r\exp(-H_{\rm SW})$, where

$$H_{\rm SW}=\frac{1}{2}\int d^3y\,\{m_0^2(\nabla_\perp\tilde{\theta})^2+A(\nabla_\parallel\tilde{\theta})^2+(\nabla_\parallel^2\tilde{\theta})^2+2u_0m_0[(\nabla_\parallel\tilde{\theta})^2]^2\}\ , \tag{2}$$

$2u_0\equiv k_BT/J$, and $\tilde{\theta}$ runs from $-\infty$ to ∞ in the trace. This model possesses the phase diagram of Fig. 6: there is, as usual in SW theory,[11] no paramagnetic phase. The phase boundary between the helical and ferromagnetic phases has been studied[9] with RG techniques. There is, of course, no long-range order on this boundary. The correlation length ξ is infinite and spin correlations decay as $r^{-\eta}(\ln r)^{-\bar{\eta}}$ at large r,

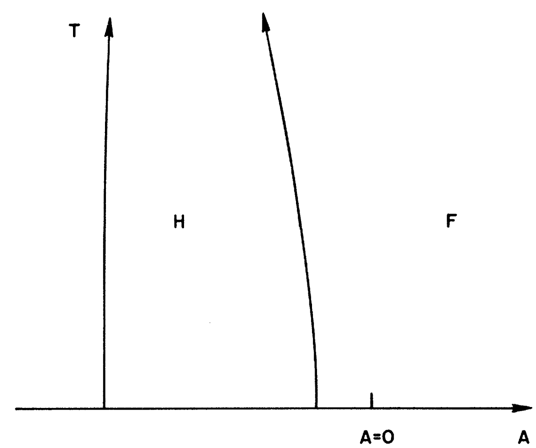

FIG. 6. Schematic phase diagram of the spin-wave approximation to the $n=m=d-1=2$ Lifshitz-point model in zero magnetic field. An infinitely long phase boundary characterized by the absence of long-range order, an infinite correlation length, and power-law decay of correlations separates a ferromagnetic phase (F) from a helical phase (H).

where η and $\tilde{\eta}$ are functions of T. Valid at low T,[11] the SW approximation provides no information about the transition into the paramagnetic phase at the LP.[24]

B. SW theory for $h \neq 0$: 3D SG model

In the presence of a uniform field term, $h(k_B T)^{-1} \int d^3y\, S_x(\vec{y})$, $H_{\rm LP}/k_B T$ takes the form

$$H_{\rm SG} = \int d^3y \Bigg[m_0^2 \frac{(\nabla_\perp \tilde{\theta})^2}{2} + \frac{A}{2}(\nabla_\parallel \tilde{\theta})^2 + \tfrac{1}{2}(\nabla_\parallel^2 \tilde{\theta})^2 + u_0 m_0 [(\nabla_\parallel \tilde{\theta})^2]^2 - \alpha_0 a^{-4} m_0^{-1} \beta_0^{-2} \cos(\beta_0 m_0^{1/2} \tilde{\theta}) \Bigg] \tag{3}$$

in the SW approximation, where a is the inverse of the ultraviolet cutoff, Λ, introduced to prevent divergences in perturbation theory, and the dimensionless parameter $\alpha_0 \equiv m_0^{1/2} a^4 h/J$.[25] The magnetization, M, and connected spin-spin correlation function, $\tilde{G}(\vec{y})$, are, respectively, defined by $\langle \exp(i\beta_0 m_0^{1/2} \tilde{\theta}) \rangle$ and $(\langle \exp\{ i\beta_0 m_0^{1/2} [\tilde{\theta}(y) - \tilde{\theta}(0)] \} \rangle - M^2)$, expectation values being computed with $H_{\rm SG}$. Hamiltonian (3) is, aside from the $[(\nabla_\parallel \tilde{\theta})^2]^2$ term, an obvious generalization of the familiar isotropic SG Hamiltonian to anisotropic situations wherein the order parameter may vary spatially in two (the "parallel") directions. The quartic SW operator $[(\nabla_\parallel \tilde{\theta})^2]^2$ has no analog in the usual 2D isotropic SG theory. The necessity for its inclusion in (3) is shown in the following section, where we point out that even if it is omitted from the original Hamiltonian it is generated under RG analysis. The *marginality*[16] of this operator with respect to the fixed line of the 3D SG problem with $\alpha_0 = 0$ has already been shown.[9]

III. RG ANALYSIS OF THE 3D SG MODEL

A. Renormalization program and dimensional considerations

It is convenient, as in the 2D SG case,[26] to employ the field-theoretic RG techniques of Brézin *et al.*[27] in analyzing $H_{\rm SG}$. The analysis is facilitated by addition to $H_{\rm SG}$ of a mass term $\frac{1}{2}\mu_0^4 \tilde{\phi}^2$, which suppresses infrared divergences in perturbation theory.[26] If the resulting model is multiplicatively renormalizable,[27] that is, if certain correlation functions can be rendered ultraviolet finite in each order of perturbation theory through renormalization of $\tilde{\theta}$ (wave-function renormalization), u_0, α_0, β_0, and m_0, one can write[27] RG equations for those correlation functions. The equations are, as we shall see, finite in the limit $\mu_0 \to 0$.[26] Specifically, a correlation function, $H(\vec{x}) \equiv \langle O(\vec{x}) O(\vec{0}) \rangle$, for some operator $O(\vec{x})$, a function of $\tilde{\theta}(\vec{x})$ and its derivatives, is multiplicatively renormalizable if there exist dimensionless renormalization constants Z, Z_u, Z_α, Z_ϕ, and Z_m such that the product $ZH(\vec{x}, u_0, \alpha_0, \beta_0, m_0, \mu_0, \Lambda)$ remains finite in the limit where the ultraviolet cutoff $\Lambda \to \infty$ when expressed in terms of the renormalized variables u, α, β, m, and μ defined by[26]

$$\alpha_0 = Z_\alpha \alpha \ , \tag{4a}$$

$$u_0 = Z_u u \ , \tag{4b}$$

$$m_0 = Z_m m \ , \tag{4c}$$

$$\beta_0^2 = (Z_\phi Z_m)^{-1} \beta^2 \ , \tag{4d}$$

$$\mu_0 = Z_\phi^{-1/4} \mu \ . \tag{4e}$$

All Z's are functions of u, α, β, Λ/m, and Λ/μ. Since our interest is the boundary between the uniform (paramagnetic) and helical phases (Fig. 5) we choose the parameter A [Eq. (3)], order by order in perturbation theory in u, and α, to locate this boundary; A is therefore treated as a renormalization constant, not as an independent coupling constant.

When $\delta \equiv -1 + \beta^2/32\pi$ is negative one can show by simple power counting[26,28] that the theory described by (3) is renormalizable. For $\delta \geq 0$, new divergences which become increasingly severe as δ increases appear and actually render the theory nonrenormalizable.[26,29] As in the 2D case,[26] however, we assume that the theory is renormalizable *order by order* in a *triple* power series in u, α, and δ,[30] verifying the assumption explicitly to second order in these parameters by computing the two- and four-point vertex functions and hence the Z's. The calculation is simplified by transformation to a new field $\theta(\vec{x})$ and new coordinates $\vec{x}$ defined by

$$\tilde{\theta}(\vec{y}_\parallel, y_\perp) = m_0^{-1/2} \theta(\vec{x}_\parallel, x_\perp) \ ,$$

$$\vec{y}_\parallel = \vec{x}_\parallel, \quad y_\perp = m_0 x_\perp \ .$$

In terms of $\theta(\vec{x})$ (note $d\vec{y} = m_0 d\vec{x}$) Hamiltonian (3), supplemented by the term $\mu_0^4 \tilde{\theta}^2/2$, takes the form

$$H_{\rm SG} = \int d\vec{x} \Bigg[\frac{(\nabla_\perp \theta)^2}{2} + \frac{A}{2}(\nabla_\parallel \theta)^2 + \frac{(\nabla_\parallel^2 \theta)^2}{2} + \frac{\mu_0^4 \theta^2}{2} + u_0 [(\nabla_\parallel \theta)^2]^2 - \alpha_0 a^{-4} \beta_0^{-2} \cos(\beta_0 \theta) \Bigg] ; \tag{5}$$

all explicit dependence of $H_{\rm SG}$ on m_0 has thus been removed. Since the two-point correlation functions $g(\vec{x}) \equiv \langle \theta(\vec{x}) \theta(\vec{0}) \rangle$ and $\tilde{g}(\vec{x}) \equiv \langle \tilde{\theta}(\vec{x}) \tilde{\theta}(\vec{0}) \rangle$ are related by $g(\vec{x}_\parallel, x_\perp) = m_0 \tilde{g}(\vec{x}_\parallel, m_0 x_\perp)$, their respective Fourier transforms are related by $g(\vec{p}_{\parallel},p_\perp)=\tilde{g}(\vec{p}_{\parallel},p_\perp/m_0)$. Note that $g(\vec{p}_{\parallel},p_\perp)$ is completely independent of m_0. Since renormalizability of the two-point function was defined as the existence of Z's such that in the limit $\Lambda\to\infty$, $Z_\phi^{-1}\tilde{g}(\vec{p}_{\parallel},p_\perp,u_0,\alpha_0,\beta_0,m_0,\mu_0)$ is finite, order by order in u, α, and δ, when expressed in terms of $\vec{p}_{\parallel}$, $p_\perp$, u, α, δ, m, and μ [see (4)], it implies that $Z_\phi^{-1}g(\vec{p}_{\parallel},Z_m p_\perp,Z_u u,Z_\alpha\alpha,(Z_\phi Z_m)^{-1/2}\beta,Z_\phi^{-1/4}\mu)$ is finite, order by order in u, α, and δ, when expressed in terms of $\vec{p}_{\parallel}$, $p_\perp$, u, α, δ, and μ. Transformation from $\tilde{\theta}$ to θ thus conveniently eliminates m_0, the renormalization of m_0 being replaced by a renormalization of $p_\perp$.[31] (Analogous statements for four-point or higher functions follow similarly.)

B. RG equations: The fixed line

The explicit computations with Hamiltonian (5) of $g^{-1}(\vec{p})$ and Γ_4, the four-point vertex, to second order in u, α, and δ are straightforward; they are summarized in Appendix B, where first-order expressions for the Z's which render the theory finite are given. The correlation function of real interest is the connected "spin-spin" function, $G(\vec{x})\equiv\langle\exp\{i\beta_0[\theta(\vec{x})-\theta(\vec{0})]\}\rangle_c$, discussed just below Eq. (3). It is tedious but routine to verify the order-by-order renormalizability of G to second order and compute the renormalization constant Z_s such that Z_sG is finite. An outline of the computation is given in Appendix C. Strictly speaking it is not $G(\vec{x})$ which is order-by-order renormalizable but $\mathcal{G}(\vec{x})$, a specific linear combination [see Eq. (C5)], of $G(\vec{x})$ and correlation functions involving all other operators, such as $(\nabla_\parallel\theta)^2$, $(\nabla_\perp\theta)^2$, and $(\nabla_\parallel^2\theta)^2$, with naive dimension less than or equal to 4.[27] It follows that $\mathcal{G}(\vec{x})$ satisfies the RG equation[27]

$$\left[\frac{\partial}{\partial l}+\beta_u\frac{\partial}{\partial u_0}+\beta_\alpha\frac{\partial}{\partial\alpha_0}+\beta_\delta\frac{\partial}{\partial\delta_0}+\gamma(u_0,\alpha_0,\delta_0)\right]\times\mathcal{G}(\vec{x}_\parallel,u_0,\alpha_0,\delta_0,\Lambda)=0\ , \quad (6)$$

where $l\equiv\ln\Lambda$, μ_0 has been sent to zero,[26,32] and we have, for simplicity, set $\vec{x}=(\vec{x}_\parallel,x_\perp=0)$. The β's and γ are functions of u_0, α_0, and δ_0 (Appendixes B and C):

$$\beta_u\equiv\partial u_0/\partial l|_{\alpha,u,\beta}=9u_0^2/4\pi+4^{-7}\pi J_2e^{4C}\alpha_0^2+\cdots\ , \quad (7a)$$

$$\beta_\alpha\equiv\partial\alpha_0/\partial l|_{\alpha,u,\beta}=4\alpha_0\delta_0+16\ln(\tfrac{32}{27})\alpha_0u_0/\pi+\cdots\ , \quad (7b)$$

$$\beta_\delta\equiv\partial\delta_0/\partial l|_{\alpha,u,\beta}=4^{-7}e^{4C}(J_1-J_2/32)\alpha_0^2+[\ln(\tfrac{4}{3})-\tfrac{7}{24}]u_0^2/(2\pi^2)+\cdots\ , \quad (7c)$$

$$\gamma\equiv\partial\ln Z_s/\partial l|_{\alpha,u,\beta}=8(1+\delta_0)+32\ln(\tfrac{32}{27})u_0/\pi+\cdots\ , \quad (7d)$$

where C is Euler's constant, $e^C\equiv1.781\ldots$, and

$$J_i\equiv\int_0^\infty dt\,t^{-2+2i}\exp\left[4\int_0^{t/4}(e^{-y}-1)\,dy/y\right]$$

for $i=1,2$.

The solution of (6) is well known[28]; noting that $\mathcal{G}$, being dimensionless, is a function only of $\Lambda x_\parallel$, u_0, α_0, and δ_0 one has

$$\mathcal{G}(x_\parallel\Lambda,u_0,\alpha_0,\delta_0)\sim\exp\left[-\int_0^l\gamma(\tilde{u},\tilde{\alpha},\tilde{\delta})\,d\tau\right]\ , \quad (8)$$

where $\tilde{u}$, $\tilde{\alpha}$, and $\tilde{\delta}$ are functions of u_0, α_0, δ_0, and τ determined by the RG flow equations[27,28]

$$\frac{\partial\tilde{u}}{\partial\tau}=-\beta_u(\tilde{u},\tilde{\alpha},\tilde{\delta})\ , \quad (9a)$$

$$\frac{\partial\tilde{\alpha}}{\partial\tau}=-\beta_\alpha(\tilde{u},\tilde{\alpha},\tilde{\delta})\ , \quad (9b)$$

$$\frac{\partial\tilde{\delta}}{\partial\tau}=-\beta_\delta(\tilde{u},\tilde{\alpha},\tilde{\delta})\ , \quad (9c)$$

and the boundary conditions $\tilde{u}(\tau=0)=u_0$, $\tilde{\alpha}(\tau=0)=\alpha_0$, and $\tilde{\delta}(\tau=0)=\delta_0$. Note that Eqs. (7a) and (9a) imply the necessity of including the $[(\nabla_\parallel\theta)^2]^2$ term in (5); even if $u_0=\tilde{u}(\tau=0)=0$, Eq. (9a) shows that $\tilde{u}$ does not remain zero for subsequent τ's unless $\alpha_0=0$.

Equations (9) have, according to (7), an obvious line of fixed points (i.e., simultaneous zeros of the three β functions) at $\tilde{u}=\tilde{\alpha}=0$. The fixed line is stable as $\tau\to\infty$ for $\tilde{\delta}>0$ (Ref. 33) and unstable for $\tilde{\delta}<0$, a reflection of the respective *irrelevance* and *relevance*[29] of the cosine operator for $\tilde{\delta}$ positive and negative. The (u_0,α_0,δ_0) space thus divides into two distinct regions separated by a 2D critical surface which includes the point $u_0=\alpha_0=\delta_0=0$ (Fig. 7). All points in region I of that figure flow under RG transformation to some point on the stable portion of the fixed line. Since the line $(u_0=\alpha_0=0)$ represents [see Eq. (5)] Gaussian SW theory wherein spin correlations decay algebraically[9,11,34] at large distance, one infers that $\mathcal{G}(\vec{x})$ decays as $|\vec{x}|^{-\eta}$ (aside from logarithmic corrections[9]) in region I. The correlation length is therefore everywhere infinite in this region. Since the value of η depends on which precise point of the critical line is reached under RG flow as $l\to\infty$, η is a continuous function of the starting point (u_0,α_0,δ_0) in region I.

In region II the RG flows do not terminate in any fixed point accessible to perturbative analysis; one concludes,[13,26] as in the 2D case, that correlations decay exponentially in this region: ξ is finite. (Being more careful we should emphasize that we have not proved ξ finite in region II. It is possible that the RG flows in this region end in a fixed point or line with $\xi=\infty$. It seems most reasonable, however, to assume that flows which do not terminate in the fixed line $\tilde{\alpha}=\tilde{u}=0$ correspond to finite ξ.)

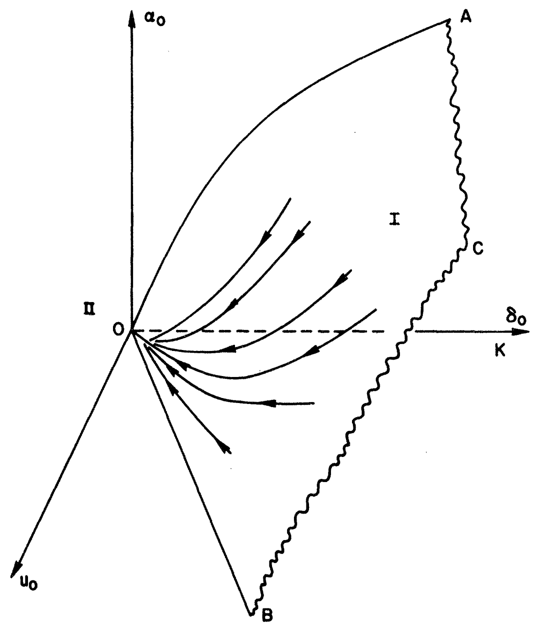

FIG. 7. Schematic drawing of RG trajectories for the anisotropic 3D SG theory in $(\alpha_0, u_0, \delta_0)$ space. Critical surface *AOBC* divides the space into two regions. In region I (behind *AOBC* in the figure) flows terminate in the fixed line *OK*. In region II flows go off to large values of the coupling constants. Points in the critical surface flow (as shown) into the critical fixed point $u_0 = \alpha_0 = \delta_0 = 0$.

C. Critical properties

Points on the critical surface flow to the critical fixed point $\tilde{u} = \tilde{\alpha} = \tilde{\delta} = 0$. [Note that, according to the RG Eqs. (7) and (9), both $\tilde{u}$ and $\tilde{\alpha}$ are coefficients of operators *marginal*[9,27,29] with respect to the critical fixed point. This behavior contrasts with the 2D situation where only a single marginal interaction term,[35] the cosine operator, occurs.] Though the RG Eqs. (9) are, even in the critical surface, difficult to solve explicitly, the obvious guess[13] is that the asymptotic approach to the critical fixed point as $\tau \to \infty$ is describable by $\tilde{u} \sim A_u/\tau$, $\tilde{\delta} \sim A_\delta/\tau$, and $\tilde{\alpha} \sim A_\alpha \tau^x$ for some numbers A_u, A_α, and A_δ and some exponent x. Substitution of this ansatz into (9) yields (Appendix D) $A_u \sim 1.39\ldots$, $A_\delta \sim -0.007\cdots$, $x = -1.1\ldots$, A_α is arbitrary. These numbers, together with expression (7d) for γ determine the large $-|\vec{x}|$ behavior of $\mathcal{G}(\vec{x})$ on the critical surface: Eq. (8) yields

$$\mathcal{G}(x_\parallel) \sim \exp\left[-8\int^{l}\{1+[A_\delta + 4\pi^{-1}\ln(\tfrac{32}{27})A_u]\tau^{-1}\}\,d\tau\right]$$

$$\sim (\Lambda x_\parallel)^{-8}[\ln(\Lambda x_\parallel)]^{-8A_\delta - 32\pi^{-1}A_u \ln(32/27)} \qquad (10)$$

for $\Lambda x_\parallel >> 1$ This result is universal—valid for *any* point on the critical surface. For $x_\perp \neq x_\parallel = 0$, Eq. (10) with $x_\parallel$ replaced by $x_\perp^{1/2}$ is similarly obtained.

The LP spin-correlation function $G(\vec{x})$ is a linear combination of $\mathcal{G}(\vec{x})$ and other connected correlation functions, one of which [see Eq. (C5)] is $L(\vec{x}) \equiv \langle [\nabla_\parallel \theta(\vec{x})]^2 [\nabla_\parallel \theta(\vec{0})]^2 \rangle_c$. Since $(\nabla_\parallel \theta)^2$ has naive dimension[27] two whereas all other operators involved in the linear combination have naive dimension four, the large-distance behavior of $G(\vec{x})$ is given by $L(\vec{x})$. In Appendix C it is shown that everywhere on the critical surface,

$$G(\vec{x}_\parallel) \sim (\Lambda x_\parallel)^{-4}[\ln(\Lambda x_\parallel)]^{-2A_u/\pi} \qquad (11)$$

for $\Lambda x_\parallel >> 1$.

The divergence of ξ as the critical surface is approached from region II is easily computed by linearization of the RG equations about the asymptotic critical trajectory. Details are relegated to Appendix D, where it is shown that

$$\xi \sim \exp(\sigma/d^\nu), \quad \nu = 1 \ , \qquad (12)$$

where d represents a small distance from an arbitrary point on the critical surface in $(u_0, \alpha_0, \delta_0)$ space, ν is a universal exponent characterizing the critical surface, and σ is a nonuniversal number which varies from point to point on the critical surface. The exponential divergence of ξ implies that the transition at any point on the critical surface is of infinite order.[13,16]

As α_0 approaches zero in region II the power-law decays characteristic of Hamiltonian (2) must be recovered. It follows that ξ must diverge in this limit, though the divergence is a trivial one, not a manifestation of real critical behavior.[36] The result

$$\xi \sim \alpha_0^{(4\delta^*)^{-1}}[(\ln\alpha_0)/4\delta^*]^{(1+x)(4\delta^*)^{-1}} \ ,$$

valid as $\alpha_0/\delta^* \to 0$, is derived in Appendix E. Here $x = -1.1\cdots$ and δ^* is a function (*negative* in region II) of δ_0 and u_0; $\delta^* \to 0$ as the critical fixed point $u_0 = \delta_0 = \alpha_0 = 0$ is approached. This divergence of the exponent $(4\delta^*)^{-1}$ at the critical fixed point signals a crossover from the power-law divergence of ξ as $\alpha_0 \to 0$ in region II to the exponential divergence of ξ along the critical surface.[36]

Since u_0, α_0, and δ_0 are related to J/k_BT and h/k_BT, the two variables of the original LP problem, by $u_0 = 32\pi(1+\delta_0) = k_BT/J$ and $\alpha_0 = m_0^{1/2}a^4h/J$ the plane $u_0 = 32\pi(1+\delta_0)$ is the locus of points in $(u_0, \alpha_0, \delta_0)$ space corresponding to all possible values of the original LP variables. The intersection of this plane with the critical surface (Fig. 8) defines the critical line $T = T_c(h)$ of the LP problem.

On the critical line the decay of the LP spin-correlation function at large distance is given by (11).

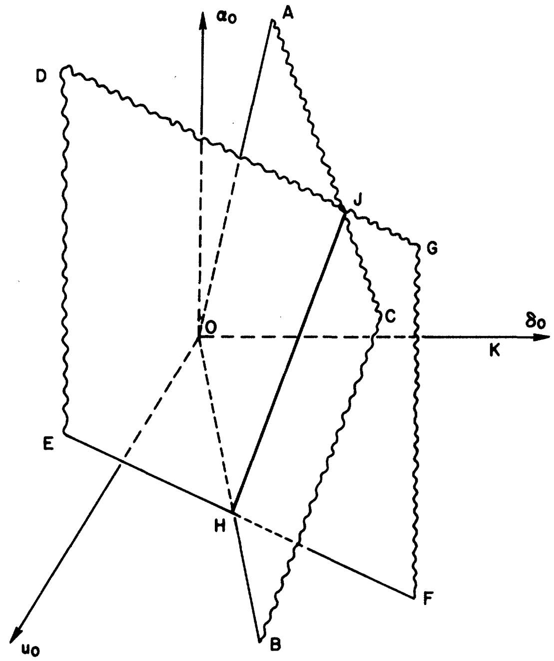

FIG. 8. Schematic drawing of the critical surface $(AOBC)$ of the 3D SG theory in $(\alpha_0, u_0, \delta_0)$ space. Points on this surface flow into the critical fixed point O under RG iteration. Points to the right of the surface flow into the critical fixed line OK. Surface $DEFG$ represents the locus of points corresponding to all possible values of the temperature and magnetic field of the $m=n=2$ LP problem on the boundary between the paramagnetic and helical regions. The intersection (HJ) of this locus and the critical surface defines the critical line of the LP problem.

As the critical line is approached from the low-temperature side ξ diverges like $\exp(\sigma/t^{\nu})$, where $t\equiv T_c(h)-T$.

IV. DISCUSSION

A. LP with $h\neq 0$

It is worth belaboring the relationship between the behavior of the 3D SG theory presented here and the real physics of the $n=m=d-1=2$ LP problem in a uniform field. While the phase diagram of the LP problem with $h\neq 0$ presumably does have the two phases shown in Fig. 4, there is no reason to believe there really is a line of critical points (infinite order or otherwise) in the 2D boundary (Fig. 5) separating the phases. Since the 3D theory describes only the SW approximation to the LP problem with $h\neq 0$, the critical line predicted by that theory need not be present in the complete LP problem. There is ample precedent in 2D for this kind of unreliability of the SW approximation. It predicts (see, e.g., Appendix A) that the 2D XY model in a field has a line of infinite-order transitions.

The 3D SG theory is interesting not because it solves the LP problem for $h\neq 0$ but because it is to our knowledge the first model[24] with any relationship to a real physical system[22] to exhibit an infinite-order[16] transition in 3D.

B. Liquid-crystal realization

That bulk liquid-crystal mixtures whose phase diagrams have (Fig. 9) nematic (N), smectic-A (A), and smectic-C (C) phases constitute a perfect realization of the zero-field $n=m=d-1=2$ LP problem has been noted by several authors.[2,6,7] The N, A, and C phases are respective analogs of the paramagnetic, ferromagnetic, and helical phases of magnetic LP terminology. The formal analogy is only perfect when a uniform magnetic field (tending to align the directors) is applied to the liquid-crystal system. In the absence of this field the Landau-Peierls theorem[37] shows that neither the A nor the C phase can have true positional long-range order (layers)[38]; this complicates the analogy since the 3D LP model [Eq. (1)] is constructed so as to have long-range order in the ferromagnetic and helical phases. In prin-

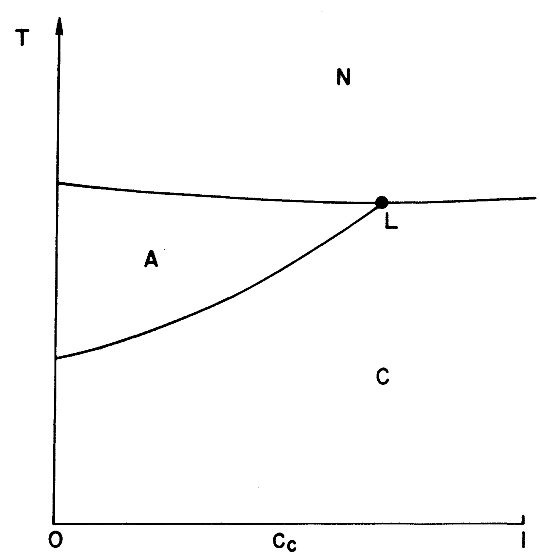

FIG. 9. Schematic phase diagram of mixtures of two different types of liquid crystals showing nematic (N), smectic-A (A), and smectic-C (C) phases meeting at a LP (L). Here C_c is the concentration of the component which undergoes a N-C transition as the temperature is lowered. The other component undergoes both a N-A and an A-C transition.

ciple[39] an arbitrarily weak field restores the long-range order in the A and C phases and resuscitates the analogy.

Since there can be no long-range order on the ferromagnetic-helical phase boundary in the $n=m=d-1=2$ LP problem, there can be no long-range (layered) positional order on the boundary between the A and C phases,[5,8,9] even in an arbitrarily strong magnetic field where the layered order in the A and C phases is well established. As the boundary is approached from either side the layers are destroyed by fluctuations: the Bragg peaks in the density correlation function must turn to power laws.[8,9]

The nature of the transition at the LP in the *NAC* system (say in finite field) is unknown. The nonexistence of long-range order on the A-C boundary just below this point argues by analogy—more seductive than substantive—to the 2D XY model[12] that the transition may be infinite order.[13,40] The fact that the (however tenuously) related anisotropic 3D SG model does indeed exhibit such a transition strengthens the allure of this idea, which also presents formidable difficulties: the relevant topological defects in 3D are presumably lines or rings. It is thoroughly unclear that such objects can, like their 2D pointlike counterparts, mediate an infinite-order (or any) transition, let alone be properly described by the 3D SG theory. Further experiments may help resolve the question of the order of the transition at the LP.

C. 3D SG theory and 3D logarithmic gases

The 2D SG theory is mathematically equivalent to the 2D Coulomb gas—a gas of point charges (or vortices) which mediate the transition in the 2D XY model.[12,13] Unfortunately, the "logarithmic gas" representation of the anisotropic 3D SG theory provides no clue as to which 3D topological defects might mediate the transition at the $n=m=d-1=2$ LP.

With $u_0=A=\mu_0=0$ in (5) an expansion of the partition function $Z_{SG}=T_r e^{-H_{SG}}$ in powers of α_0 (Ref. 41) can indeed be interpreted (as in the 2D case[14]) as the grand canonical partition function for an *anisotropic* 3D logarithmic gas—an overall neutral system composed of positively and negatively charged point particles interacting with anisotropic logarithmic interactions. Unfortunately this 3D anisotropic logarithmic gas[22] does *not* undergo a phase transition. To see this, recall from Sec. III that for given α_0 one must choose the parameter $A=A(\alpha_0)$ to locate the paramagnetic-helical phase boundary where the infinite-order transition occurs. Moreover, A is always negative on this boundary. Enforcing $A=0$ moves one off the phase boundary into a single (the uniform) phase. Since the anisotropic logarithmic gas is equivalent to the 3D SG model with $A=0$, one concludes that the anisotropic gas does not experience a transition[22]: it is always in the "plasma" phase. For A negative the expansion of Z_{SG} in powers of α_0 breaks down; the connection to the logarithmic gas is lost. We conclude that the infinite-order transition in the anisotropic 3D SG theory has no obvious interpretation as a "metal-insulator" or "plasma-dielectric" phase transition in a system of charged, interacting particles.[42]

ACKNOWLEDGMENTS

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APPENDIX A: 2D XY MODEL AND SG THEORY: A REVIEW

The 2D XY model in a uniform magnetic field h is defined by the Hamiltonian

$$\frac{H}{k_B T}=-K\sum_{\langle ij\rangle}\vec{S}_i\cdot\vec{S}_j-\frac{h}{k_B T}\sum_i S_i^x\ , \tag{A1}$$

where the classical spin $\vec{S}_i$ at each site i of a 2D lattice has unit magnitude and is free to rotate in a (say the XY) plane. Nearest-neighbor spins interact with exchange strength $J\equiv k_B TK$.

A. SW theory (Ref. 11): $h=0$

With $h=0$ (A1) can be written $-K\sum_{\langle ij\rangle}\times\cos(\theta_i-\theta_j)$ with the substitution $S_i^x+iS_i^y=e^{i\theta_i}$. Here $-\pi<\theta_i\leq\pi$ for each i. At low temperatures, $K>>1$, the spins on nearest-neighbor sites are very nearly aligned; one can write $\cos(\theta_i-\theta_j)\sim 1-(\theta_i-\theta_j)^2/2$ and let each θ_i run from $-\infty$ to ∞ with negligible error. The resulting Gaussian SW approximation is easily solved.[11] The spin-spin correlation function $G(\vec{x})=\langle\exp\{i[\theta(\vec{x})-\theta(\vec{0})]\}\rangle$ decays like $|\vec{x}|^{-(2\pi K)^{-1}}$ at large $|\vec{x}|$; $\xi=\infty$, therefore. This result is correct for $K>>1$ and wrong for $K<<1$ where one expects the finite ξ and exponential decays of an orthodox paramagnetic state which the SW theory is too crude to describe.

B. SW plus vortices: $h=0$

The addition of vortices to the SW approximation provides a complete description of the 2D XY phase transition.[12,13] Since the vortex-vortex interaction is logarithmic the vortex degrees of freedom can be mathematically represented as a 2D Coulomb gas[12,13]—a system of positive and negative charges with overall charge neutrality interacting via the (logarithmic) 2D Coulomb potential. The 2D Coulomb gas is in turn mathematically equivalent to the 2D SG theory.[14] The "SW plus vortices" approximation is thus describable by the partition function $Z_{SG} = \mathrm{Tr}\exp(-\mathcal{H}_{SG})$, where

$$\mathcal{H}_{SG} = \int d^2x\,[\tfrac{1}{2}(\nabla\varphi)^2 - \alpha\beta^{-2}a^{-2}\cos\beta\varphi] \qquad (A2)$$

is the classical 2D SG Hamiltonian. Here a is the short-distance cutoff of the associated 2D Coulomb interaction, $\varphi(\vec{r})$ runs from $-\infty$ to ∞ for each $\vec{r}$, $\beta^2 \equiv (2\pi)^2K$, and α (proportional to the fugacity in the Coulomb gas representation) is a complicated function of K not known in closed form. This function is schematically shown as the dotted line in Fig. 10.

The free energy, F, and spin-spin correlation function, $G(\vec{x})$, of the 2D XY model are, respectively, given by[15,26,35]

$$F = -k_BT\ln\mathrm{Tr}\exp(-\mathcal{H}_{SG}) \;,$$
$$G(\vec{x}) = \left\langle \exp\left[\frac{2\pi}{\beta}\left(\int_{-\infty}^{x_1}\phi_2(z,x_2)\,dz - \int_{-\infty}^{0}\phi_2(z,0)\,dz\right)\right]\right\rangle \;, \qquad (A3)$$

in the SG representation. Here $\vec{x} = (x_1,x_2)$ and $\varphi_2(x_1,x_2) \equiv \partial\phi/\partial x_2$.

C. RG analysis of the 2D SG theory

Hamiltonian (A2) has been analyzed with standard RG techniques.[13,15,26] The resulting RG trajectories in the (α,β^2) plane are schematically drawn in Fig. 10. Each temperature (or value of K) in the XY model corresponds to a point in the (α,β^2) plane. The locus of these points for all possible values of K is schematically shown as the dashed line in the figure. Trajectory (separatrix) OA divides the figure naturally into two regions. In region I, the low-T regime (recall $\beta^2 \sim K$), flows terminate in a fixed line at $\alpha = 0$. Since when $\alpha = 0$ $\mathcal{H}_{SG}$ becomes a pure SW Hamiltonian[11] this region (and hence the low-T phase of the 2D XY model) is characterized by the algebraic decay of correlations, $G(\vec{x}) \sim |\vec{x}|^{-\eta(\alpha,\beta)}$, and the infinite ξ predicted by SW theory.

For $\beta^2 < 8\pi$ the fixed line at $\alpha = 0$ is no longer stable; region II (the high-T phase of the XY model) is not, therefore, described by SW theory. Finite ξ and consequent exponential decay of correlations characterize this region. As any point on the critical line is approached from a distance d, ξ diverges[13,15,26] like $e^{ad^{-1/2}}$ as $d \to 0$, where the number a depends on

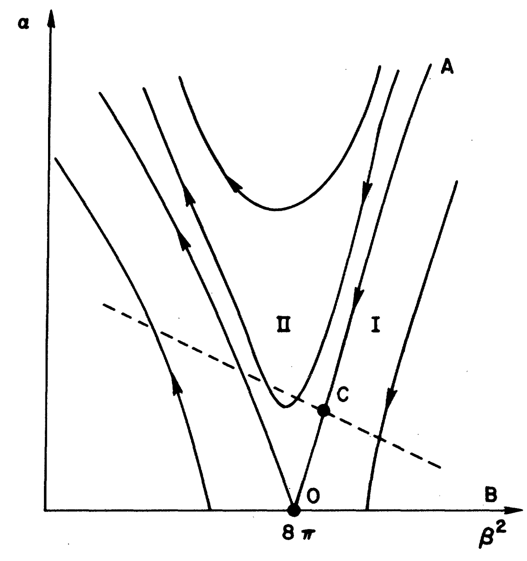

FIG. 10. Schematic diagram of RG trajectories of 2D SG model [Eq. (A2)]. Critical trajectory AO divides the diagram into two phases or regions (I and II). Trajectories in region I flow into the stable fixed line OB. Trajectories in region II flow off to large values of the coupling constant α. The dashed curve schematically represents the locus of points corresponding to different values of temperature in the zero-field 2D XY model. The intersection of this curve and the critical trajectory OA is C, the critical point of the XY model. Starting at C one flows into the critical fixed point, O.

the point in question. In particular ξ diverges as $\exp(at^{-1/2})$ as the critical point c (Fig. 10) of the model is approached along the dotted line from the high-T side. The free energy, whose singular part behaves like ξ^{-d} thus has only an unobservably weak essential singularity[13] on the critical line; the transition is therefore of "infinite order."

The critical fixed point O (Fig. 10) controls the decay of correlations at all points on the critical line OA. One finds a critical exponent η of $\frac{1}{4}$ along OA, where[13,26]

$$G(\vec{x}) \sim x^{-1/4}(\ln x)^{1/8}\left[1 + \frac{1}{16}\frac{\ln(\ln x)}{\ln x}\right]$$

for large x.

Finally, it is straightforward to show[35] that the interactions neglected in deriving the SG representation of the 2D XY model are *irrelevant* in the usual RG sense.[16]

D. SW theory: Finite h

For $h \neq 0$ (A1) becomes[17]

$$\frac{H}{k_BT} = +\frac{K}{2}\sum_{\langle ij\rangle}(\theta_i - \theta_j)^2 - \frac{h}{k_BT}\sum_i\cos\theta_i \qquad (A4)$$

in the SW approximation. Simple rescaling of the variable θ and use of continuum notation makes this Hamiltonian identical to $\mathcal{H}_{SG}$ of (A2), provided one makes the identifications $\alpha/\beta^2 = h/k_B T$ and $\beta^{-1} = K^{1/2}$. The spin variable $S_x(\vec{x}) + iS_y(\vec{x})$ is given by $e^{i\beta\varphi(\vec{x})}$ in this representation. It is trivial to rewrite Sec. IV results of RG analysis of the SG theory in terms of h and T. A line of infinite-order phase transition in the h-T (i.e., $\alpha - \beta^2$) plane separates a *high-T* phase with power-law decay of correlations and ξ infinite from a *low-T* phase with finite ξ and exponential decay. Both phases have finite magnetization except at $h = 0$. As the critical line is approached from the low-T side, ξ diverges exponentially. The spin-spin correlation function at large distance decays like x^{-4} (Ref. 43) (apart from logarithmic corrections) everywhere on the critical line. Note that the phase with power-law decay occurs at high T in this finite-field SW approximation and at low T in zero field when vortices are included.

It is worth emphasizing that these finite-field results are artifacts of the SW approximation; the 2D XY model does *not* undergo a phase transition for nonzero h. Inclusion of vortices would make this point clear.

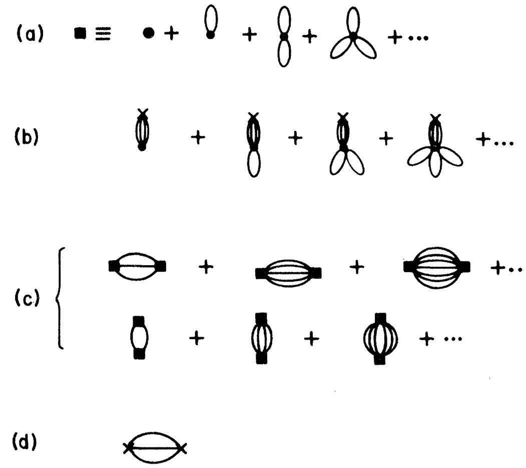

FIG. 11. Diagrams giving ultraviolet-divergent contributions to the self-energy Σ. Each solid line represents a bare propagator $g_0(\vec{p})$. Each dot (●) represents an α_0 vertex. Each cross (+) represents a u_0 vertex. The square (■) is defined in diagram (a).

APPENDIX B: SECOND-ORDER COMPUTATION OF THE Z's

The computations of $g^{-1}(\vec{p}_0)$, the inverse Fourier transform of $\langle\theta(\vec{x})\theta(\vec{0})\rangle$, and Γ_4, the four-point vertex, to second order in u_0, α_0, and δ_0 are summarized here.

A. $g^{-1}(\vec{p}_0)$

Writing Dyson's equation, $g^{-1}(\vec{p}_0) = g_0^{-1}(\vec{p}_0) - \Sigma(\vec{p}_0)$, with $g_0^{-1}(\vec{p}) \equiv p_\perp^2 + (p_\parallel^2)^2 + \mu_0^4$, and $\cos\beta_0\theta$ as $\sum_{n=0}^{\infty} (-\beta_0^2\theta^2)^n/(2n)!$ one generates from (5) the diagrammatic perturbation series for Σ shown in Fig. 11. Only diagrams divergent as $a \to 0$ have been included; the Z's are chosen to remove these divergences in each order of the triple expansion. Diagrams of Figs. 11(a) and 11(b) are independent of the external momentum $\vec{p}_0$; they are readily summed[26] to yield the divergent quantity

$$\Sigma_{a+b}(\vec{p}_0) = -\alpha_0(\mu_0 K_1^{-4\pi})^{4(1+\delta_0)} a^{4\delta_0}(1 - \beta_0^4 I_1 u_0) \ , \tag{B1}$$

where

$$I_1 \equiv \int d^3x \{[\nabla_\parallel g_0(\vec{x})]^2\}^2 \sim -\ln(\tfrac{32}{27}) \ln a/(4\pi)^3 \ , \tag{B2}$$

and $\ln K_1 \equiv (3\ln 2 - 2C)(8\pi)^{-1}$.

Expanding (B1) in powers of δ_0 we have, to second order,

$$\Sigma_{a+b}(\vec{p}_0) = -\alpha_0(\mu_0 K_1^{-4\pi})^{4(1+\delta_0)} \times \{1 + [4\delta_0 + 16\pi^{-1}\ln(\tfrac{32}{27})u_0]\ln a\} \ . \tag{B3}$$

In deriving this result we have used the fact that $g_0(\vec{x})$, the Fourier transform of $g_0(\vec{p})$, is given, for small $|\vec{x}|$, by

$$g_0(\vec{x}) \sim [-\ln|x_\perp/2| + f((x_\parallel^2 + a^2)/|x_\perp|) - C]/8\pi \ , \tag{B4}$$

where $f(t) \equiv \int_0^{|t|/4} dx\,(e^{-x} - 1)/x$.

The $O(\alpha_0^2)$ diagrams of Fig. 11(c) are evaluated in a straightforward manner:

$$\Sigma_c(\vec{p}_0) = \int d^3x e^{-i\vec{p}_0\cdot\vec{x}} \alpha_0^2\beta_0^{-2}(\mu_0 K_1^{-4\pi})^{8(1+\delta_0)} \left\{\sinh[\beta_0^2 g_0(\vec{x})] - \delta^3(x)\int d^3y \cosh[\beta_0^2 g_0(\vec{y})]\right\} \ . \tag{B5}$$

Expanding $e^{-i\vec{p}_0\cdot\vec{x}}$ in powers of $\vec{p}_0\cdot\vec{x}$ we find divergent terms proportional to $p_{0_\perp}^2$, and $(\vec{p}_{0_\parallel}^2)^2$. (The term independent of $\vec{p}_0$ is finite[26] as $a \to 0$.) We choose the parameter A in (5) to cancel the $O(p_{0_\parallel}^2)$ term exactly; this choice locates the boundary between the regions with uniform and nonuniform magnetization in LP terminology. Using (B4) one finds the following divergent $O(p_{0_{\parallel}}^4)$ and $O(p_{0_{\perp}}^2)$ contributions to $\Sigma(\vec{p}_0)$:

$$\Sigma_c(\vec{p}_0) \sim \frac{\alpha_0^2 e^{4C}}{2\cdot 4^6}(\ln a)\left[J_1 p_{0_{\perp}}^2 - \frac{J_2 p_{0_{\parallel}}^4}{32}\right], \tag{B6}$$

with J_i defined below Eq. (7).

The $O(u_0^2)$ term in Fig. 11(d) produces divergent terms of $O(p_{0_{\parallel}}^2)$ and $O(p_{0_{\parallel}}^4)$. We again choose A to cancel the former; the latter is readily evaluated:

$$\Sigma_d(\vec{p}_0) \sim -u_0^2[\tfrac{7}{3} - 8\ln(\tfrac{4}{3})]p_{0_{\parallel}}^4(\ln a)/8\pi^2 . \tag{B7}$$

Finally, then, to second order in u_0, α_0, and δ_0:

$$g^{-1}(\vec{p}_0) \sim \mu_0^4(1+\alpha_0 K_1^{-16\pi}\{1+[4\delta_0+16u_0\ln(\tfrac{32}{27})/\pi]\ln a\}) + p_{0_{\perp}}^2[1-e^{4C}J_1\alpha_0^2(\ln a)/2(4^6)]$$
$$+(\vec{p}_{0_{\parallel}}^2)^2(1+\{4^{-9}e^{4C}J_2\alpha_0^2+(8\pi^2)^{-1}[\tfrac{7}{3}-8\ln(\tfrac{4}{3})]u_0^2\}\ln a) . \tag{B8}$$

The renormalizability assumption demands that the substitutions (4) [with (4c) replaced by $p_{0_{\perp}} = Z_m p_{\perp}$ since m_0 has been absorbed in $p_{0_{\perp}}$] and appropriate definitions of the Z's render the quantity $Z_\phi g^{-1}(\vec{p}_0)$ finite. Thus is one is led to the following definitions of the Z's:

$$Z_\alpha = 1+[4\delta+16u\ln(\tfrac{32}{27})/\pi]\ln\Lambda ,$$
$$Z_\phi = 1+\{4^{-9}e^{4C}J_2\alpha^2+(8\pi^2)^{-1}[\tfrac{7}{3}-8\ln(\tfrac{4}{3})]u^2\}\ln\Lambda , \tag{B9}$$
$$(Z_\phi Z_m)^{-1} = 1+\{4^{-7}e^{4C}(J_1-J_2/32)\alpha^2+(2\pi^2)^{-1}[\ln(\tfrac{4}{3})-\tfrac{7}{24}]u^2\}\ln\Lambda .$$

B. Γ_4

The vertex $\Gamma_4(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4)$ is the amputated Fourier transform of the connected, single-particle irreducible four-point correlation function. To compute Z_u one need only consider the part of $\Gamma_4(\vec{p},\vec{p},-\vec{p},-\vec{p})$ proportional to $(\vec{p}_{\parallel}^2)^2$. Diagrams contributing to this quantity are shown in Fig. 12. Since only divergent contributions need be considered, diagrams 12(a), finite as $a\to 0$, can be ignored, as can diagrams 12(b), which are independent of $p_{\parallel}$ (and whose divergent parts are in any event cancelled by renormalization of α_0). Diagrams 12(c) are easily found to contribute

$$\Gamma_4^c(\vec{p}) \sim -4(\vec{p}_{\parallel}^2)^2 u_0(1-9u_0\ln\Lambda/4\pi) . \tag{B10}$$

Diagrams 12(d) produce divergences of $O(1)$, $O(\vec{p}_{\parallel}^2)$, and $O((\vec{p}_{\parallel}^2)^2)$. The first two of these are removed by our previous renormalizations, leaving

$$\Gamma_4^d(\vec{p}) \sim +6\pi e^{4C}J_2(\vec{p}_{\parallel}^2)^2\alpha^2\ln\Lambda/4^6 . \tag{B11}$$

It follows that the definition $u_0 = Z_u u$ with

$$Z_u = 1+[(4\pi)^{-1}9u+4^{-7}\pi e^{4C}J_2(\alpha^2/u)]\ln\Lambda \tag{B12}$$

renders Γ_4 finite to second order.

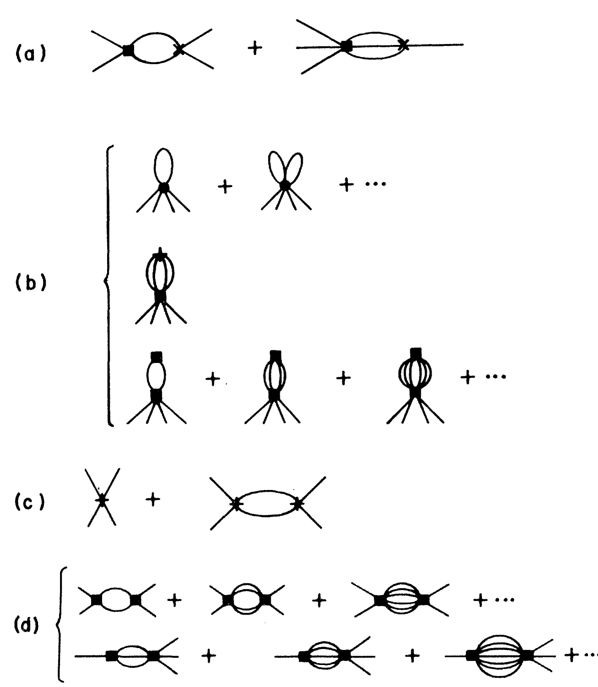

FIG. 12. Diagrams contributing to the four-point vertex, Γ_4. Symbols are defined in the caption of Fig. 11.

APPENDIX C: RENORMALIZATION OF $G(\vec{x})$

A. Z_s to $O(u,\alpha,\delta)$

Multiplicative renormalizability of the connected spin-spin correlation function $\tilde{G}(\vec{x}) = \langle \exp\{ i\beta_0 m_0^{1/2}[\tilde{\theta}(\vec{x}) - \vec{\theta}(\vec{0})]\}\rangle_c$ [in the $\tilde{\theta}$ variables of (3)] implies the existence of a renormalization constant Z_s (independent of $\vec{x}$) such that $Z_s\tilde{G}(\vec{x})$ is finite when expressed in terms of the renormalized quantities u, α, β, and m. Since, in terms of the field $\theta(\vec{x})$, $\tilde{G}(\vec{x}_\parallel, x_\perp) = G(\vec{x}_\parallel, x_\perp/m_0)$, where $G(\vec{x}) \equiv \langle \exp\{ i\beta_0[\theta(\vec{x}) - \theta(\vec{0})]\}\rangle_c$, renormalizability means that $Z_s G(\vec{x}_\parallel, x_\perp/Z_m)$ is finite when expressed in terms of u, α, β, $\vec{x}_\parallel$, and $x_\perp$. Perturbative study of $G(\vec{x})$ gives Z_s; the calculation to $O(u,\alpha,\delta)$ follows.

To zeroth order, $G(\vec{x}) = \exp\{\beta_0^2[g_0(\vec{x}) - g_0(\vec{0})]\}$. Putting $x_\perp = 0$ for simplicity and using (B4) one finds

$$G_0(\vec{x}) = (x_\parallel/a)^{-8(1+\delta_0)} \quad . \tag{C1}$$

In $O(\alpha_0)$ all contributions to $G(\vec{x})$ vanish in the $\mu_0 \to 0$ limit and so can be ignored. In $O(u_0)$ there is a single class (Fig. 13) of diagrams divergent as $a \to 0$, evaluation of which yields

$$G_a(\vec{x}) = -2u_0\beta_0^4 I_1 G_0(\vec{x}) \sim 32u_0 \ln(\tfrac{32}{27})(\ln a) G_0(\vec{x})/\pi \quad . \tag{C2}$$

To first order in u_0, α_0, and δ_0, therefore,

$$G(\vec{x}) = (x_\parallel/a)^{-8}\{1 + [8\delta_0 + 32\pi^{-1}u_0 \ln(\tfrac{32}{27})]\ln a\} \quad , \tag{C3}$$

whereupon,

$$Z_s = (a/\mu)^{-8}\{1 + [8\delta + 32\pi^{-1}\ln(\tfrac{32}{27})u]\ln\Lambda\} \quad . \tag{C4}$$

Since (Sec. III) the critical fixed point in this problem occurs at $u = \alpha = \delta = 0$, this first-order expression for Z_s is sufficient for a complete computation of the large-distance behavior of $G(\vec{x})$. [Indeed, even the terms of $O(u,\delta)$ in (C4) would not be needed for such a computation were it not that both u and δ are *marginal*[16] variables with respect to the critical fixed point and so determine the logarithmic corrections to the power-law decay at large $|\vec{x}|$.]

FIG. 13. Diagrams contributing singularly to $G(\vec{x})$, the spin-spin correlation function, to first order. The wavy line (●〰〰●) represents the factor $G_0(\vec{x})$

B. Renormalizability in second order

Although still higher-order terms in Z_s are irrelevant to the determination of $G(\vec{x})$ at large $|\vec{x}|$ it is important in principle that there exist a Z_s (independent of $\vec{x}$) which renormalizes the theory in *every* order. No $\vec{x}$-dependent divergent terms should appear in any order. An attempt to verify this assumption leads to (superficial) difficulties even in second order: one finds that while most of the $\vec{x}$-dependent divergences are neatly cancelled by the renormalizations of u_0, α_0, and δ_0 defined in Appendix B, some are not. This is not surprising. Since the operator $\exp[i\beta_0\theta(\vec{x})]$ has [Eq. (C3)] naive dimension four when $\delta_0 = 0$, one expects that connected correlation functions involving not $\exp[i\beta_0\theta(\vec{x})]$ alone but rather $O(\vec{x})$, a linear combination of $\exp[i\beta_0\theta(\vec{x})]$ and all other operators with naive dimension less than or equal to 4, should be multiplicatively renormalizable. It is tedious but straightforward to verify that in fact the operator $O(\vec{x})$ defined by

$$O(\vec{x}) = \cos[\beta_0\theta(\vec{x})] + a_1[\nabla_\parallel\theta(\vec{x})]^2 + a_2\{[\nabla_\parallel\theta(\vec{x})]^2\}^2 + a_3[\nabla_\parallel^2\theta(\vec{x})]^2 + a_4[\nabla_\perp\theta(\vec{x})]^2 + a_5\nabla_{i_\parallel}\theta(\vec{x})\nabla_{i_\parallel}\nabla_\parallel^2\theta(\vec{x}) + a_6[\nabla_{i_\parallel}\nabla_{j_\parallel}\theta(\vec{x})][\nabla_{i_\parallel}\nabla_{j_\parallel}\theta(\vec{x})] \quad , \tag{C5}$$

where repeated indices i and j are summed over the two parallel directions, *is* multiplicatively renormalizable to second order, provided the a_i are chosen appropriately in $O(\alpha)$. [The particular $O(\alpha)$ choices for the a_i are not sufficiently interesting to list here.] The coefficients a_i must be chosen in each order of the triple perturbation series to ensure the renormalizability of $O(\vec{x})$ in that order.

C. Asymptotic behavior of $L(\vec{x})$

To compute the asymptotic behavior of $L(\vec{x}) \equiv \langle[\nabla_\parallel\theta(\vec{x})]^2[\nabla_\parallel\theta(\vec{0})]^2\rangle_c$, one need only know the renormalization constant Z_L which renders $L(\vec{x})$ finite in low orders. To first order in u_0 and α_0 only one diagram (Fig. 14) produces a divergence. This diagram represents the analytic expression

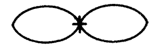

FIG. 14. Diagram contributing singularly to $L(\vec{x})$ in first order.

$$-8u_0\int d^3x_1[\nabla_i\nabla_j g_0(\vec{x}-\vec{x}_1)][\nabla_i\nabla_k g_0(\vec{x}-\vec{x}_1)][\nabla_l\nabla_m g_0(\vec{x}_1)][\nabla_l\nabla_n g_0(\vec{x}_1)](\delta_{jk}\delta_{mn}+2\delta_{jm}\delta_{kn}) ,$$

where repeated indices are summed over the two parallel directions. The divergent part of this expression is

$$-16u_0[\nabla_i\nabla_j g_0(\vec{x})][\nabla_i\nabla_k g_0(\vec{x})](\delta_{jk}\delta_{mn}+2\delta_{jm}\delta_{kn})\int d^3x_1[\nabla_l\nabla_m g_0(\vec{x}_1)][\nabla_l\nabla_n g_0(\vec{x}_1)] .$$

Use of (B4) then yields the divergent contribution $\{-(4u_0/\pi)[\nabla_i\nabla_j g_0(\vec{x})]^2\ln\Lambda\}$. It follows that $Z_L L(\vec{x})$ is finite to first order as $\Lambda\to\infty$, where

$$Z_L=1+2u_0(\ln\Lambda)/\pi . \tag{C6}$$

$L(\vec{x})$ then satisfies a RG equation like (6), with γ given by $\partial\ln Z_L/\partial l=2u_0/\pi$, whereupon (8) and the fact that $\tilde{u}$ approaches zero, its critical-fixed-point value, like A_u/τ for large τ imply that on the critical surface,

$$L(\vec{x}_{\parallel},x_\perp=0)\sim|\vec{x}_{\parallel}|^{-4}(\ln|\vec{x}_{\parallel}|)^{-2A_u/\pi} \tag{C7}$$

for $x_{\parallel}\Lambda>>1$. The large-distance behavior of $L(\vec{x})$ and $G(\vec{x})$ [see (C5)] are identical.

APPENDIX D: THE CORRELATION LENGTH, ξ

The RG flow Eqs. (9) have the structure

$$\frac{\partial\alpha_i}{\partial\tau}=-\sum_{j\le k-1}^{3}C_{ijk}\alpha_j\alpha_k \quad (i=1,2,3) , \tag{D1}$$

where $\alpha_1=\tilde{\alpha}$, $\alpha_2=\tilde{\delta}$, $\alpha_3=\tilde{u}$, and the $\{C_{ijk}\}$ are numerical coefficients given in (7). Suppose that $\tilde{\alpha}(\tau=0)\equiv\alpha_0$ is much smaller than $\tilde{u}(\tau=0)\equiv u_0$. Then, at least for sufficiently small τ, the $\tilde{\alpha}^2$ term can be neglected with respect to the $\tilde{u}^2$ term in the $\tilde{u}$ and $\tilde{\delta}$ equations, which take the form

$$\frac{\partial\tilde{u}}{\partial\tau}=-C_{333}\tilde{u}^2 , \quad \frac{\partial\tilde{\delta}}{\partial\tau}=-C_{233}\tilde{u}^2 . \tag{D2}$$

These equations have the solution $\tilde{u}=(u_0^{-1}+C_{333}\tau)^{-1}$, $\tilde{\delta}=\delta^*+C_{233}C_{333}^{-1}\tilde{u}$, where δ^* is an arbitrary integration constant. The $\tilde{\alpha}$ equation,

$$\frac{\partial\tilde{\alpha}}{\partial\tau}=(-C_{112}\tilde{\delta}-C_{113}\tilde{u})\tilde{\alpha} , \tag{D3}$$

then yields [recall $C_{112}=4$ from Eq. (7)]

$$\tilde{\alpha}=\alpha_0(1+C_{333}u_0\tau)^x e^{-4\delta^*\tau} , \tag{D4}$$

with $x\equiv-(4C_{233}+C_{113}C_{333})/C_{333}^2\sim-1.1\ldots$. When $\delta^*=0$ the critical fixed point $\tilde{\alpha}=\tilde{\delta}=\tilde{u}=0$ is reached as $\tau\to\infty$. Asymptotically, then, $\tilde{u}\sim A_u/\tau$, $\tilde{\delta}\sim A_\delta/\tau$, and $\tilde{\alpha}\sim A_\alpha\tau^x$, where $A_u=C_{333}^{-1}$, $A_\delta=C_{233}C_{333}^{-2}$, and $A_\alpha=\alpha_0(C_{333}u_0)^x$. Since $x<-1$, $\tilde{\alpha}<<\tilde{u}$ for all τ if $\alpha_0<<u_0$; this justifies the neglect of the $\tilde{\alpha}^2$ terms in the $\tilde{u}$ and $\tilde{\delta}$ equations. It is clear from these solutions that the critical surface is indeed two dimensional: for given u_0 and α_0 one need only adjust δ_0 so as to make $\delta^*=0$ in order to achieve flow into the critical field point.

For $\delta^*<0$ (i.e., in region II) $\tilde{\alpha}$ grows with τ; no fixed point accessible to perturbation theory is achieved under RG flow. For infinitesimal $\delta^*<0$ the flows stay very close to the critical surface until τ becomes large, or, more precisely, until $\tilde{\alpha}(\tau)$ becomes comparable to $\tilde{u}(\tau)$, that is, until $\tau\sim1/\delta^*$. Recalling that $\tau\sim\ln(|\vec{x}|/a)$ one concludes that only at distances $|\vec{x}|$ for which $\ln(|\vec{x}|/a)\gtrsim1/\delta^*$ is one out of the critical region. This defines ξ as

$$\xi/a\sim\exp[\sigma(\delta^*)^{-\nu}] , \tag{D5}$$

where σ is a nonuniversal number—it typically varies from point to point on the critical surface—and the exponent $\nu=1$.

APPENDIX E: DIVERGENCE OF ξ AS $\alpha_0\to0$

In Sec. II A it was remarked that (provided μ_0 is properly zero) ξ is infinite when $\alpha_0=0$ in Hamiltonian (5). It follows that ξ diverges as $\alpha_0\to0$ in the $(\delta<0)$ phase where ξ is finite. The exponent characterizing the divergence is readily computed from Eq. (D4). When $\alpha_0<<1$ for fixed $\delta^*<0$, $\tilde{\alpha}$ becomes comparable to $\tilde{u}$ when $\alpha_0\tau^x e^{-4\delta^*\tau}\sim\tau^{-1}$, or

$$4\delta^*\tau\sim\ln\alpha_0+(x+1)\ln[(\ln\alpha_0)/4\delta^*] . \tag{E1}$$

With (see Appendix D) $\tau\sim\ln\xi/a$ this yields

$$\xi/a\sim\alpha_0^{(4\delta^*)^{-1}}[(\ln\alpha_0)/4\delta^*]^{(x+1)(4\delta^*)^{-1}}$$

for small α_0 and $\delta^*<0$. The divergence of the power $1/4\delta^*$ as $\delta^*\to0$ signals the exponential divergence of ξ at the critical surface $\delta^*=0$. Note that to leading order in u_0, δ^* is related to δ_0 via $\delta^*=\delta_0-C_{233}u_0/C_{333}$.

[1]R. M. Hornreich, M. Luban, and S. Shtrikman, Phys. Rev. Lett. 35, 1678 (1975).
[2]The smectic-A–smectic-C liquid-crystal transition in a magnetic field constitutes one of several good physical realizations of the ferromagnetic-helical transition in the LP model. See Sec. IV and, e.g. (the following list is by no means exhaustive), D. Johnson, D. Allender, R. de Hoff, C. Maze, E. Oppenheim, and R. Reynolds, Phys. Rev. B

16, 470 (1977); G. Sigaud, F. Hardouin, and M. F. Achard, Solid State Commun. 23, 35 (1977). It has been known both experimentally [e.g., R. A. Wise, D. H. Smith, and J. W. Doan, Phys. Rev. A 7, 1366 (1973)] and theoretically [e.g., P. G. de Gennes, Mol. Cryst. Liq. Cryst. 21, 49 (1973); R. M. Hornreich and S. Shtrikman, Phys. Lett. 63A, 39 (1977)] for some time that the *A-C* transition is second order. A. Michelson [Phys. Rev. B 16, 585 (1977)] argues, however, that the ferromagnetic-helical transition for the $m=1$ LP problem can be first order in the presence of a symmetry-breaking field. See also A. D. Bruce, R. A. Cowley, and A. F. Murray, J. Phys. C 11, 3591 (1978).

[3]S. A. Brazovskii, Zh. Eksp. Teor. Fiz. 68, 175 (1975) [Sov. Phys. JETP 41, 85 (1975)].

[4]See, e.g., Hornreich *et al.*, Ref. 1; D. Mukamel, J. Phys. A 10, L249 (1977); R. M. Hornreich, Phys. Rev. B 19, 5914 (1979), and references therein.

[5]G. S. Grest and J. Sak, Phys. Rev. B 17, 3607 (1978).

[6]See, e.g., J.-h. Chen and T. C. Lubensky, Phys. Rev. A 14, 1202 (1976); K. C. Chu and W. C. McMillan, *ibid.* 15, 1181 (1977).

[7]I am grateful to R. J. Birgeneau for pointing out to me the liquid-crystal realization of the LP problem.

[8]T. A. Kaplan, Phys. Rev. Lett. 44, 760 (1980).

[9]G. Grinstein, J. Phys. A 13, L201 (1980).

[10]Strictly speaking, there are logarithmic corrections to these power laws. See Ref. 9.

[11]F. J. Wegner, Z. Phys. 206, 465 (1967); V. L. Berezinskii, Zh. Eksp. Teor. Fiz. 59, 907 (1971) [Sov. Phys. JETP 32, 493 (1971)]; J. Zittartz, Z. Phys. B 23, 55, 63 (1976); N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).

[12]J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973); J. M. Kosterlitz, *ibid.* 7, 1046 (1974).

[13]Kosterlitz, Ref. 12.

[14]The equivalence between the 2D Coulomb gas and the 2D SG theory has been pointed out by many authors. See, e.g., (a) R. Heidenreich, B. Schroer, R. Seiler, and D. Uhlenbrock, Phys. Lett. A 54, 119 (1975); (b) J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B 16, 1217 (1978).

[15]P. B. Wiegmann, J. Phys. C 11, 1583 (1978); T. Ohta, Prog. Theor. Phys. 60, 968 (1978).

[16]For a good, general discussion of infinite-order transitions see F. J. Wegner, in *Phase Transitions and Critical Phenomena* (Academic, London, 1976), Vol. 6.

[17]E. Brézin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976); J. Zittartz, Ref. 11.

[18]I am grateful to B. I. Halperin for a helpful remark about this phase.

[19]If, following S. Coleman, Phys. Rev. D 11, 2088 (1975), one writes the cosine interaction term in the SG theory as $\cos\beta\phi$, then the phase with $\xi=\infty$ occurs at *large* β. In the equivalence between the "SW plus vortices" approximation to the 2D *XY* model in zero field and the 2D SG theory, $T\sim\beta^{-2}$; the $\xi=\infty$ phase thus occurs at *low T*. In the equivalence between the SW version of the 2D *XY* model in finite field and the SG, $T\sim\beta^{+2}$; the $\xi=\infty$ phase thus occurs at *high T*. See Appendix A for details.

[20]A similar power-law divergence of ξ occurs in the 2D SG problem. See, e.g., P. Minnhagen, A. Rosengren, and G. Grinstein, Phys. Rev. B 18, 1356 (1978). The logarithmic term here is a consequence of the extra *marginal* operator which, as will be shown, occurs in the 3D problem.

[21]This divergence of the exponent $1/4\delta^*$ signals the *exponential* divergence of ξ at the critical line. See, e.g., P. Minnhagen, A. Rosengren, and G. Grinstein, Ref. 20, for discussion of the analogous situation in 2D.

[22]The classical "logarithmic gas," a system consisting of equal numbers of positive and negative point charges whose mutual interactions vary logarithmically with separation, undergoes an infinite-order transition in *any* dimension. When $d=2$ the logarithmic gas is simply a Coulomb gas and is equivalent to the 2D SG theory, whose interactions are purely local. When $d=3$ the (isotropic) logarithmic gas is not a Coulomb gas, has no obvious equivalence to a local field theory, and no obvious relation to an interesting physical system. I am grateful to V. J. Emergy for instructive comments on this subject.

[23]In writing $[q_\perp^2+(q_\parallel^2)^2]^{-1}$ on the boundary one implicitly assumes that the ferromagnetic-helical transition occurs continuously; see Ref. 2. If a first-order transition to the helical state occurs then, for small $\vec{q}$, $G_\perp(\vec{q})$ is described by an expression characteristic of the ferromagnetic state, i.e., $(q_\perp^2+q_\parallel^2)^{-1}$, on the phase boundary, whereupon long-range order *can* exist. We cannot rigorously rule out this possibility, but rather study the implications of a continuous transition.

[24]Kaplan (Ref. 8) argues that because the suspectibility computed in the SW approximation diverges below a certain temperature the SW theory actually predicts a phase transition at that temperature. Since SW theory gives $M=0$, $\xi=\infty$, the algebraic decay of correlations at *all* temperatures, it seems more natural to regard it as predicting but a single phase. The real transition is presumably mediated, as in the 2D *XY* case, by topological defects and has little to do with spin waves.

[25]Why the coefficient of the cosine term is written in such a cumbersome fashion is made clear in Sec. III.

[26]D. J. Amit, Y. Y. Goldschmidt, and G. Grinstein, J. Phys. A 13, 585 (1980). Most of the calculational details of the present 3D analysis have close analogs in this 2D computation. See also S. Coleman, Ref. 19, for a nice discussion of the 2D SG theory.

[27]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena* (Academic, London, 1976), Vol. 6.

[28]See, e.g., D. J. Amit, *Field Theory, The Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).

[29]That the 3D SG theory is renormalizable only for $\delta<0$ simply means that the cosine operator in (3) is *relevant* and *irrelevant* for $\delta<0$ and $\delta>0$, respectively. See Wegner, Ref. 16, and Brézin *et al.* Ref. 27, for general discussions of this terminology.

[30]The renormalization properties of the 3D SG theory are analogous to those of the familiar ϕ^4 theory in $4+\epsilon$ dimensions with δ playing the role of ϵ. For ϵ, respectively, <0, $=0$, and >0 the ϕ^4 theory is superrenormalizable, renormalizable, and nonrenormalizable. For $\epsilon>0$ it can be made finite order by order in powers of ϵ and the ϕ^4 coupling constant.

[31]Note that $p_\perp$ in $g(\vec{p}_\parallel,p_\perp)$ has dimensions of (momentum)^2, since $x_\perp\equiv y_\perp/m_0$ has dimensions of (length)^2.

[32]In perturbation theory μ_0 must be kept finite to avoid infrared divergences in the correlation functions. As shown in Appendix B, however, the β and γ functions of (6)

are, even in perturbation theory, independent of μ_0, allowing the limit $\mu_0 \to 0$ to be taken in that equation.

[33]It is easy to see that for given small α_0, the fixed line at $\tilde{\delta} > 0$ is only reached if u_0 is, roughly, $\gtrsim \alpha_0$. For insufficiently large values of u_0, $\tilde{u}$ becomes negative under RG iteration: the flows run off to infinity. Noting that for $u_0 < 0$ H_{SG} of (5) is unstable and that [see (9a)] the $\alpha_0 \cos(\beta_0 \theta)$ term generates via the RG a *negative* $\tilde{u}$ if $u_0 = 0$ originally, one interprets the runoff for insufficiently large u_0 as a signal of the instability of H_{SG}. The situation is similar to $u_0 \phi^4 + v_0 \phi^6$ theory in 4D. Though the Gaussian fixed point is stable in that case, v_0 must be sufficiently large and positive for given negative u_0 for the fixed point to be reached.

[34]A. Caillé, C. R. Acad. Sci. Paris B 274, 891 (1972).

[35]See José *et al.*, Ref. 14(b), for a nice discussion of this point.

[36]See Minnhagen *et al.* Ref. 20.

[37]L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, London, 1958).

[38]The lack of positional long-range order has been observed in the *A* phase. See J. Als-Nielsen, R. J. Birgeneau, M. Kaplan, J. D. Litster, and C. R. Safinya, Phys. Rev. Lett. 39, 1668 (1977).

[39]In practice the field required actually to see the layered order experimentally may be very large. See, e.g., J. Als-Nielsen, J. D. Litster, R. J. Birgeneau, M. Kaplan, C. R. Safinya, A. Lindegaard-Andersen, and S. Mathiesen, Phys. Rev. B 22, 312 (1980).

[40]The experiments of Johnson *et al.* (Ref. 2) show a tantalizing weakening of the specific-heat anomaly as the Lifshitz point is approached along the *N-A* phase boundary. This is consistent with the LP transition being of infinite-order, but, since the *N-C* transition is observed to be first order (see also Ref. 3), other explanations of the data are easy to concoct.

[41]The generation of this expansion is, as in the 2D SG case, trivial, requiring nothing more than the evaluation of Gaussian integrals. See Coleman (Ref. 19) for a concise discussion of the arithmetic involved.

[42]For *A* positive, the expansion of Z_{SG} in powers of α_0 shows that the anisotropic 3D SG theory with $u_0 = 0$ is equivalent to an anisotropic 3D *Coulomb* (not logarithmic!) gas; this gas is always in its plasma phase of course, the Coulomb forces in 3D being too weak at large distance to effect a transition to the dielectric phase.

[43]Note that the critical value of η in the SW approximation to the 2D *XY* model in finite field is 4, not $\frac{1}{4}$.