Crossover functions by renormalization-group matching: $O(\epsilon^2)$ results

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By considering the relationship of the matching techniques of Bruce and Wallace to the differential renormalization-group generators, we find that a restatement of the former gives improved results with the same number of perturbative terms. In particular, the vertex functions and specific heat of a n-component spin system are given exactly in the spherical limit $n \to \infty$ even at first order in perturbation theory ($T > T_c$). The nature of the nonlinear scaling variables is clarified, and the results are generally expressed in a more compact form. The general n-component disordered phase functions are rederived to $O(\epsilon^2)$, where $\epsilon \equiv 4-d$. The crossover equations for the $n = 1$ Ising-like case are derived for the Helmholtz potential $A(M)$, the magnetic field h/M, the inverse susceptibility Γ_2, and the correlation length ξ to $O(\epsilon^2)$.

I. INTRODUCTION

The possibility of calculating the complete, and not merely the critical, form of the thermodynamic functions using the renormalization group was indicated even in the early papers of Wilson.[1] This approach relies on the often-forgotten fact that the original Wilson renormalization-group approach represents a stepwise evaluation of the partition function. A particularly simple formulation of this is given by the use of differential renormalization-group generators.[2,3] The generators supply differential equations for the renormalized Hamiltonian. The complete nonlinear solution of these equations provides the physical free energy in the limit of infinite l, where $\exp(-l)$ describes the scale of the fluctuations not yet incorporated in the Wilson elimination of degrees of freedom. For example, using the one-particle-irreducible (1PI) generator,[3] the renormalized N-spin coupling constant $U_N(l)$ can be used to calculate the N-point 1PI vertex function Γ_N (at all wave vectors $k = 0$):

$$\Gamma_N = \lim_{l\to\infty} \mathfrak{D}^{N/2} U_N(l) \exp(-\lambda_N l) \quad . \tag{1.1}$$

In Eq. (1.1) λ_N is the canonical dimension of the N-point vertex in dimension d, $\lambda_N = d + \frac{1}{2}N(2-d)$, and $\mathfrak{D}$ is the anomalous dimension crossover function (Nicoll and Chang,[4] referred to as Ia):

$$\mathfrak{D} = \exp\left[\int_0^l \eta(l')\,dl'\right] \quad . \tag{1.2}$$

In Eq. (1.2), $\eta(l')$ is an l'-dependent effective value of the critical-point exponent η. Without the $\mathfrak{D}^{N/2}$ factor, Γ_N would be an exact nonlinear scaling field (Wegner[5]) of canonical dimension. For example

$$\Gamma_2 = \mathfrak{D}\,\xi^{-2} \quad , \tag{1.3}$$

where ξ is the (second moment) correlation length and an exact scaling field of dimension (or eigenvalue) -1 (in momentum units). If we allow the renormalization-group equation to act on the parameters on which ξ depends, its transformation properties are simple: $\xi \to \exp(-l)\xi$. The function $\mathfrak{D}$ has no such simple property, having an effective dimension of $-\eta$ close to the critical point and a dimension equal to zero far from it (where "dimension" refers to its renormalization-group behavior). Therefore, the inverse susceptibility $(=\Gamma_2)$ is not itself a nonlinear scaling field. Crossover equations of state can be obtained for a variety of systems with this method (Nicoll and Chang[6] henceforth referred to as Ib).

However, the solution of coupled nonlinear differential equations of the sort provided by the differential generators becomes extremely difficult beyond lowest order in perturbation theory. The essential difficulty lies in the fact that the equations must not only embody the renormalization-group property that the renormalized Hamiltonian has the same free energy as the original system; it must also be capable of evaluating that free energy complete in every detail. Even if one is interested simply in thermodynamic functions, the renormalization-group equations are in terms of the full wave-vector dependent correlation functions. This coupling is in turn a consequence of the expression of the evolution of the partition function in closed form by the differential generator without perturbation theory.

An alternative approach is that of Bruce and Wallace,[7] henceforth referred to as II. (A related approach is considered by Lawrie.[8]) This relies on the existence of a renormalized theory in the pre-Wilson sense of a cutoff parameter $\Lambda \to \infty$ (Brézin *et al.*[9]) The Λ independence of the renormalized theory leads to renormalization-group equations for the unrenormalized theory reflecting that independence. This corresponds to the part of the Wilson view relating the coupling constants of a renormalized Hamiltonian to an original or bare Hamiltonian with the same thermodynamic properties. In their usual form, these equations cannot evaluate the thermodynamic functions; they simply relate the properties of one system to another system with a different value of the cutoff Λ. Specializing for the remainder of this paper to the n-component s^4 theory with coupling constant $\Lambda^\epsilon u$ and reduced temperature $\Lambda^2 t$ [$t \propto (T-T_c)/T_c$] the renormalization-group equations take the form

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(u)\frac{\partial}{\partial u}-\frac{1}{\nu(u)}t\frac{\partial}{\partial t}-\tfrac{1}{2}N\eta(u)\right]\Gamma_N = \Delta\Gamma_N \simeq 0 \quad . \qquad (1.4)$$

The inclusion of the Λ^2 in the definition of t changes the form of the renormalization-group equations from that given in II and increases the resemblance to the generator equations. The dropping of the inhomogeneous term $\Delta\Gamma_N$, which is presumed negligible in the critical region, is the step that simplifies the renormalization-group equations; the functions $\beta(u)$, $\nu(u)$, and $\eta(u)$ can be calculated perturbatively. The solution to this is

$$\Gamma_N(u,t,\Lambda=1)=\mathfrak{D}^{N/2}\Gamma_N(u(l),t(l),\Lambda=\exp(-l)) \quad , \qquad (1.5)$$

where

$$\mathfrak{D}=\exp\left[\int_0^l \eta(u(l'))\,dl'\right] \quad , \qquad (1.6)$$

and $u(l)$ and $t(l)$ are the solutions of the renormalization-group flow equations

$$\frac{du}{dl}=-\beta(u) \quad , \quad \frac{dt}{dl}=\frac{1}{\nu(u)}t \quad . \qquad (1.7)$$

These equations are quite similar to those of the differential generator formulation. However, in this case there is no natural choice of l corresponding to the limit $l \to \infty$. Equations (1.5)–(1.7) must be supplemented by concrete perturbative expressions for the vertex functions themselves. If we use the device of expanding around $d=4$ in an ϵ expansion, $\epsilon = 4-d$, these will be of the form

$$\Gamma_2(u,t,\Lambda)=\Lambda^2 t\left[1+\sum a_{mr}u^r(\ln t)^m\right] \quad ,$$
$$\Gamma_4(u,t,\Lambda)=\Lambda^\epsilon u\left[1+\sum b_{mr}u^r(\ln t)^m\right] \quad . \qquad (1.8)$$

Following Nelson and Domany,[10] Bruce and Wallace have chosen the value of l in II such that $L \equiv \ln t = 0$ at $l = l^*$, giving the results

$$\Gamma_2=\mathfrak{D}(l^*)e^{-2l^*}t(l^*)\left[1+\sum u^r(l^*)a_{0r}\right] \quad ,$$
$$\Gamma_4=\mathfrak{D}^2(l^*)e^{-\epsilon l^*}u(l^*)\left[1+\sum u^r(l^*)b_{0r}\right] \quad . \qquad (1.9)$$

As we will see below, $u(l^*)$ is a renormalization-group invariant with a temperature dependence which varies from the bare value u to the fixed-point value u^* [defined by $\beta(u^*)=0$], as $t \to 0$. The principle nonanalytic temperature dependence of Γ_2 and Γ_4 is carried by the behavior of the explicit exponential terms. The terms in the summation therefore represent correction to scaling terms.

This particular choice of matching puts a double burden of interpretation on the potential user of these results. In general we cannot expect to have exact closed form expressions for $t(l)$ and $u(l)$. To the uncertainty of their perturbation expansions we must add the interpretation of the series of correction-to-scaling terms given in the summations. In one case however, the exact renormalization-group trajectories of $t(l)$ and $u(l)$ are known: the spherical limit $n \to \infty$. In this case, the matching condition $L=0$ does not properly represent the exactly known solutions unless the entire perturbation series is used. The choice of L which does give the spherical limit exactly even for a truncated perturbation series is that L for which the perturbation series cancels exactly. With this L the results are (for all n)

$$\Gamma_2=\mathfrak{D}(l_2)e^{-2l_2}t(l_2) \quad , \qquad (1.10a)$$

$$\Gamma_4=\mathfrak{D}^2(l_4)e^{-\epsilon l_4}u(l_4) \quad , \qquad (1.10b)$$

where we have indicated that the matching point for each vertex function is in general different. It will also change as more terms of the sum are used. This particular choice of matching condition and no other makes the resemblance between the generator and field-theoretic approaches strongest. Further it is difficult to propose any other equally simple condition which will recover the spherical limit as n becomes large. Of course, any choice of L which is a renormalization-group invariant and reduces to a constant as $t \to 0$ is formally equivalent in perturbation theory to the order that the calculation includes. We can only hope that the alternate resummation of the correction-to-scaling terms represented by this choice is an improvement for all values of n. For example, with any other matching condition the error in a second-order calculation is $O(\epsilon^3)$. With the spherical limit given exactly the error is presumably reduced for all n. An examination of the form of the ϵ expansion suggests that the error is reduced by a factor of $n+8$ to $O(\epsilon^3/(n+8))$. Thus even for $n=1$, 2, and 3, a considerable improvement may be gained.

This can be checked explicitly by comparing the first-order results with the second, since this new matching condition gives the exact spherical result even at first order. Even if the actual improvement is small, there are advantages in anchoring the calculation at the spherical limit, thereby allowing some direct $d=3$ spherical results to be used for comparison. For example, a clear distinction is made between cutoff-form-dependent and independent terms which is lost in the $L=0$ match.

In Sec. II, the results of II are reinterpreted in this light and a similar calculation used to give a corresponding expression for the specific heat which may be compared with Theumann.[11] In Sec. III, the Ising-model equation of state is considered; with the free energy $A(M)$, h/M, Γ_2, and $\Gamma_2\xi^2$ given to $O(\epsilon^2)$. Finally in Sec. IV, the application of the results to three dimensions will be discussed. Appendix A contains a discussion of the effects of varying cutoffs on the crossover in the spherical limit. Appendix B gives a compilation of the various perturbation series used in the text.

II. DISORDERED PHASE RESULTS FOR n-COMPONENT SPIN SYSTEMS

In this section we will rederive $O(\epsilon^2)$ crossover functions in the light of the proposed matching condition. We will see that the results are expressible in more compact, physically meaningful, and suggestive forms, as well as recovering the spherical limit. The notation of II will be used throughout with a few changes to show the relationship to the generator approach given in I.

The expressions for $\beta(u)$ and $1/\nu(u)$ are given in II and summarized in Appendix B. To $O(u^3)$ the solution to Eq. (1.7a) is given by

$$\left[\frac{1-p}{1-\bar{u}}\right]^{\epsilon/\omega}\frac{\bar{u}}{p}=e^{-\epsilon l}\ , \tag{2.1}$$

where $p=u(l)/u^*$, $\bar{u}=u/u^*$. The fixed point value u^* and the corrections to scaling exponent ω have the following ϵ expansions:

$$u^*=\frac{3\epsilon}{(1+\frac{1}{2}\epsilon)(n+8)}\left[1+\frac{9n+42}{(n+8)^2}\epsilon+O(\epsilon^2)\right]$$

$$=\frac{1}{1+\frac{1}{2}\epsilon}\hat{u}\ , \tag{2.2a}$$

$$\omega=\epsilon\left[1-\frac{9n+42}{(n+8)^2}\epsilon\right]\ . \tag{2.2b}$$

The combination $(1-p)/(1-\bar{u})$ occurs frequently and in correspondence to the notation of I will be denoted by Y. Equation (2.1) is a statement that U $[\equiv\bar{u}/(1-\bar{u})^{\epsilon/\omega}]$ is an exact nonlinear scaling field of dimension $\epsilon=4-d$. As we will see, the value of p assigned by the matching condition is a renormalization-group invariant changing in value from $p=\bar{u}$ to $p=1$ as the critical point is approached. We have factored out $B_0\equiv 1+\frac{1}{2}\epsilon$ which represents the exact ϵ dependence for $n\to\infty$. B_0 depends on the nature of the cutoff (cf. Appendix A), while $\hat{u}$ is independent of the cutoff form. Where appropriate we will use $\hat{u}$ to show cutoff invariant expressions.

The corresponding solutions of Eqs. (1.6) and (1.7b) are

$$e^{-2l}t(l)=tY^{(2-\nu^{-1})/\omega}\times\exp\left[\frac{(n+2)(13n+44)}{6(n+8)^2}u^*(p-\bar{u})\right]\ , \tag{2.3a}$$

$$\mathfrak{D}=Y^{-\eta/\omega}\exp[-(p-\bar{u})\eta/\omega]\ . \tag{2.3b}$$

These expressions are equivalent in the ϵ expression to those given in II but are rewritten to exhibit the resemblances to I and to clarify the nonlinear scaling behavior. The numerical factor in Eq. (2.3a) can be rewritten in a cutoff-form-independent way to the same order as

$$\frac{(n+2)(13n+44)u^*}{6(n+8)^2}=\frac{2-1/\nu}{\omega}-\frac{1}{3}(n+2)\frac{\hat{u}}{\epsilon}\equiv D_n\ . \tag{2.3c}$$

The critical-point exponents ν and η have the ϵ expansions

$$\nu^{-1}=2-\epsilon\frac{(n+2)}{(n+8)}-\frac{1}{2}\epsilon^2\frac{(n+2)(13n+44)}{(n+8)^3}+O(\epsilon^3)\ ,$$
$$\eta=\frac{n+2}{2(n+8)^2}\epsilon^2+O(\epsilon^3)\ . \tag{2.4}$$

Note, however, that the forms of Eqs. (2.1) and (2.3) may be used with critical-point exponents determined to higher order in the ϵ expansion or with experimentally determined exponents (this is termed exponent improvement, see I and Brezin *et al.*[9]). Equations (2.1)–(2.4) are exact for $n\to\infty$. Equation (2.3) shows that

$$T[\equiv t(1-\bar{u})^{(1/\nu-2)/\omega}\exp(\bar{u}D_n)]$$

is an exact nonlinear scaling field of dimension 2. Note in this regard that the fields defined in II are not *global* nonlinear scaling fields in this sense. This is immediately clear from Eq. (2.35) of II since, if the indicated scaling fields were global nonlinear scaling fields the susceptibility would be a global nonlinear scaling field.

Using the diagrammatic expansion for Γ_2 given in Eq. (2.5) of II, the matching condition is

$$L_2 \equiv lnt(l_2) = -1 + \tfrac{1}{4}\epsilon + \tfrac{1}{2}u^*p(f-1) \quad , \qquad (2.5)$$

where $f = 4 + \pi^2 - 8\lambda$, $\lambda \simeq 1.17$. We can now use Eqs. (2.1)–(2.3) to determine p and hence the entire function; alternatively, we may eliminate l by defining $\kappa_2^2 = \exp(-2l_2)t(l_2)$

$$Y^{\epsilon/\omega}\frac{\bar{u}}{p} = \kappa_2^\epsilon \exp\tfrac{1}{2}(-\epsilon L_2) \quad . \qquad (2.6)$$

Since L_2 is only determined in an ϵ expansion, the final results should be written

$$\Gamma_2 = \mathfrak{D}\exp(-2l_2)t(l_2) = \mathfrak{D}\kappa_2^2 \quad , \qquad (2.7a)$$

$$Y_2^{\epsilon/\omega}\frac{\bar{u}}{p_2} = \kappa_2^\epsilon B_0[1 - \tfrac{1}{4}p_2\hat{u}\epsilon(1-f) + O(\epsilon^3)] \quad . \qquad (2.7b)$$

Although this is of the same form as Eq. (1.13) we may not deduce that $\kappa_2 = \xi^{-1}$. The series for Γ_2/ξ^{-2} can be derived to give (for general L)

$$\Gamma_2\xi^2 = \mathfrak{D}(l)\{1 - \eta p^2[\tfrac{1}{2}(L+1) - \tfrac{1}{3}I]\} \quad , \qquad (2.8)$$

where $I = -2.349$. Therefore, with $\kappa^2 = \xi^{-2}$

$$\frac{\kappa_2^2}{\kappa^2} = (1 + \eta p_2^2 \tfrac{1}{3} I) \quad . \qquad (2.9)$$

There is a small weak distinction which is $O(\eta)$. Therefore in equations such as Eq. (2.7b) κ_2 can be replaced by κ to the same order. The equation for κ_2^2 is simply Eq. (2.3a)

$$\kappa_2^2 = tY_2^{(2-\nu^{-1})/\omega}\exp[D_n(p_2 - \bar{u})] \qquad (2.10)$$

providing a complete parametric representation.

The subscript 2 reminds us that this value of p is that appropriate to the matching condition for Γ_2. Note that κ_2 considered as a function of its arguments (t,u) is an exact nonlinear scaling field of dimension 1; combining this with the exact nonlinear scaling properties of U given in Eq. (2.1) we see that p_2 is an exact nonlinear renormalization-group invariant. There is only one invariant for this problem so that any other renormalization-group invariant is a function of p_2.

The above results are exact in the spherical limit, both in form and in detail even at one loop order, $L_2 = -1$. As shown in Appendix A the constant factor in Eq. (2.7b) for the cutoff used in II is exactly B_0 to all orders in ϵ. The proper value of $\exp(-\frac{1}{2}\epsilon L)$ is obtained for all cutoffs if the one-loop integral is evaluated exactly. The difference between this matching condition and any other is also seen most clearly in the spherical limit, as detailed in Appendix A. Any renormalization-group invariant choice of L gives the same asymptotic critical behavior and will provide a crossover to mean-field behavior. However, the details of this crossover do depend on the choice of L. The point to note is that any choice of L other than that adopted here gives rise to a series of corrections (polynomials in p) which in fact are properly represented by the adjustment of the constant amplitude appearing in Eq. (2.7b). For $n \neq \infty$, we cannot be certain that this matching condition is optional, but the relationship to the generator expressions is suggestive. We see in these matching conditions the cutoff dependent scale factor B_0 explicitly appearing in Eq. (2.7b). This term is nonuniversal even for fixed $\bar{u}$, but it can be removed by a change of scale. As shown in Appendix A, the use of the $L = 0$ or any other matching condition does not generally isolate this nonuniversal factor, so that the use of different cutoffs would give different crossovers in the ϵ expansion. The presently employed matching technique correctly gives a universal crossover curve.

Turning now to Γ_4, the matching condition gives a different value of the invariant L:

$$L_4 = -2 + u^*p\tfrac{1}{6}(n+2) + \frac{1}{3}\left[\frac{5n+22}{n+8}\right]u^*pf \quad . \qquad (2.11)$$

Note that in the spherical limit L_4 does not reduce to a p-independent constant. However, it is also true that $\exp(-2l_4)$ is no longer exactly equal to $\Gamma_2/\mathfrak{D}$; the series for Γ_2 is *not* canceled by this choice of L. The best physically transparent answer is given by expressing the p for Γ_4 in terms of ξ not $t(l_4)$. To this order we do not need the $O(\epsilon^2)$ corrective factor given in Eq. (2.8) and may set $\kappa^2 = \Gamma_2/\mathfrak{D}$:

$$\kappa^2 = e^{-2l_4}t(l_4)[1 - u^*p_4\tfrac{1}{6}(n+2)] \quad . \qquad (2.12)$$

Therefore, Γ_4 is given by $\Gamma_4 = \mathfrak{D}^2(l_4)Y_4^{\epsilon/\omega}u$ where

$$Y_4^{\epsilon/\omega}\frac{\bar{u}}{p_4} = \kappa^\epsilon\left[1 + \epsilon + \tfrac{1}{2}\epsilon^2 - \tfrac{1}{6}\epsilon u^* p_4\frac{5n+22}{n+8}f\right] \quad . \qquad (2.13a)$$

The p dependence of L_4 in the large n limit is canceled by the corresponding p dependence of the relationship between κ^2 and $t(l_4)$. In this case, the amplitude given in Eq. (2.13a) is not given exactly for $n \to \infty$. However, considerations of the exact result allow it to be written as

$$Y_4^{\epsilon/\omega}\frac{\bar{u}}{p_4} = \frac{\kappa^\epsilon}{1 - \frac{1}{2}\epsilon}B_0\left[1 - \tfrac{1}{6}\epsilon\hat{u}p_4\frac{5n+22}{n+8}f\right] \quad , \qquad (2.13b)$$

which is exact in the spherical limit. The invariants p_2 and p_4 are, course, related:

$$\left[\frac{1-p_4}{1-p_2}\right]^{\epsilon/\omega}\frac{p_2}{p_4} = \frac{\left[1 - \frac{1}{6}\epsilon\hat{u}p_4\frac{5n+22}{(n+8)}f\right]}{[1 - p_2\frac{1}{4}\epsilon\hat{u}(1-f)](1 - \frac{1}{2}\epsilon)} \quad . \qquad (2.14)$$

The specific heat C also satisfies a renormalization-group equation. In the renormalized field theory, there is an additional additive renormalization constant resulting in an inhomogeneous term even in the critical regime

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(u)\frac{\partial}{\partial u}-\frac{1}{\nu(u)}\left(2+t\frac{\partial}{\partial t}\right)\right]C=\Lambda^{+d}B(u) \ . \tag{2.15}$$

Again, there are small differences in notation from that of Ref. 11 to increase the resemblance to generator methods. The solution of this equation is given by

$$C(u,t,\Lambda=1)=C(u(l),t(l),\Lambda=e^{-l})\frac{t^2(l)}{t^2}+\int_0^l \exp(-dl')B(u(l'))\frac{t^2(l')}{t^2}dl' \ . \tag{2.16}$$

For the specific heat there is no bare term to which the diagram series is added; the Landau-Ginzburg model has no specific heat in the disordered phase. Moreover, the original idea of Wilson[1] for calculating the free energy (and hence C) was simply to add up the constant terms in the Hamiltonian produced by each iteration of the renormalization-group reduction of degrees of freedom (Nauenberg and Neinhuis,[12] Nelson and Rudnick[13]). These terms correspond to the inhomogeneous term in Eqs. (2.15)–(2.16); for these reasons we choose to match by canceling the entire perturbation expansion leaving only the inhomogeneous term. To the order needed we have $B=nB_0$ and

$$L_C=-2+\tfrac{1}{6}(n+2)u^*p_C \ . \tag{2.17}$$

The integration is performed by transforming the integral over l to one over Y

$$C=nB_0\int_0^l \exp(\epsilon l)\,Y^{2(2-\nu^{-1})/\omega}\,[1+2D_n(p-\bar{u})]dl \ . \tag{2.18}$$

Evaluating the integral (again terms $\propto p^2$ in the integral are dropped) we find

$$C=+\frac{n}{\epsilon\bar{u}}B_0\left[\frac{Y^{-\alpha/\omega\nu}-1}{\alpha/\epsilon\nu}+\frac{Y^{1-\alpha/\omega\nu}-1}{1-\alpha/\omega\nu}\left(\frac{\epsilon}{\omega}-1+2D_n\right)(1-\bar{u})\right]\exp 2D_n(1-\bar{u}) \ . \tag{2.19}$$

This is exact for $n=\infty$ and -2 which correspond to the most negative and most positive values of α. (This result for the inhomogeneous term differs from that of Ref. 11.) Finally the choice of matching condition gives

$$Y_C^{\epsilon/\omega}\frac{\bar{u}}{p_C}=\kappa^{\epsilon}\frac{B_0}{(1-\frac{1}{2}\epsilon)} \ . \tag{2.20}$$

We have shown that a simple change in the matching condition allows us to express the results of field theoretic perturbative calculations in a more compact form which more closely resembles the differential generator approach and recovers the exact spherical limit for large n. In general, this matching condition still requires the use of ϵ expansions both in the solution of the renormalization-group equations and in the matching conditions for L. However, the nontrivial dependence of the results on L [essentially in the amplitude terms of Eqs. (2.7b) and (2.13c)] is second order in the ϵ expansion rather than entering at first order for any other matching value.

III. ISING-LIKE $n=1$ EQUATION OF STATE

In this section we apply the matching method to the equation of state. Only the $n=1$ case will be considered because such a simple matching approach is suitable for resumming perturbation series with only a single singularity. The general n-component model has both transverse and longitudinal susceptibility and thus has two types of logs. In the differential generator approach there are two values of l at which the longitudinal and transverse fluctuations are essentially suppressed (cf. I). In the $l\to\infty$ limit these mark breaks in the behavior of the l-dependent functions as portions of their forms reach their asymptotic values. No single matching point can properly represent such behavior; other methods can be employed such as an analysis and resummation of important diagrams (Schäfer and Horner[14]). We will give the crossover equations for the free energy $A(M)$, h/M, Γ_2/κ^2 (where h is the magnetic field, $\Gamma_2=\chi^{-1}$, $\kappa=\xi^{-1}$) to $O(\epsilon^2)$.

The renormalization-group equations for finite magnetization M are

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(u)\frac{\partial}{\partial u}-\frac{1}{\nu(u)}t\frac{\partial}{\partial t}-\tfrac{1}{2}\eta(u)\left(N+M\frac{\partial}{\partial M}\right)\right]\Gamma_N\simeq 0 \ . \tag{3.1}$$

The solution is simple:

$$\Gamma_N(u,t,M^2,\Lambda=1)=\mathfrak{D}^{N/2}\Gamma_N(u(l),t(l),\mathfrak{D}M^2,\Lambda=\exp(-l)) \ . \quad (3.2)$$

Thus each explicit factor of M^2 is associated with an anomalous dimension crossover factor $\mathfrak{D}$.

The perturbation series for h/M can be easily obtained to $O(\epsilon^2)$ from those for $A(M)$ given in Appendix B or from Wallace and Zia.[15] We choose to match so that h/M has its Landau form

$$\frac{h}{M}=\mathfrak{D}\{t(l)\exp(-2l)+u(l)[\tfrac{1}{6}\exp(-\epsilon l)]\mathfrak{D}M^2\} \ . \quad (3.3)$$

The matching point is

$$L_1\equiv\ln\kappa_1^2\exp(2l)$$
$$=-1+\tfrac{1}{4}\epsilon+\tfrac{1}{2}u^*p[(f-1)+\tfrac{1}{2}q(1+f)] \ . \quad (3.4)$$

In Eq. (3.4)

$$\kappa_1^2=\exp(-2l)\{t(l)+\exp(2-\epsilon)l[\tfrac{1}{2}u(l)\mathfrak{D}]M^2\} \ , \quad (3.5a)$$

$$q\equiv\frac{u(l)\exp(-\epsilon l)\mathfrak{D}M^2}{\kappa_1^2}=\frac{Y^{\epsilon/\omega}uM^2\mathfrak{D}}{\kappa_1^2} \ . \quad (3.5b)$$

The M dependence is carried by the exact renormalization-group invariant q. The present matching condition has the additional benefit that $q=3$ on the coexistence surface and $q=2$ for $t=0$ to all orders in ϵ. The solution is completed by specifying Y

$$Y^{\epsilon/\omega}\frac{\bar{u}}{p}=\kappa_1^\epsilon B_0\{1-\tfrac{1}{4}\epsilon\hat{u}p[(f-1)+q(\tfrac{1}{2}+\tfrac{1}{2}f)]\} \ . \quad (3.6)$$

The exact scaling field κ_1^2 has no direct physical significance. Γ_2 is given by

$$\Gamma_2=\mathfrak{D}\kappa_1^2[1+q\tfrac{1}{2}\hat{u}p(1-\tfrac{1}{2}\hat{u}p)$$
$$-\tfrac{1}{4}(\hat{u}p)^2q(1-\tfrac{1}{2}q)(f+1)] \ . \quad (3.7)$$

Note that this matching implies that the terms in the square brackets are identically equal to $\frac{1}{3}\delta$ at $q=2$, $p=1$. This allows κ_1^2 to be solved for in terms of Γ_2 to complete the equation of state in physical terms. Equation (3.3) can, of course, be rewritten as

$$\frac{h}{M}=\mathfrak{D}[tY^{(2-1/\nu)/\omega}\exp D_1(p-\bar{u})+\tfrac{1}{6}uY^{\epsilon/\omega}\mathfrak{D}M^2] \ , \quad (3.8)$$

which again stresses Landau-like nature of the result in this form.

We now turn to the Helmholtz potential $A(M)$. It obeys the renormalization-group equation

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(u)\frac{\partial}{\partial u}-\frac{t\partial}{\nu(u)\partial t}-\tfrac{1}{2}\eta M\frac{\partial}{\partial M}\right]A$$
$$=-\tfrac{1}{2}B(u)t^2\Lambda^d \ . \quad (3.9)$$

The solution is

$$A(u,t,M^2,\Lambda=1)=A(u(l),t(l),\mathfrak{D}M^2,\Lambda=e^{-l})$$
$$-\int_0^l e^{-dl}\tfrac{1}{2}B(u(l))t^2(l) \ . \quad (3.10)$$

We could choose l such that the diagram series for A vanished leaving only the Landau-Ginzburg and trajectory integral terms. However, by using the same L as for the equation of state we preserve the useful normalization of the invariant q. Therefore there are three contributions to A. First the Landau-Ginzburg part:

$$A_{\rm LG}=\tfrac{1}{2}tY^{(2-1/\nu)/\omega}[\exp D_1(p-\bar{u})]\mathfrak{D}M^2$$
$$+\frac{u}{4!}Y^{\epsilon/\omega}\mathfrak{D}^2M^4 \ ; \quad (3.11a)$$

second, the trajectory integral

$$-\frac{t^2}{2\epsilon\bar{u}}B_0\exp[2D_1(1-\bar{u})]\left[\frac{(Y^{-\alpha/\omega\nu}-1)}{(\alpha/\epsilon\nu)}+\frac{(Y^{1-\alpha/\omega\nu}-1)}{1-\alpha/\omega\nu}\left(\frac{\epsilon}{\omega}-1+2D_1\right)(1-\bar{u})\right] \ ; \quad (3.11b)$$

and finally, the remainder of the diagrams

$$-\frac{B_0}{8}\frac{(\kappa_1^2)^2Y^{-\epsilon/\omega}}{\bar{u}}p\left\{\frac{1}{1-\tfrac{1}{4}\epsilon}-\hat{u}p[(f-1)+\tfrac{1}{2}q(3-f)]\right\} \ . \quad (3.11c)$$

Matching to cancel the whole series would eliminate Eq. (3.11c) but h/M and Γ_2 would have to be evaluated with L_A:

$$L_A=-\tfrac{1}{2}+\tfrac{5}{16}\epsilon-\tfrac{1}{8}u^*p[1-q(1-4f)] \ , \quad (3.12a)$$

where p and q are defined at L_A. We note that in the spherical limit a single match point is again appropriate; matching for h/M gives the exact result for A as well when the one-loop term is evaluated exactly. That is, the one-loop match point cancels all the higher-order terms as well. Since $Y-1$ is formally $O(\epsilon)$ the trajectory integral is determined to one less order than the mean-field and diagram terms.

Finally, to complete the survey of the $O(\epsilon^2)$ Ising-like result we compute the quantity $\Gamma_2\xi^2 = \mathfrak{D}\Gamma_2(l)\xi^2(l)$. The series for $\Gamma_2\xi^2$ is given in Appendix B. At the h/M match point this gives

$$\Gamma_2\xi^2 = \mathfrak{D}\{1 + \tfrac{1}{12}\hat{u}qp(1-\tfrac{1}{2}\epsilon) + \tfrac{1}{18}(\hat{u}p)^2[I + \tfrac{1}{2}q(1-4I) + \tfrac{1}{2}q^2(\tfrac{1}{4}11 + I)]\} \quad . \tag{3.13}$$

We may use this to compare our matching length scale κ_1 with $\kappa \equiv \xi^{-1}$

$$\kappa_1^2 = \kappa^2 \frac{\{1 + \tfrac{1}{12}\hat{u}pq(1-\tfrac{1}{2}\epsilon) + \tfrac{1}{18}(\hat{u}p)^2[I + \tfrac{1}{2}q(1-4I) + \tfrac{1}{2}q^2(\tfrac{1}{4}11 + I)]\}}{[1 + \tfrac{1}{2}\hat{u}pq(1-\tfrac{1}{2}\hat{u}p) - \tfrac{1}{4}(\hat{u}p)^2 q(1-q/2)(f+1)]} \quad . \tag{3.14}$$

It is easy to show that these expressions Eqs. (3.13)–(3.14) give the correct amplitude ratios obtained by other methods.[9,16]

IV. DISCUSSION

In this section we will discuss several aspects of the present calculations which bear on their applicability to three dimensions. No attempt to be exhaustive is intended, but the general lines will be indicated.

We first consider the fact that even at $O(\epsilon)$ the spherical limit is recovered exactly. We may understand this by noting that for $n \to \infty$ all the diagrams are simply related to the one-loop diagram and sum geometrically. Thus when we cancel the diagram series for $n \neq \infty$ we are defining an effective "mass" $m^\epsilon \propto \kappa_1^\epsilon \exp(-\tfrac{1}{2}\epsilon L)$ in terms of which we can make an equivalent simple summation. The actual summation is of course, given by the solution to the renormalization-group equations, but we can imagine this mass as containing all the information not embodied in a "screened interaction." This gives some physical insight into the present matching technique.

Conservatively speaking, one lesson is that the choice of L should not be a passive act; each choice carries with it a decision about the details of the crossover behavior. Unfortunately, the spherical examples also show that it is not enough to consider various constant values of L; the renormalization-group invariants enter when the best $n = \infty$ L is used. Therefore, we require at least partly objective criteria for L. Aside from the $n = \infty$ limits being given exactly, the present choice has the advantage of preserving to the greatest degree possible the original Landau-Ginzburg form.

This has several consequences which are in part technical. First, for the Ising-like equation of state the invariant $q\,(\equiv u\Lambda^\epsilon \mathfrak{D}M^2/\kappa_1^2)$ can be normalized to all orders in ϵ to its $\epsilon = 0$ values: $q = 2$ for $t = 0$ and $q = 3$ for the coexistence surface. This makes q a good candidate for a parametric description of the equation of state. Second, if we wish to blend the crossover-critical equation of state with a more general background we may do so simply by appending the desired terms; for example

$$\frac{h}{M} = \mathfrak{L} + \sum_{p=3}^{\infty} a_p(t)M^p \quad , \tag{4.1a}$$

where $\mathfrak{L}$ represents Landau-Ginzburg terms. In fact, the a_p should also contain crossover terms. To leading order we can always express them in terms of Y

$$a_p \to Y^{\lambda_p} a_p \quad , \tag{4.1b}$$

with $\lambda_p = \tfrac{1}{6}p(p-1) + O(\epsilon)$. Finally, by compactly expressing the equation of state in terms of Y, it is relatively simple to correct the crossover equations for cutoff effects.

The need for such a correction can be seen even at $O(\epsilon)$. The expression for Y in terms of a matching l^* is

$$Y^{-1} = 1 + \bar{u}(e^{\epsilon l^*} - 1) \quad . \tag{4.2}$$

For $t \to 0$, $l^* \to \infty$; but for $t \sim 1$, $l^* \sim 0$, Eq. (4.2) makes little sense if $l^* \leq 0$, since it represents the one-loop integral, which can generally be expressed as (see Appendix A)

$$I_1 = \frac{1}{b(0)}\left[a\left(\frac{\kappa^2}{\Lambda^2}\right)\left(\frac{\kappa}{\Lambda}\right)^{-\epsilon} - b\left(\frac{\kappa^2}{\Lambda^2}\right)\right] \quad , \tag{4.3}$$

where a and b are analytic functions of κ^2/Λ^2 which depend on the nature of the cutoff employed. The renormalization-group equations apply as written in this work only if a and b are set equal to their $\kappa = 0$ limits, converting Eq. (4.3) into something resembling Eq. (4.2). The magnitude of the difference is calculated at $d = 3$ for the spherical model in Appendix A and is shown in Fig. 1. The differences are not large until one is well out of the critical regime, but if a smooth transition to high-temperature behavior is desired, we must somehow account for the differences between Eqs. (4.2) and (4.3).

One method that is exact for $n = \infty$, and correct to one-loop (resumed) for all n is to define a new length scale $\tilde{\kappa}$

$$\tilde{\kappa}^\epsilon \frac{b(0)}{a(0)} = \kappa^\epsilon \exp(-\tfrac{1}{2}\epsilon L) \quad , \tag{4.4}$$

where κ and L as usual depend on the function

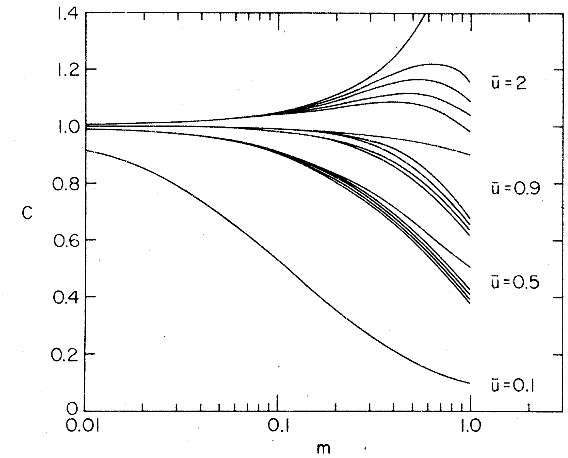

FIG. 1. Family of crossover curves is given for the spherical model at $d=3$, for four different cutoffs. The function plotted is $C\equiv\bar{u}m/t$ as a function of m where $m=\xi^{-1}=(\Gamma_2)^{1/2}$. Each cutoff result has been normalized to have the same Wegner corrections. The asymptotic form incorporating only the Wegner terms is also shown. For each value of $\bar{u}=u/u^*$, the extrapolated asymptotic (Wegner terms) curve lies highest; beneath it are the sharp cutoff, the $k=2$ [cf. (A 6)] cutoff, the exponential loop cutoff, and, finally, the $k=1$ cutoff (A5).

matched. Then the matching condition could be reexpressed as

$$Y^{\epsilon/\omega}\frac{p}{\bar{u}}=\left[\frac{a(\tilde{\kappa}^2)}{b(0)}\tilde{\kappa}^{-\epsilon}-\frac{b(\tilde{\kappa}^2)}{b(0)}+1\right]^{-1} . \qquad (4.5)$$

To higher than one-loop order this is only heuristic but, since it is not expected that cutoff effects will be large, an approach motivated by one-loop and spherical results should suffice. An alternate approach is to properly include the cutoff in the renormalization-group equations as is done in differential generator approaches or in the manner of Lawrie.[17] However, in either case the solution of the nonlinear differential equations is rendered far more complex, while Eq. (4.5) is in the spirit of the effective bubble resummation discussed above. It is encouraging that the various cutoff forms give similar results (cf. Fig. 1). Any modification of the purely renormalization-group results should be considered primarily as a smoothing algorithm for the matching technique to avoid the unphysical consequences of Eq. (4.2) as $l^*\to 0$. A consistent approach would require the inclusion of ϕ^6 and related terms in the Hamiltonian as well as the cutoff effects on the renormalization-group flows for the parameters and inhomogeneous term.

We now turn to the nature of the dependence of these $O(\epsilon^2)$ results and their arbitrary ϵ generalizations on the form of the cutoff employed. The cutoff used by Nickel[18] and followed by many others, including II, is to replace a simple k^2 propagator with k^2+k^4/Λ^2. Many other forms such as $k^2(1+k^2/\Lambda^2)^m$ are possible. As mentioned above, the usual renormalization-group formulation discards most of the consequences of a cutoff choice; however, some do remain. In particular, the value of the fixed point u^* depends on the cutoff form as do the details of various amplitudes related to u^*. Discarding all effects not related to the Wegner expansion in powers of κ^ω we may show that the entire Wegner crossover function is independent of cutoff form and is thus universal. This implies "universal amplitude ratios" not only for the first Wegner connection-to-scaling term but for all its powers.

The demonstration is simple and will be given for one function, κ^2 for $M=0$ as an example. Using a κ^2 match point L_κ, then $\kappa^2=t(l^*)$

$$\kappa^2=t\exp\left[\int_0^{l^*}(1/\nu-2)\,dl\right] . \qquad (4.6a)$$

This can always be written as

$$\kappa^2=t\left[\frac{1-p}{1-\bar{u}}\right]^{(2-1/\nu)/\omega}\frac{F(p)}{F(\bar{u})} , \qquad (4.6b)$$

where $F(p=0)=1$. To $O(\epsilon^2)$, $F(p)=\exp(D_n p)$. The match point is determined from the u equation

$$\left[\frac{1-p}{1-\bar{u}}\right]^{\epsilon/\omega}\frac{G(p)}{G(\bar{u})}\frac{\bar{u}}{p}=\frac{\kappa^\epsilon}{\Lambda^\epsilon}\exp(-\tfrac{1}{2}\epsilon L_\kappa) , \qquad (4.7a)$$

where $G(p)=1$ to $O(\epsilon^2)$, $G(0)=1$ and we have restored a value for the cutoff Λ (usually set equal to 1). The behavior of $\exp(-\frac{1}{2}\epsilon L_\kappa)$ is known trivially to one loop; we write it as

$$\left[\frac{1-p}{1-\bar{u}}\right]^{\epsilon/\omega}\frac{G(p)}{G(\bar{u})}\frac{\bar{u}}{p}=\frac{\kappa^\epsilon}{\Lambda^\epsilon}\frac{b(0)}{a(0)}H(p) , \qquad (4.7b)$$

where $H(p=0)=1$. Clearly, the factor $\Lambda^{-\epsilon}b(0)/a(0)$ depends on the form of the cutoff. To each order in perturbation theory F, G, and H can be written as polynomials.

The existence of an infinite Λ limit implies that the parametric system Eqs. (4.6)–(4.7) has a unique cutoff independent limit if $u=u_0\Lambda^{-\epsilon}$ and the limit $\Lambda\to\infty$ is taken. Then we have

$$\kappa^2=t(1-p)^{(2-1/\nu)/\omega}F(p) , \qquad (4.8a)$$

$$(1-p)^{\epsilon/\omega}\frac{G(p)}{p}u_0=\kappa^\epsilon\hat{u}H(p) . \qquad (4.8b)$$

Equation (4.8) defines $\kappa^2(t,u_0)$ parametrically and is independent of the cutoff. This applies not only to the critical limit $p\to 1$ but also to the "mean-field" region $p\to 0$. The complete system Eqs. (4.6)–(4.7) differs from Eq. (4.8) only in the choice of nonuniversal [$\bar{u}$- and $\Lambda^{\epsilon}b(0)/a(0)$-dependent] scale factors, and is therefore also cutoff independent (within scales). A similar demonstration is possible for any other function and for $M \neq 0$ (the use of the invariant q is useful in that context). While each of the component parts of a general parametric expression of the form Eq. (4.8) need not be cutoff independent, they (and all of the corresponding expressions given here) are so to $O(\epsilon^2)$. This highlights another disadvantage of the $L = 0$ match, since it does not separate the cutoff dependent and independent factors into multiplicative factors as in Eq. (4.7b). Note that for fixed $\bar{u}$, all the cutoff dependence resides in the $\Lambda^{-\epsilon}b(0)/a(0)$ factor.

Strickly speaking, the infinite Λ limit guarantees cutoff invariance only in the region $0 \leq p \leq 1$. Because p ranges from $\bar{u}$ (at $l^* = 0$) to 1 (at the critical point), the region $p > 1$ is accessible for $\bar{u} > 1$ (which appears to be the case for the $d = 3$ Ising model[19]). However, at each order in the ϵ expansion, F, G, and H and the function N defined by

$$\exp\left[\int_0^{l^*} \eta \, dl\right] = Y^{-\eta/\omega} N(p)/N(\bar{u})$$

are expressible as polynomials or their exponentials. We therefore expect no difficulty in extending the invariance of these functions to $p > 1$.

A final topic concerns the convergence of the crossover forms in the ϵ expansion. This is highlighted by the recent work of Aharony and Ahlers[20] and Chang and Houghton[21] on the universal properties of correction-to-scaling amplitudes. Aharony and Ahlers point out that the lowest order, the ratio of the correction-to-scaling amplitudes of any two vertex functions, is given by the ratio of their anomalous dimensions (shift in eigenvalue from Gaussian). In the present context, this follows from the fact that to lowest order the corrections to scaling are carried entirely by the Y function. If this were true beyond lowest order, then the ratios would continue to be given by the anomalous dimensions. Further, this would guarantee exponent relations among the effective critical-point exponents obtained by local fits to the crossover curves; e.g., if only Y occured, $\alpha + 2\beta + \gamma = 2$ would hold everywhere in the crossover region.

However, not all the crossover is in the function Y. Chang and Houghton have calculated several amplitudes two orders beyond lowest order, corresponding to an $O(\epsilon^3)$ crossover function. One can easily show that the results given in the present work duplicate the first correction to the leading amplitude. The ϵ expansions are incredibly poorly convergent, yielding very little definitive information at first glance. Part of this poor convergence may be attributable to the correspondingly poor behavior of the critical exponents at $O(\epsilon^3)$. By calculating the full crossover functions, these exponents are isolated in the exponents of the Y's; this may allow a more useful determination of these amplitudes. Explicit $O(\epsilon^3)$ calculations of the full crossover equations are in progress to determine if this is the case. Conservatively, the work of Chang and Houghton is a warning that the ϵ expansions which remain in the matched crossover functions given here [such as for $\exp(-\frac{1}{2}\epsilon L)$] need to be treated cautiously.

In summary, we have shown that by considering the spherical limit and the structure of the differential generators, an alternative matching technique can be found which should give a better resummation. Moreover, it yields results in a convenient form for blending in additional terms in the equation of state or including the non-Wegner corrections due to the form of the cutoff. Renormalization-group nonlinear scaling fields and invariants are stressed in the approach; this should provide reliable expressions since exact nonlinear principles are correctly embodied.

APPENDIX A: CUTOFF CORRECTIONS AND MATCHING-POINT EFFECTS

If only the form of a Wegner expansion is needed, then we need not calculate a full crossover equation; in this case the amplitudes of the leading and nonleading singularities can be considered as free parameters. If more than a single term is necessary, either because $\bar{u} << 1$ or because a complete passage to mean field is desired, then the second and higher powers of the Wegner correction will be needed and their amplitudes are not free. The nth Wegner term is $\xi^{-n\omega}$ smaller than the leading singularity. In the ϵ expansion where $\omega \simeq \epsilon + O(\epsilon^2)$ these are indeed the most important corrections; but as $\epsilon \to 1$, 2ω (and 3ω) become comparable to the corrections which arise from the neglected details of the cutoff mechanism.

In this Appendix we explore the nature of the latter effects by means of the spherical limit, $n = \infty$, for which exact results may be obtained; in particular $\omega \equiv \epsilon$. They have previously been considered for the n-vector model to $O(\epsilon)$ by Lawrie[17] for several cutoffs. Lawrie's results are essentially included here since the $O(\epsilon)$ results is a one-loop result and for $n = \infty$ the one-loop result represents all the information. These having been examined, the degree of approximation incumbent in these matching-point methods is examined. If the propagator is taken to be $g(p^2) + m^2$, then in the spherical limit, the inverse susceptibility is

$$\Gamma_2 \equiv m^2 = \frac{t}{1 + (u/\epsilon)I} , \quad \frac{I}{\epsilon} = \int \frac{d^d p}{g(p^2)[g(p^2) + m^2]} . \tag{A1}$$

(The corresponding expression for Γ_4 involves

$\tilde{I} = \partial/\partial m^2 m^2 I$.) In general I is of the form

$$I = a(m^2)m^{-\epsilon} - b(m^2) \ , \quad (A2)$$

where a and b are analytic in m^2 [$b(0) \neq 0$] and $a(0)$ is universal

$$a(0) = \Gamma(1+\tfrac{1}{2}\epsilon)\Gamma(1-\tfrac{1}{2}\epsilon) \ .$$

The function b and the m^2 dependence of a depend on the cutoff. For a smooth cutoff $g(p^2) = p^2(1+p^2)^k$ we find

$$b(0) = \Gamma(2k+\tfrac{1}{2}\epsilon)\Gamma(1-\tfrac{1}{2}\epsilon)/\Gamma(2k) \ ;$$

on the other hand, for $g(p^2) = p^2(1+p^{2k})$,

$$b(0) = \Gamma(1-\epsilon/2k)\Gamma(2+\epsilon/2k) \ .$$

The full forms of a and b can be calculated in a few cases. For example, a sharp cutoff $p^2 \leq 1$, $g(p^2) = p^2$ gives $a(m^2) = a(0)$ and

$$b_s = {}_2F_1(1, \tfrac{1}{2}\epsilon, 1+\tfrac{1}{2}\epsilon, -m^2) \ . \quad (A3)$$

On the other hand, associating a factor of $\exp(-p^2)$ with the loop integral gives

$$a_L = \Gamma(1-\tfrac{1}{2}\epsilon)\Gamma(1+\tfrac{1}{2}\epsilon)e^{m^2} \ ,$$
$$b_L = \Gamma(1-\tfrac{1}{2}\epsilon)\,{}_1F_1(1, 1+\tfrac{1}{2}\epsilon, m^2) \ . \quad (A4)$$

For $d = 3$, $\epsilon = 1$, the smooth cutoff cases $g(p^2) = p^2(1+p^{2k})$ can be evaluated for $k = 1, 2$. For $k = 1$ we find the simple result

$$I_1 = \frac{\pi}{2m^2}\left[1 - \frac{1}{\sqrt{2m+1}}\right] \ . \quad (A5)$$

The result for $k = 2$ is much more complicated:

$$I_2 = \frac{\pi}{2m^2}\left[\frac{1}{\sqrt{2}} + \frac{4}{3}\frac{[C_-^{1/2} - (\sqrt{3}C_-D_+ + C_+D_-)/C_+]}{(3C_-^2 + C_+^2)}\right] ,$$
$$D_\pm = \tfrac{1}{2}[(C_-^2 + 3C_+^2)^{1/2} \pm C_-]^{1/2} \ , \quad (A6)$$
$$C_\pm = 2^{-1/3}\{[(m^4 + \tfrac{4}{27})^{1/2} + m^2]^{1/3} \pm [(m^4 + \tfrac{4}{27})^{1/2} - m^2]^{1/3}\} \ .$$

The rather complicated form of I_2 serves as an example of a result true for general cutoffs and all ϵ: If $g(p^2) = p^2 + O(p^2 p^{2k})$, then $a(m^2) = a(0) + O(m^{2k})$. This is a consequence of the fact that the nonquadratic parts of $g(p^2)$ may be considered to be irrelevant corrections to scaling of the two-point correlation function.

In three dimensions the sharp and loop cutoffs results can be written

$$I_s = \frac{1}{m}(\tfrac{1}{2}\pi - \tan^{-1}m) \ , \quad (A7)$$

$$I_L = \tfrac{1}{2}\pi\frac{1}{m}e^{m^2}\operatorname{erfc}(m) \ . \quad (A8)$$

The different values of $b(0)$ lead to different values of the fixed-point value of u [$u^* \equiv 1/b(0)$]. In terms of $\bar{u} \equiv u/u^*$ and a rescaled m each of the above results at $d = 3$ can be written for small m as

$$m^2 = \frac{t}{1 + \bar{u}(m^{-1}-1)\Theta(1-m)} \ . \quad (A9)$$

This last expression contains the first Wegner correction to all orders, but is valid only for $m < 1$.

Figure 1 compares the values of $C \equiv \bar{u}m/t$ for the cutoffs Eqs. (A5)–(A9) as a function of $\bar{u}$ and m. As is plainly shown, the inclusion of the non-Wegner terms changes the deviation from asymptotic behavior. However, for $\bar{u} \ll 1$ the case of a small asymptotic regime, the complete Wegner crossover is sufficient. In this case, a single Wegner correction term is not enough. Note also the close agreement of the various cutoff forms. These deviations are maximal among n-vector models because they are corrections to the function Y which appears with the power $(n+2)/(n+8) + O(\epsilon)$. Thus for $n = 1$, the effects will be roughly the cube root of that shown.

As noted in the text, the match point described in the present work allows a simple inclusion of the cutoff effects into the crossover function in a manner exact for $n = \infty$ and correct to resummed one loop for general n.

Now let us examine (within the spherical model) the effects of match-point choice. For any cutoff but ignoring r/Λ^2 terms, the exact answer is ($r \equiv \Gamma_2 = \chi^{-1} = \xi^{-2}$)

$$r = \frac{t}{1 + u(cr^{\epsilon/2} - 1)} \ , \quad (A10)$$

where

$$c = a(0)/b(0) = 1 + \lambda\epsilon + O(\epsilon^2) \ .$$

The matching condition of the text would, of course, recover this exactly. In a one-loop $O(\epsilon)$ approximation with $L = 0$ match we would have

$$r = \frac{t}{1 + \bar{u}(e^{\epsilon l_0} - 1)}\left[1 - \bar{u}\frac{\lambda\epsilon e^{\epsilon l_0}}{1 + \bar{u}(e^{\epsilon l_0} - 1)} + O(\epsilon^2)\right] ,$$

$$1 = \frac{te^{2l_0}}{1 + \bar{u}(e^{\epsilon l_0} - 1)} \ . \quad (A11)$$

Evaluating Eqs. (A10) and (A11) as written for $\epsilon = 1$ we have the exact result

$$\frac{t}{\bar{u}^2 c^2[\tfrac{1}{2}(1 + \sqrt{1+4tx}/c^2)]^2} \ , \quad (A12)$$

where $x = (1-\bar{u})/\bar{u}^2$. The one-loop $O(\epsilon)$ result for $L = 0$

$$r = \frac{t^2}{\bar{u}^2[\tfrac{1}{2}(1 + \sqrt{1+4tx})]^2}\left[1 - \frac{2\lambda}{1 + \sqrt{1+4xt}}\right] \ . \quad (A13)$$

As an approximation to Eq. (A12), Eq. (A13) is quite poor. For the $k=1$ cutoff used here and in II, $\lambda=-\frac{1}{2}$, $c=\frac{2}{3}$, and the asymptotic amplitudes differ by a factor of $\frac{3}{2}$. Moreover, if a different cutoff were used, c and λ would change. The effects on Eq. (A12) can be eliminated by a change of scale; Eq. (A13) has lost this property. Of course, our primary interest is in crossover functions. Defining $C(y)$ by $r=At^2C(y)$ with $C(0)=1$ and $y=\text{const}\ t$, the exact crossover function can be written

$$C(y)=\frac{4}{(1+\sqrt{1+y}\,)^2}\ . \tag{A14}$$

In general the one-loop $O(\epsilon)$ crossover functions cannot be made to agree with the exact result for both $0<y\ll 1$ and $y\gg 1$. Let us consider the case of $\bar{u}\ll 1$ so that the full crossover regime can be represented by the Wegner terms alone without the full cutoff functions. Then, if we scale y for the one-loop approximation

$$C_1=\frac{4}{(1+\sqrt{1+\alpha y}\,)^2}\left(1+\frac{\lambda}{1-\lambda}\,\frac{\sqrt{1+\alpha y}-1}{\sqrt{1+\alpha y}+1}\right)\ . \tag{A15}$$

The rescaling factor α is determined by forcing agreement between C_1 and C. If we require that the term linear in y agree (the first Wegner correction) then α must be chosen as $\alpha=\alpha_<$,

$$\alpha_<=\frac{2(1-\lambda)}{2-3\lambda}\ .$$

Note that the regime $\frac{2}{3}<\lambda<1$ is excluded. If we require agreement for large y, $\alpha=\alpha_>$,

$$\alpha_>=1/(1-\lambda)\ .$$

Here we require $\lambda<1$.

In Figs. 2(a) and 2(b), the ratio $C_1(y)/C(y)$ is given for various values of λ using $\alpha_<$ [Fig. 2(a)] and $\alpha_>$ [Fig. 2(b)].

The value of λ given by the ϵ expansion is determined by the cutoff. For $q^2(p^2)=p^2(1+p^2)^k$

$$\lambda=-\frac{1}{2}\sum_{j=1}^{2k-1}\frac{1}{j}\ , \tag{A16}$$

while for $g(p^2)=p^2(1+p^{2k})$

$$\lambda=\frac{-1}{2k}\ . \tag{A17}$$

Thus in the ϵ expansion λ is fixed and Fig. 2 represents the error in the crossover function for different choices of cutoff; if we allow λ to vary somewhat from its calculated value, we may improve our result. In fact, $\lambda=0$ gives us the exact result, while $\lambda=+\frac{1}{2}$ gives a crossover function good to 2%(!) over the entire region (only $\lambda=0$ and $\frac{1}{2}$ permit agreement at both large and small y).

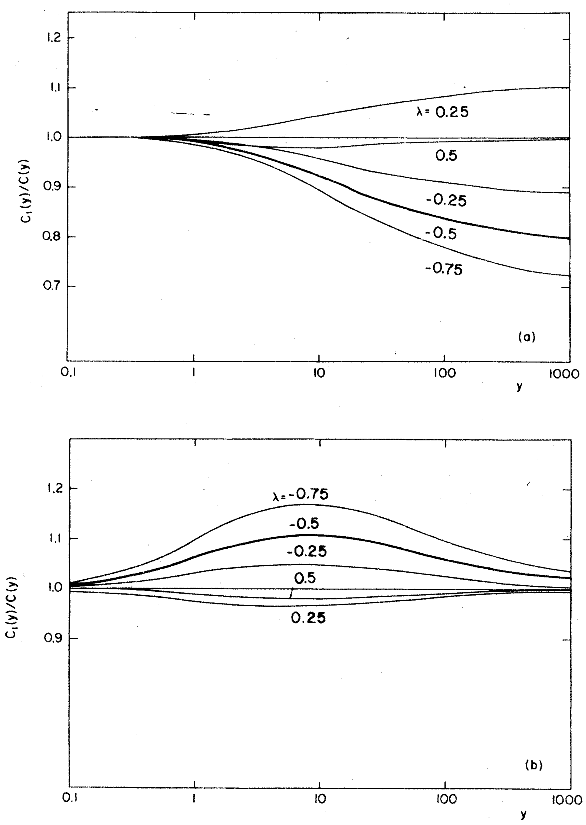

FIG. 2. $O(\epsilon)$ $L=0$ match-point crossover functions $C_1(y)$ are compared to the exact result $C(y)$ for various values of the cutoff-dependent parameter λ [cf. Appendix A (A16)–(A17)] The curve for $\lambda=-0.5$ used in II is shown with a heavy line. (a) The crossover functions scaled to give the same first Wegner correction. (b) The crossover functions scaled to agree with the far from critical regime.

The ability to pick a value of λ which gives the exact result is an accident of the spherical model. The model calculations for finite n will be useless if the ϵ-expansion estimates are allowed to vary arbitrarily. It is to be hoped that the present technique, which separates cutoff-dependent and independent quantities and yields compact expressions, will yield a sharp test of the details of the renormalization-group model.

APPENDIX B: PERTURBATION SERIES

First, we list the results for the Ising-like $n=1$ case and the general n disordered phase specific heat. For completeness, the general n results for the disordered phase Γ_2 and Γ_4 series will be quoted from Bruce and Wallace.[7]

Discarding the non-Wegner terms, the corrections to $A(M)$ to two loops are

$$\Delta A = -\tfrac{1}{4}\Lambda^{-\epsilon}\frac{(\kappa^2)^2}{\epsilon}\left(\frac{(\kappa/\Lambda)^{-\epsilon}}{1-\frac{1}{4}\epsilon}-B_0\right)+\tfrac{1}{8}u\Lambda^{-\epsilon}(\kappa^2)^2\left(\frac{(\kappa/\Lambda)^{-\epsilon}-B_0}{\epsilon}\right)^2-\tfrac{1}{12}u^2M^2\kappa^2[-\tfrac{3}{8}(ln^2\kappa^2/\Lambda^2-f)] \quad \text{(B1)}$$

The first two terms are exact for all ϵ, the third can be obtained from the work of Nickel.[18] A factor of $a(0)K_d$ where

$$a(0)=\Gamma(1+\tfrac{1}{2}\epsilon)\Gamma(1-\tfrac{1}{2}\epsilon)$$

and

$$K_d=2\pi^{d/2}/(2\pi)^d\Gamma(\tfrac{1}{2}d)$$

has been absorbed into A, u, and M^2. The constant $B_0=b(0)/a(0)=1+\frac{1}{2}\epsilon$ exactly with a cutoff of the form p^4/Λ^2 added to the propagator. All higher-order bubble diagrams include a factor of $(\kappa/\Lambda)^{-\epsilon}-B_0$ so that the matching condition $\exp(-\frac{1}{2}\epsilon L)=1+\frac{1}{2}\epsilon$ would cancel all the bubbles in both $A(M)$ and h/M.

The equation for $\Delta h/M$ is obtained by differentiation

$$\Delta h/M=-\tfrac{1}{2}u\kappa^2\left(\frac{(\kappa/\Lambda)^{-\epsilon}-B_0}{\epsilon}\right)+\tfrac{1}{4}u^2\kappa^2\left(\frac{(\kappa/\Lambda)^{-\epsilon}-B_0}{\epsilon}\right)^2-\tfrac{1}{8}u^2\kappa^2\left(\frac{(\kappa/\Lambda)^{-\epsilon}-B_0}{\epsilon}\right)(\kappa/\Lambda)^{-\epsilon}$$
$$-\tfrac{1}{6}u^2\kappa^2[-\tfrac{3}{8}(ln^2\kappa^2/\Lambda^2-f)]+\tfrac{1}{32}\Lambda^{\epsilon}M^2u^3(ln^2\kappa^2/\Lambda^2+2ln\,\kappa^2/\Lambda^2-f) \quad \text{(B2)}$$

Unfortunately, Bruce and Wallace[7] and Wallace and Zia[15] use different normalizations of u. In this work we have adopted the conventions of Bruce and Wallace for this replacement:

$$\Delta A\to\tfrac{1}{2}\Delta A\ ,\ u\to 2u\ ,\ uM^2\to uM^2\ . \quad \text{(B3)}$$

For comparison with Wallace and Zia:

$$\Delta A\to 2\Delta A\ ,\ u\to\tfrac{1}{2}u\ ,\ uM^2\to uM^2\ . \quad \text{(B4)}$$

We may also calculate the series for $\Gamma_2\xi^2$. With the Bruce-Wallace normalization

$$\Gamma_2\xi^2=1+\tfrac{1}{12}uq[1-\tfrac{1}{2}\epsilon(L+1)]$$
$$-\tfrac{2}{3}u^2[\tfrac{1}{8}(L+1)-\tfrac{1}{12}I]$$
$$+\tfrac{1}{36}u^2q[\tfrac{15}{2}(1+L)+(1-4I)]$$
$$+\tfrac{1}{36}u^2q^2[-\tfrac{3}{2}(L+1)+\tfrac{11}{4}I]\ . \quad \text{(B5)}$$

The specific heat to two loops is given by (exactly) in Bruce-Wallace normalization

$$C=n\Lambda^d\left(\frac{(1-\frac{1}{2}\epsilon)t^{-\epsilon/2}-B_0}{\epsilon}\right)-n\tfrac{1}{3}(n+2)u\Lambda^d\left[\left(\frac{(1-\frac{1}{2}\epsilon)t^{-\epsilon/2}-B_0}{\epsilon}\right)^2+\frac{(1-\frac{1}{2}\epsilon)}{2\epsilon}(B_0-t^{-\epsilon/2})\right]\ . \quad \text{(B6)}$$

For completeness we may add the series for the disordered phase Γ_2 and Γ_4 from Ref. 7:

$$\Gamma_2=\Lambda^2t\{1+\tfrac{1}{6}(n+2)u(lnt+1-\tfrac{1}{4}\epsilon ln^2t)+\tfrac{1}{36}(n+2)u^2[(n+5)ln^2t+3(n+2)lnt+2(n+2)-3f]\}\ , \quad \text{(B7)}$$

$$\Gamma_4=\Lambda^{\epsilon}u(1+\tfrac{1}{6}(n+8)u\{[1+\tfrac{1}{6}u(n+2)]lnt+\tfrac{1}{6}u(n+2)+2-\epsilon(\tfrac{1}{4}ln^2t+\tfrac{1}{2}lnt)\}$$
$$+\tfrac{1}{36}(n^2+6n+20)u^2(ln^2t+4lnt+4)+\tfrac{1}{9}(5n+22)u^2(\tfrac{1}{2}ln^2t+lnt-\tfrac{1}{2}f))\ . \quad \text{(B8)}$$

This implies (cf. II)

$$\beta(u)=-\epsilon u+\tfrac{1}{3}(n+8)u^2(1+\tfrac{1}{2}\epsilon)-\tfrac{1}{3}(3n+14)u^3\ ,\quad \eta(u)=\tfrac{1}{18}(n+2)u^2\ ,$$
$$1/\nu(u)=2-\tfrac{1}{3}(n+2)u(1+\tfrac{1}{2}\epsilon)+\tfrac{1}{3}(n+2)u^2+\tfrac{1}{18}(n+2)u^2\ . \quad \text{(B9)}$$

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