First- and second-order transitions in the Potts model near four dimensions

Amnon Aharony* and E. Pytte
IBM T. J. Watson Research Center, P. O. Box 218, Yorktown Heights, New York 10598
(Received 28 July 1980)

The continuum generalization of the p-state Potts model is analyzed in the ordered phase. Renormalization-group iterations in $d=4-\epsilon$ dimensions are followed by an elimination of the transverse modes and a mapping onto an effective Ising model. This model is then used to show that the transition is first order for $p > p_c(d)$ and continuous for $p < p_c(d)$. We find that $p_c(d)=2$ for $d>4$ and $p_c(4-\epsilon)=2+\epsilon+O(\epsilon^2)$.

I. INTRODUCTION

The order of the phase transition of the p-state Potts model[1] has been of much recent interest. It is believed that the transition is second order for $p < p_c(d)$ and first order for $p > p_c(d)$, where the critical value $p_c(d)$ varies with the dimensionality of the system, d. At $d=2$, Baxter[2] showed exactly that $p_c(2)=4$. Recent approximate real-space renormalization-group calculations[3,4] reproduced this result, and gave details of the behavior for different values of p. Some of these results have also been confirmed experimentally.[5] As d approaches one from above, Migdal-type recursion relations applied by Stephen[6] showed that the transition is second order for all finite p; i.e., $p_c(1)=\infty$. At the other end, one expects the Potts model to be correctly described by mean-field theory above the upper critical dimensionality, $d_c=6$.[7] Mean-field theory predicts that all Potts models with $p>2$ should have a first-order transition, due to the presence of cubic terms in the appropriate Ginzburg-Landau expansion. Thus, $p_c(d)=2$ for $d>6$. In fact, a recent study[8] of the Potts model in $6-\epsilon$ dimensions showed that the transition remains first order for all $p>2$; i.e., $p_c(6-\epsilon)=2$. The critical value $p_c(d)$ thus seems to be a monotonically decreasing function of d, changing from 4 at $d=2$ to 2 at some $d<6$.[9]

The actual dependence of p_c on d is of great importance, as the properties of the Potts model at $d=3$ are still far from being settled. For $p=3$ and $d=3$, experiments[10,11] exhibit a first-order transition. A first-order transition is also found in a Monte Carlo renormalization-group calculation[12] and in earlier field-theoretical renormalization-group calculations.[13,14] Thus it seems that $p_c(3)<3$. However, earlier series-expansion results[15,16] may be indicating a second-order transition for $p=3$, $d=3$, so that $p_c(3)$ is probably quite close to 3.

In the present paper we investigate the dependence of p_c on d. Extending the earlier work,[8] we show that p_c remains equal to 2 for all $d \geq 4$. We then generalize earlier calculations[14,17] in $d=4-\epsilon$ to obtain our main new result; i.e.,

$$p_c(4-\epsilon)=2+\epsilon+O(\epsilon^2) \ . \tag{1}$$

Our calculation is based on the continuum generalization of the Potts model.[7,8] Since at $p=2$ the Potts model becomes an Ising model, we are basically expanding in powers of $(p-2)$ about this model. Technically, we argue that a sufficient number of iterations of the renormalization group in the ordered phase will eliminate most of the fluctuations in the transverse modes. Integrating the remaining transverse modes out of the partition function then leaves us with an effective Ising-model Hamiltonian in the longitudinal component of the order parameter. The properties of this Hamiltonian are then used to determined p_c.

The integration over the transverse modes is described in Sec. II. The explicit integration of the recursion relations in $d=4-\epsilon$ dimensions is then described in Sec. III, and the results are combined with those of Sec. II to yield the final Ising-model parameters in Sec. IV, where these parameters are also used to identify $p_c(4-\epsilon)$. The situation for $4<d<6$, and the results in general, are then discussed in Sec. V.

II. EFFECTIVE ISING HAMILTONIAN

Following Priest and Lubensky[7] the effective reduced Hamiltonian is written as

$$H=-\tfrac{1}{4}\int (r+k^2)\sum Q_{ij}(k)Q_{ij}(-k)+\omega\int\sum Q_{ij}(k)Q_{jk}(k')Q_{kl}(-k-k')$$
$$-u\int\sum Q_{ij}(k)Q_{ij}(k')Q_{kl}(k'')Q_{kl}(-k-k'-k'')-v\int\sum Q_{ij}(k)Q_{jk}(k')Q_{kl}(k'')Q_{li}(-k-k'-k'') \ , \tag{2}$$

where r is linear in the temperature T and Q_{ij} are symmetric diagonal traceless p-dimensional tensors. The tensor components Q_{ij} are related to the components A_α of the p-state Potts model

$$H=-\tilde{J}\sum_{\langle xx'\rangle} A(x)\cdot A(x') \tag{3}$$

by

$$Q_{ii}=\sum_{\alpha=1}^{p} A_\alpha a_{ii}^\alpha \ , \tag{4}$$

where

$$a_{ii}^\alpha=\left[\frac{p-\alpha}{p-\alpha+1}\right]^{1/2}\times\begin{cases}0 & \text{if } i<\alpha\\ 1 & \text{if } i=\alpha\\ -1/(p-\alpha) & \text{if } i>\alpha\ .\end{cases} \tag{5}$$

We now assume uniaxial ordering, i.e., that only one component, A_1, orders. For reasons that will become clear below, it is then convenient to shift A_α via

$$A_\alpha(x)=Q\delta_{\alpha,1}+L_\alpha \ . \tag{6}$$

The parameter Q will be determined later. The corresponding expression for Q_{ii} is given by Eq. (4),

$$Q_{11}=\left[\frac{p-1}{p}\right]^{1/2}(Q+L)\ , \tag{7}$$

$$Q_{ii}=\frac{-1}{p-1}\left[p-\frac{1}{p}\right]^{1/2}(Q+L)+q_{ii}\quad\text{for } i\neq 1\ , \tag{8}$$

where q_{ii} is a traceless diagonal tensor of dimension $(p-1)$ and where the subscript on L_1 has been dropped.

We add a fictitious field $-A_1(x)h$ to the Hamiltonian and separate it into its "fluctuating" part[8]

$$\begin{aligned}H={}&-\tfrac{1}{4}\int(r_L+k^2)L(k)L(-k)-\tfrac{1}{4}\int(r_T+k^2)\sum_{i\neq 1}q_{ii}q_{ii}-\tilde{h}L(0)\\&+[(p-2)c\omega-4Q(u+bv)]\int L(k)L(k')L(-k-k')\\&-[3\omega c+4Q(u+3vc^2)]\int L(k)\sum_{i\neq 1}q_{ii}(k')q_{ii}(-k-k')\\&+(\omega+4cvQ)\int\sum_{i\neq 1}q_{ii}(k)q_{ii}(k')q_{ii}(-k-k')-(u+bv)\int L(k)L(k')L(k'')L(-k-k'-k'')\\&-2(u+3vc^2)\int L(k)L(k')\sum_{i\neq 1}q_{ii}(k'')q_{ii}(-k-k'-k'')+4vc\int L(k)\sum_{i\neq 1}q_{ii}(k')q_{ii}(k'')q_{ii}(-k-k'-k'')\\&-u\sum_{\substack{i\neq 1\\ j\neq 1}}q_{ii}(k)q_{ii}(k')q_{jj}(k'')q_{jj}(-k-k'-k'')-v\sum_{i\neq 1}q_{ii}(k)q_{ii}(k')q_{ii}(k'')q_{ii}(-k-k'-k'')\end{aligned} \tag{9}$$

and its fluctuation-independent (mean-field) part

$$H_{\rm MF}=-\tfrac{1}{4}rQ^2+(p-2)c\omega Q^3-hQ-(u+bv)Q^4\ . \tag{10}$$

In these equations

$$\begin{aligned}r_L&=r-12(p-2)\omega cQ+24(u+bv)Q^2\ ,\\ r_T&=r+12\omega cQ+8(u+3vc^2)Q^2\ ,\\ \tilde{h}&=h-\tfrac{1}{2}rQ+3\omega c(p-2)Q^2-4(u+bv)Q^3\ ,\\ c&=[p(p-1)]^{-1/2}\ ,\\ b&=(p^2-3p+3)c^2\ .\end{aligned} \tag{11}$$

The Hamiltonian (9) contains fluctuations in both L and q_{ii}. We shall find, however, that for $p\sim 2$, r_L is less than r_T by a factor of $(p-2)$. We thus iterate the renormalization-group equations out of the critical region until $r_T(l^*)\simeq 1$.[8,17,18] We note, however, that any choice of $r_T(l^*)$ will yield the same final universal result, Eq. (1). At this stage, we can integrate the transverse modes $\{q_{ii}\}$ out of the partition function, using standard perturbation expansions in $\omega(l^*)$, $u(l^*)$, and $v(l^*)$. The result, apart from a constant, is

$$\begin{aligned}H_{\rm eff}={}&-\tfrac{1}{4}\int(r_{\rm eff}+k^2)L(k)L(-k)-h_{\rm eff}L(0)\\&+\omega_{\rm eff}\int L(k)L(k')L(-k-k')\\&-u_{\rm eff}\int L(k)L(k')L(k'')L(-k-k'-k'')\ ,\end{aligned} \tag{12}$$

with

$$u_{\rm eff}=u(l^*)+bv(l^*)\ , \tag{13}$$

$$\omega_{\rm eff}=(p-2)c\omega(l^*)-4Q(l^*)[u(l^*)+bv(l^*)]\ , \tag{14}$$

$$h_{\rm eff}=\tilde{h}(l^*)-6(p-2)c\,\omega(l^*)I_1 -2^3(p-2)Q(l^*)[u(l^*)+3c^2v(l^*)]I_1 \ , \qquad (15)$$

$$r_{\rm eff}=r_L(l^*)-2^4 3^2(p-2)c^2\omega^2(l^*)I_2 +2^4(p-2)[u(l^*)+3c^2v(l^*)]I_1 \ , \qquad (16)$$

where

$$I_m=K_d\int_0^1 \frac{k^{d-1}dk}{[r_T(l^*)+k^2]^m} \ ,$$
$$K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2) \ , \qquad (17)$$

and where we have kept only the leading terms in an expansion in both the cubic and quartic coefficients.

Having generated the Ising-like Hamiltonian (12), we can now use existing knowledge of this Hamiltonian. As usual, it is convenient to eliminate the cubic term by an appropriate shift in the order parameter. In our case this is simply achieved by an appropriate choice of $Q(l^*)$, as determined by Eq. (14)

$$Q(l^*)=\frac{(p-2)c\,\omega(l^*)}{4[u(l^*)+bv(l^*)]} \ . \qquad (18)$$

Once $\omega_{\rm eff}=0$, Eq. (12) reduces to the usual Ising model in a field. Below the critical point, this model has a first-order transition from negative $\langle L\rangle$ to positive $\langle L\rangle$ as $h_{\rm eff}$ goes through zero. Thus, the first-order point is identified via

$$h_{\rm eff}=0 \ . \qquad (19)$$

Since $h_{\rm eff}$ is explicitly dependent on the temperature, as seen from Eqs. (11), this determines the first-order transition temperature. Finally, we note that this transition is first order only below the critical point, i.e., for[19]

$$r_{\rm eff}<r_{0c} \ . \qquad (20)$$

If we set the temperature at the transition value determined from Eq. (19), $r_{\rm eff}$ will then depend only on p. The equality $r_{\rm eff}=r_{0c}$ will thus determine the critical value p_c.

III. RECURSION RELATIONS

For the Hamiltonian given by Eq. (9) the recursion relations are[7,8]

$$\frac{dr_L}{dl}(l)=[2-\eta(l)]r_L(l)+2^4 3K_d[u(l)+bv(l)]g_L(l)+2^4K_d(p-2)[u(l)+3c^2v(l)]g_T(l)$$
$$-2^4 3^2K_d\{(p-2)c\,\omega(l)-4Q(l)[u(l)+bv(l)]\}^2g_L^2(l)$$
$$-2^4K_d(p-2)\{3c\,\omega(l)+4Q(l)[u(l)+3c^2v(l)]\}^2g_T^2(l) \ , \qquad (21)$$

$$\frac{dr_T}{dl}(l)=[2-\eta(l)]r_T(l)+2^4K_dp[u(l)+3c^2(p-2)v(l)]g_T(l)+2^4K_d[u(l)+3c^2v(l)]g_L(l)$$
$$-2^5K_d\{3c\,\omega(l)+4Q(l)[u(l)+3c^2v(l)]\}^2g_T(l)g_L(l)$$
$$-2^4 3K_dc^2p(p-3)[\omega(l)+4cv(l)Q(l)]^2g_T^2(l) \ , \qquad (22)$$

$$\frac{d\tilde{h}}{dl}(l)=\tfrac{1}{2}[d+2-\eta(l)]\tilde{h}(l)+6K_d(p-2)c\,\omega(l)[g_L(l)-g_T(l)]$$
$$-2^3 3K_dQ(l)[u(l)+bv(l)]g_L(l)-2^3K_d(p-2)Q(l)[u(l)+3c^2v(l)]g_T(l) \ , \qquad (23)$$

where

$$g_L(l)=\frac{1}{1+r_L(l)}, \quad g_T=\frac{1}{1+r_T(l)} \qquad (24)$$

For the higher-order coefficients the recursion relations for the disordered phase will suffice,

$$\frac{d\omega}{dl}(l)=\tfrac{1}{2}[6-d-3\eta(l)]\omega(l)+288K_d\left[1-\frac{3}{p}\right]\frac{\omega^3(l)}{[1+r(l)]^3}-2^43K_d\left[2u(l)+3\left[1-\frac{2}{p}\right]v(l)\right]\frac{\omega(l)}{[1+r(l)]^2}\;,$$

$$\frac{du}{dl}(l)=[4-d-2\eta(l)]u(l)+3^22^6K_du(l)\omega^2(l)\left[1-\frac{4}{p}\right]\frac{1}{[1+r(l)]^3}+3^32^6K_dv(l)\omega(l)^2\frac{1}{p^2}\frac{1}{[1+r(l)]^3}$$
$$-2^4K_d\left[(p+7)u^2(l)+6\left[1-\frac{1}{p}\right]u(l)v(l)+(9/p^2)v^2(l)\right]\frac{1}{[1+r(l)]^2}\;, \tag{25}$$
$$\frac{dv}{dl}(l)=[4-d-2\eta(l)]v(l)+3^22^7K_du(l)\omega^2(l)\frac{1}{[1+r(l)]^3}+3^32^6K_dv(l)\omega^2(l)\left[1-\frac{3}{p}\right]\frac{1}{[1+r(l)]^3}$$
$$-2^43K_d\left[4u(l)v(l)+3\left[1-\frac{2}{p}\right]v^2(l)\right]\frac{1}{[1+r(l)]^2}\;,$$

where

$$\eta(l)=48K_d\left[1-\frac{2}{p}\right]\omega^2(l)\;. \tag{26}$$

For $d=4-\epsilon$ these have the solutions

$$r_L(l)=T_L(l)-2^3K_4\left[(p+1)u(l)+3\left[1-\frac{1}{p}\right]v(l)\right]+2^33K_4[u(l)+bv(l)]T_L(l)\ln[1+T_L(l)]$$
$$+2^3K_4(p-2)[u(l)+3c^2v(l)]T_T(l)\ln[1+T_T(l)]$$
$$+2^33^2K_4\{(p-2)c\omega(l)-4Q(l)[u(l)+bv(l)]\}^2\left[\ln[1+T_L(l)]+\frac{T_L(l)}{1+T_L(l)}\right]$$
$$+2^3(p-2)K_4\{3c\omega(l)+4Q(l)[u(l)+3c^2v(l)]\}^2\left[\ln[1+T_T(l)]+\frac{T_T(l)}{1+T_T(l)}\right]\;, \tag{27}$$

$$r_T(l)=T_T(l)+O(u,\omega^2)\;, \tag{28}$$

and

$$\tilde{h}(l)=\tilde{h}_0(l)+2^23K_4Q(l)[u(l)+bv(l)]+2^2K_4(p-2)Q(l)[u(l)+3c^2v(l)]$$
$$+3K_4(p-2)c\omega(l)\{T_L(l)\ln[1+T_L(l)]-T_T(l)\ln[1+T_T(l)]\}$$
$$-2^23K_4Q(l)[u(l)+bv(l)]T_L(l)[1+T_L(l)]$$
$$-2^2K_4(p-2)Q(l)[u(l)+3c^2v(l)]T_T(l)\ln[1+T_T(l)]\;, \tag{29}$$

where

$$T_L(l)=t(l)-12(p-2)c\omega(l)Q(l)+24[u(l)+bv(l)]Q^2(l)\;, \tag{30}$$
$$T_T(l)=t(l)+12c\omega(l)Q(l)+8[u(l)+3c^2v(l)]Q^2(l)\;, \tag{31}$$
$$\tilde{h}_0(l)=h(l)-\tfrac{1}{2}t(l)Q(l)+3(p-2)c\omega(l)Q^2(l)-4[u(l)+bv(l)]Q^3(l)\;, \tag{32}$$

and where $t(l)$ satisfies the equation

$$\frac{dt}{dl}(l)=2t(l)-2^4K_4\left[(p+1)u(l)+3\left[1-\frac{1}{p}\right]v(l)\right]t(l)-2^43^2K_4\left[1-\frac{2}{p}\right]\omega^2(l)\;. \tag{33}$$

Further

$$Q(l)=Qe^{(1-\epsilon/2)l}\ ,\quad h(l)=he^{(3-\epsilon/2)l}\ . \tag{34}$$

Here h is the external uniaxial field applied to the Potts model, which we set equal to zero.

We shall require only approximate solutions for $u(l)$, $v(l)$, $\omega(l)$, and $t(l)$. Following Priest and Lubensky,[7] we start with the recursion relations for u and v in the absence of ω. For $p<5$, they find that the $(p-1)$-component Heisenberg fixed point is stable, i.e.,

$$u(l)\to u^*=\epsilon/16(p+7)K_4,\quad v(l)\to 0 \tag{35}$$

for $l\to\infty$. Note that for $p\to 2$ this becomes the usual Ising fixed point. Assuming that $v(l)$ is small we indeed find

$$\begin{aligned} u(l)&=ue^{\epsilon l}/U(l)\ ,\\ v(l)&=ve^{\epsilon l}/U(l)^{12/(p+7)}\ , \end{aligned} \tag{36}$$

where

$$U(l)=1+(e^{\epsilon l}-1)u/u^* \tag{37}$$

and where u and v are the initial values $u=u(0)$ and $v=v(0)$. We note that $v(l)$ indeed decays to zero for l sufficiently large and $p<5$. We now introduce a small cubic term ω. Neglecting terms of order ω^3, the recursion relation for $\omega(l)$ is

$$\frac{d\omega(l)}{dl}=\left(1+\frac{\epsilon}{2}\right)\omega(l)-2^5 3K_4u(l)\omega(l)+\cdots\ , \tag{38}$$

yielding

$$\omega(l)=\omega e^{(1+\epsilon/2)l}/U(l)^{6/(p+7)}\ . \tag{39}$$

To the same approximation, the solution of Eq. (33) for $t(l)$ is given by

$$t(l)=\frac{\tilde{t}e^{2l}}{U(l)^{(p+1)/(p+7)}}-\frac{3^2}{2p}\frac{\omega^2(l)}{u(l)}\ , \tag{40}$$

$$\tilde{t}=r+2^3K_4(p+1)u+\frac{3^2}{2p}\frac{\omega^2}{u}\ . \tag{41}$$

IV. EVALUATION OF $p_c(4-\epsilon)$

We return now to the evaluation of the effective parameters of the Ising model as given by Eqs. (13)–(16). To leading order in ϵ, the integrals I_1 and I_2 may be evaluated at $d=4$ and $r_T(l^*)=T_T(l^*)$, yielding

$$\begin{aligned} I_1&=\frac{1}{2}\left[1+T_T(l^*)\ln\left(\frac{T_T(l^*)}{1+T_T(l^*)}\right)\right]\ ,\\ I_2&=\frac{1}{2}\left[\ln\left(\frac{1+T_T(l^*)}{T_T(l^*)}\right)-\frac{1}{1+T_T(l^*)}\right]\ . \end{aligned} \tag{42}$$

Combining Eqs. (15) and (29) (with the latter evaluated at $l=l^*$) then gives

$$h_{\rm eff}=\tilde{h}_0(l^*)-3K_4\{(p-2)c\omega(l^*)-4[u(l^*)+bv(l^*)]Q(l^*)\}\{1-T_L(l^*)\ln[1+T_L(l^*)]\}\ . \tag{43}$$

When we substitute Eq. (18) for $Q(l^*)$ we find

$$\tilde{h}_{\rm eff}=\tilde{h}_0(l^*)\ , \tag{44}$$

where $\tilde{h}_0(l^*)$ is given by Eq. (32), *without any corrections* from the diagrams. Setting $\tilde{h}_{\rm eff}=0$, and making use of Eq. (18), gives

$$t(l^*)=\frac{(p-2)^2c^2\omega^2(l^*)}{[u(l^*)+bv(l^*)]}\ . \tag{45}$$

As outlined in Sec. II we iterate the recursion relations until $r_T(l^*)\approx 1$. In practice, it is more convenient to choose $T_T(l^*)=1$. Thus, from Eq. (31) we set

$$t(l^*)+12c\omega(l^*)Q(l^*)+8[u(l^*)+3c^2v(l^*)]Q^2(l^*)=1\ . \tag{46}$$

Substituting Eqs. (18) and (45) for $Q(l^*)$ and $t(l^*)$ gives

$$\frac{[u(l^*)+bv(l^*)]}{c^2\omega^2(l^*)}=3(p-2)+O((p-2)^2)\ . \tag{47}$$

Hence by Eq. (45) the first-order transition occurs at

$$t(l^*)=(p-2)/3+O((p-2)^2)\ . \tag{48}$$

From Eq. (30) we now obtain

$$T_L(l^*)=-(p-2)/6+O((p-2)^2)\ . \tag{49}$$

This shows that our assumption that $|r_L|<r_T$ does indeed remain true up to $l=l^*$.

Next we need to calculate $r_{\rm eff}$ as given by Eq. (16). Substituting Eqs. (27) and (42) we find again that all the diagrams cancel to leading order in $(p-2)$. Thus we obtain simply

$$r_{\rm eff}=T_L(l^*)=-(p-2)/6+O((p-2)^2)\ . \tag{50}$$

At this stage we can iterate the recursion relations for the effective Ising Hamiltonian (12). The recursion relation for the temperature variable is[19]

$$\frac{dr_{\rm eff}}{dl}=2r_{\rm eff}+2^4 3K_4\frac{u_{\rm eff}}{1+r_{\rm eff}} , \quad (51)$$

so that to leading order the temperature scaling field is[19]

$$t_{\rm eff}=r_{\rm eff}+2^3 3K_4 u_{\rm eff} . \quad (52)$$

Thus finally we need to calculate $u_{\rm eff}$ as given by Eq. (13). From Eq. (47) it follows that for $p\sim 2$

$$e^{-2l^*}\sim(p-2) . \quad (53)$$

From Eqs. (36) it then further follows that

$$u(l^*)=u^*[1+O((p-2)^{\epsilon/2})] ,$$
$$v(l^*)=0+O((p-2)^{\epsilon/6}) . \quad (54)$$

Thus

$$u_{\rm eff}=u(l^*)+bv(l^*)=u^*+O((p-2)^{\epsilon/6}) , \quad (55)$$

with u^* as given by Eq. (35). Finally, from Eqs. (50) and (52) we obtain

$$t_{\rm eff}=-\frac{(p-2)}{6}+\frac{\epsilon}{6} . \quad (56)$$

The transition (as function of $h_{\rm eff}$) is first order for $t_{\rm eff}<0$, i.e., for $p>2+\epsilon$. This concludes the derivation of Eq. (1).

V. DISCUSSION

In the preceding sections we mainly concentrated on the solutions for $d<4$, i.e., $\epsilon>0$. In fact, the same solutions apply for $d>4$. The only difference is that $u(l)$, according to Eq. (36), will decay to zero, as $e^{-|\epsilon|l}$, and, therefore, $u(l^*)\propto(p-2)^{\epsilon/2}$. Thus, both terms in Eq. (52) will vanish at $p=2$. This shows that $p_c(d)=2$ for all $d>4$.

One could also repeat the above calculations in $6-\epsilon$ dimensions. The solutions of Sec. III must then be replaced by solutions in which $\omega(l)$ approaches its fixed-point value,[8]

$$\omega^2\rightarrow\epsilon/144K_6\left(\frac{10}{p}-3\right) . \quad (57)$$

Again, a detailed analysis shows that all the diagrams contributing to Eq. (19) cancel, so that Eqs. (47)–(49) are still correct. However, using Eq. (57) in Eq. (47) yields $u(l^*)\propto(p-2)$, so that finally Eq. (52) gives $t_{\rm eff}\propto(p-2)$, again showing that $p_c=2$. This is consistent with the results obtained in Ref. 8.

To summarize, we have been able to show that $p_c(d)=2$ for all $d>4$, and to evaluate the deviation from $d=4-\epsilon$ dimensions. A simple extrapolation of Eq. (1) to lower dimensions approaches the correct answer at $d=2$, i.e., $p_c(2)=4$. The same extrapolation goes through $p_c(3)=3$. Corrections of higher order in ϵ are thus essential in order to determine whether in fact $p_c(3)$ is smaller than 3. The extrapolated result $p_c(3)=3$ does seem to confirm the general belief that $p_c(3)$ is not very far from 3, explaining the conflicting evidence in the literature.

ACKNOWLEDGMENTS

A.A. acknowledges useful conversations with D. J. Scalapino, J. L. Cardy, E. K. Riedel, J. Rudnick, and D. Blankschtein. This collaboration was started while A.A. was visiting the physics department at Harvard University, where he enjoyed the hospitality of B. I. Halperin and the support of the NSF (under Grant No. DMR-77-10210). Work supported by the U.S.-Israel Binational Science Foundation.

---

*Permanent address: Tel Aviv University, Israel.

[1]R. B. Potts, Proc. Cambridge Philos. Soc. 48, 106 (1952).
[2]R. J. Baxter, J. Phys. C 6, 445 (1973).
[3]B. Nienhuis, A. N. Berker, E. K. Riedel, and M. Schick, Phys. Rev. Lett. 43, 737 (1979).
[4]J. L. Cardy, M. Nauenberg, and D. J. Scalapino (unpublished).
[5]J. G. Dash, Phys. Rep. 38C, 177 (1978).
[6]M. J. Stephen, Phys. Lett. 56A, 149 (1976).
[7]R. G. Priest and T. C. Lubensky, Phys. Rev. B 13, 4159 (1976).
[8]E. Pytte, Phys. Rev. B 20, 4450 (1980).
[9]This monotonicity is also confirmed by recent real-space calculations at intermediate values of d; E. K. Riedel (private communication); and A. N. Berker, D. Andelman, and A. Aharony, J. Phys. A (in press).
[10]A. Aharony, K. A. Müller, and W. Berlinger, Phys. Rev. Lett. 38, 33 (1977).
[11]B. Barbara, M. F. Rossignol, and P. Bak, J. Phys. C 11, L183 (1978).
[12]H. W. J. Blöte and R. H. Swendsen, Phys. Rev. Lett. 43, 799 (1979).
[13]G. R. Golner, Phys. Rev. B 8, 3419 (1973).
[14]J. Rudnick, J. Phys. A 8, 1125 (1975).
[15]R. V. Ditzian and J. Oitmaa, J. Phys. A 7, L61 (1974).
[16]J. P. Straley, J. Phys. A 7, 2173 (1974).
[17]D. Blankschtein and A. Aharony, J. Phys. C 13, 4635 (1980).
[18]See also E. Domany and M. E. Fisher, Phys. Rev. B 15, 3510 (1977).
[19]See, e.g., J. Rudnick and D. R. Nelson, Phys. Rev. B 13, 2208 (1976).