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Critical properties of random Potts models

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The critical properties of Potts models with random bonds are considered in two dimensions. A position-space renormalization-group procedure, based on the Migdal-Kadanoff method, is developed. While all previous position-space calculations satisfied the Harris criterion and the resulting scaling relation only approximately, we found conditions under which these relations are exactly satisfied, and constructed our renormalization-group procedure accordingly. Numerical results for phase diagrams and thermodynamic functions for various random-bond Potts models are presented. In addition, some exact results obtained using a duality transformation, as well as an heuristic derivation of scaling properties that correspond to the percolation problem are given.

I. INTRODUCTION

Phase transitions in two-dimensional models and physical systems have been the subject of intensive theoretical and experimental studies.[1] In particular, order-disorder transitions in adsorbed monolayers have provided realizations of various models of theoretical interest.[2] Investigation of the critical behavior of these systems are often hindered by finite-size effects and substrate imperfections. The inhomogeneities that appear in experimental situations are in most cases too complicated for simple theoretical modeling. Therefore theoretical studies have concentrated mainly on the problems of quenched bond and site dilution.[3] Most detailed calculations are limited to the effects of randomness on Ising models. In particular, the dilute-bond Ising model has been studied throughout the years by a variety of methods; only recently have random Potts models been considered.[4]

While detailed calculations provide information about the phase diagram, critical exponents, and thermodynamic functions of random systems, some important qualitative aspects can be understood on the basis of general, model-independent arguments. For example, if one considers the problem of bond dilution, it is quite obvious that when p, the bond concentration, is below the value for percolation p_c, the system consists of an aggregate of finite connected clusters, and therefore no phase transition is expected. The free energy will contain singularities associated with percolation at $p=p_c$ and $T=0$ K. For finite temperatures, however, the singularities are expected to be different; or, in renormalization-group language, the percolation point is a multicritical point, from which the system "crosses over" to some thermal critical behavior.[5,6]

In another limit, for $p=1$ the free energy is that of the pure system. As dilution is introduced, a transition line appears in the concentration-temperature plane. A question of central importance, namely, under what conditions is this line characterized by the same exponents as the pure system, was addressed by Harris.[7] He showed that randomness does not change the critical behavior of the pure system provided α, the specific-heat exponent is negative. The Harris criterion can also be derived by the replica method,[8] combined with scaling arguments.[9] Such analysis yields that the crossover exponent that corresponds to the perturbation associated with randomness must have the value $\phi=\alpha/\nu$.

We note that none of the existing approximate (position-space) renormalization-group calculations of the bond dilute Ising model satisfies this scaling relation. This is the first issue addressed in the present work; namely, the construction of a (position-space) renormalization-group procedure that satisfies the Harris criterion. One hopes that by using a procedure that correctly reproduces scaling near the pure system fixed point, the behavior of the bond diluted Ising model can be studied in a quantitative manner. In particular, since $\alpha=0$ for this model, the operator that corresponds to randomness is *marginally irrelevant* at the pure system fixed point. The presence of such an operator is known to induce logarithmic corrections to the critical behavior.[10]

Another case of interest is that in which $\alpha_{\text{pure}}>0$. Randomness is then relevant, and the critical behavior may be dominated by a "random fixed point."[11] This could be the situation, for example, for the random-bond three- and four-state Potts models. To our knowledge, no previous work employing position-space methods has studied the existence of such a fixed point, the exponents associated with it or the bond distribution that characterizes it.[12]

In experiments it is difficult to introduce bond *dilution*, i.e., a certain fraction of zero bonds in a matrix of nonzero interactions. However, it is possible to make systems with different nonzero bond strengths distributed at random. For instance, when the magnetic interaction is induced by nonmagnetic atoms between magnetic ones, one can use different nonmagnetic atoms and obtain different superexchange interactions. For this reason we studied phase diagrams and thermodynamic properties of such systems.

We succeeded in constructing a position-space renormalization procedure that satisfies exactly the scaling relation $\phi = \alpha/\nu$. The general considerations that led to our procedure, and its implementation for the bond dilution problem are discussed in Sec. II; details of the method are presented in Sec. III. In Sec. IV we use duality transformation and geometrical arguments to give relations between different random systems. We obtain an exact critical line and deduce some characteristics of the percolation network from known thermal properties. We also discuss the influence of negative bonds in Potts models; in particular we note that frustration has different effects in Ising and Potts models.

The numerical work concentrates on bond-random q-state Potts models, in which q, the number of states in the Potts model, is being used as a parameter to adjust α for the pure system to have its known values,[13] so that the effect of $\operatorname{sgn}(\alpha)$ on crossover can be studied. Results for the phase diagrams and critical behavior are given in Sec. V. Section VI summarizes our study.

II. HARRIS CRITERION

In this section we consider how the Harris criterion[7] can be incorporated in a position-space renormalization-group calculation. First, the criterion is rederived on the basis of the original heuristic argument. Then an exponent relation resulting from scaling is given (in the context of the replica method). Finally, we show how the criterion can be exactly incorporated in a position-space technique.

The problem, as originally posed by Harris,[7] concerns the critical behavior of a system into which a small amount of randomness has been introduced. For example, one considers a situation where a randomly selected fraction p of the (nearest-neighbor) bonds has been removed, and asks under what conditions can the critical behavior of the dilute system be identical to that of the pure system. Harris assumes that the system exhibits one sharp transition at temperature $T_c(p)$. Then various subsystems l of size L_l are considered. If $L_l \gtrsim \xi_l$, where ξ_l is the local correlation length, such a subsystem can be viewed as independent of the rest of the system, and $\xi_l \sim |T - T_c(p_l)|^{-\nu}$, with p_l the local impurity concentration. The concentration p_l fluctuates about its mean value p with deviation $\Delta p_l \sim L_l^{-d/2}$ and therefore

$$\xi_l \sim |T - T_c(p) \pm a_l \xi_l^{-d/2}|^{-\nu} \tag{2.1}$$

with some coefficient a_l. This picture of a sharp transition and the assumption of a single underlying length scale is self-consistent if

$$|T - T_c(p)| >> \xi_l^{-d/2} \ , \tag{2.2}$$

and, therefore

$$|T - T_c(p)| >> |T - T_c(p)|^{d\nu/2} \ . \tag{2.3}$$

This relation, in conjunction with the scaling law $d\nu = 2 - \alpha$, leads to the condition $\alpha < 0$.

The critical behavior of random systems can be addressed via the replica trick[8] and scaling arguments. When bond randomness is introduced into a model with say, nearest-neighbor interactions between spins (Ising, n vector, Potts, etc.), the replica method yields an effective Hamiltonian that couples n replicas of the basic interacting system.[14] In particular, if the pure system is characterized by the Hamiltonian

$$H = \sum_{\vec{r}} \epsilon(\vec{r}) \tag{2.4}$$

the effective Hamiltonian of the replicated system is given by

$$H_{\text{eff}} = \sum_{\vec{r}} \left[\sum_{\alpha=1}^{n} \epsilon_\alpha(\vec{r}) + \Delta \sum_{\alpha;\beta} \epsilon_\alpha(\vec{r}) \epsilon_\beta(\vec{r}) + \cdots \right] , \tag{2.5}$$

where $\epsilon_\alpha(\vec{r})$ is the energy density operator of the original nonrandom system. The parameter Δ is proportional to some measure of the randomness. For vanishing randomness $\Delta = 0$, and H_{eff} reduces to n decoupled systems. Since the coupling between the replicas has, to lowest order in the local variables, the form $\epsilon_\alpha \epsilon_\beta$ (energy-energy coupling) the scaling index of this operator is given by[9]

$$x_\Delta = 2x_\epsilon \ . \tag{2.6}$$

The eigenvalue associated with randomness near the decoupled (pure) system critical point is given by

$$y_r = d - 2x_\epsilon = d - 2(d - y_\epsilon) = \alpha/\nu \ . \tag{2.7}$$

Therefore the eigenvalue that corresponds to randomness, near the pure system fixed point, is given by α/ν, where α, ν are the pure system exponents.

This scaling argument applies near the pure-system limit, and is pertinent to the relevance of randomness near the pure-system fixed point. If $\alpha > 0$, randomness is relevant, and the behavior of the random system will in general differ from that of the pure system. In this case the renormalization-group trajectories can take the system to a stable random fixed point.[11,14] Note that at this fixed point the replicas are coupled, and therefore the scaling analysis presented here is not applicable there. Therefore, we cannot conclude, on the basis of scaling, that the stability of a random fixed point is determined by the sign of the respective specific-heat exponent. If there is a stable random fixed point, it is in general characterized by a fixed distribution of couplings. This means that at each length scale there is a continuous *distribution* of interactions between spin regions or block spins. It is not obvious that close to such a critical system a description in terms of a *single* correlation length is appropriate. We would rather expect a distribution of local correlation lengths. Without the assumption of a single local correlation length the arguments of Harris do not hold any longer. We conclude that the pure system is stable against randomness if $\alpha < 0$. However, we cannot see that at a stable random fixed point α will always be negative.

The scaling of randomness [Eq. (2.7)] at the pure system is not necessarily satisfied by approximate calculations. Indeed, in the literature we have not found any real-space renormalization methods which give this scaling exactly. Thus one of our aims was to restore the scaling relation $y_r = \alpha/\nu$ in the context of an approximate position-space renormalization-group calculation. To achieve this we assume that (a) our renormalization-group recursion relations replace $n = b^d$ bonds K_i by a single new bond K', where b is the standard length rescaling factor. An additional assumption is that (b) all bonds are equivalent; i.e., the recursion relation that determines the new coupling has the form

$$K' = F(K_1, K_2, \ldots, K_n) \quad , \tag{2.8}$$

with F invariant under permutation of the indices $i = 1, \ldots, n$.

The bonds K_i are assumed to be taken from a random, independent distribution, characterized by average value K and standard deviation σ_K,

$$\langle K_i \rangle_{\text{av}} = K \quad , \quad \langle (K_i - K)(K_j - K) \rangle_{\text{av}} = \sigma_K^2 \delta_{i,j} \quad . \tag{2.9}$$

If the distribution is narrow, that is only those values for which

$$|\delta K_i|/K = |K_i - K|/K \ll 1$$

appear with sizable probability, one can expand Eq. (2.8) to first order in δK_i

$$K' \cong F(K, \ldots, K) + \sum_{i=1}^{n} \left. \frac{\partial F}{\partial K_i} \right|_{K_i = K} \delta K_i + O(\delta K_i^2) \quad . \tag{2.10}$$

The first term is the recursion relation of the pure (nonrandom) system, e.g.,

$$K' = f(K) = F(K, \ldots, K) \quad . \tag{2.11}$$

Near the pure-system fixed point K^* one has

$$\left. \frac{df}{dK} \right|_{K^*} = b^y \quad . \tag{2.12}$$

Also, since we assumed all bonds to be equivalent, we have

$$\left. \frac{\partial F}{\partial K_i} \right|_{K_j = K^*} = \frac{1}{n} b^y = b^{y-d} \quad . \tag{2.13}$$

Therefore near the pure-system fixed point (i.e., $K_i = K^* + \delta K_i$) the recursion relations for the random system have the form

$$K' \cong K^* + b^{y-d} \sum_{i=1}^{n} \delta K_i \tag{2.14}$$

thereby generating a new, renormalized bond distribution. The variance of the new distribution

$$(\sigma_K')^2 = \langle (K' - K^*)^2 \rangle_{\text{av}} \quad , \tag{2.15}$$

can be found from Eq. (2.14), using Eq. (2.9), and is given by

$$(\sigma_K')^2 = b^{2y-d} (\sigma_K)^2 \quad . \tag{2.16}$$

Therefore, viewing the variance σ_K^2 as the scaling variable that measures randomness, this analysis yields

$$y_r = 2y - d = \alpha/\nu \tag{2.17}$$

and the scaling relation (2.7) is satisfied. It is important to note that we expanded the recursion relation about the average value of K_i. This is valid only if the width of the K_i distribution is narrow; in particular, if no bond takes, with large probability, a value that deviates significantly from the average.

This condition is *not* satisfied when bond dilution is considered.[6] In that case a system which is almost pure is characterized by a small concentration of bond impurities, which, however, take values that are not close to the average value K. To obtain the scaling relation (2.7) for the dilute case a further assumption has to be made about the function $F(K_1, \ldots, K_n)$, namely (c) for $K > K_1 > 0$:

$$F(K, K, \ldots, K) > F(K_1, K, \ldots, K) > K_1 \quad . \tag{2.18}$$

Consider now the dilute-bond case where each bond can have the value K with probability p, or the value zero with probability $q=1-p$. For small q, we can follow the bond distribution to lowest (first) order in q, under consecutive applications of the renormalization-group transformation.

We now show, that under repeated iterations of a properly defined renormalization procedure, this bond distribution maps onto one which is sufficiently narrow, so that an expansion such as Eq. (2.10) is possible, and the analysis that follows is applicable. To see this, note that after one iteration we have

$$K'=\begin{cases}F(K,K,\ldots,K), & \mathcal{P}=1-nq+O(q^2)\\ F(O,K,\ldots,K), & \mathcal{P}=nq+O(q^2)\\ \text{other values}, & \mathcal{P}=O(q^2)\ ,\end{cases} \tag{2.19}$$

where $\mathcal{P}$ is the probability.

Using assumption (c) we note that subsequent applications of the transformation will drive the two largest values of the coupling K' towards each other, while the relative weight of the smaller coupling increases. After l iterations, this weight is given by $q_l=n^lq+O(q^2)$. For small enough q, this weight is still small enough so that values with probabilities of $O(q_l^2)$ can be neglected. So after l iterations we have, to leading order in q_l, the distribution

$$K\cong\begin{cases}K_1, & \mathcal{P}=1-q_l\\ K_2, & \mathcal{P}=q_l\ .\end{cases} \tag{2.20}$$

Using our assumption (c), we note that K_2 will approach K_1 from below, so that after l iterations

$$\Delta K=K_1-K_2<<K\ . \tag{2.21}$$

Therefore, under l repeated applications of a properly defined renormalization-group procedure, a system with a small fraction of missing bonds maps onto the regime of a narrow bond distribution that was discussed above. Once in this regime, expansions such as Eq. (2.10) can be used, with

$$\sigma_K^2\cong q_l\Delta K^2\ . \tag{2.22}$$

Further iterations reproduce the correct scaling behavior; one gets

$$q'=b^dq_l\ ,\quad \Delta K'=b^{y-d}\Delta K\ , \tag{2.23}$$

where the second relation is obtained by expanding $F(K_2,K_1,\ldots,K_1)$ about $K_2=K_1$, as was done for the case of a narrow distribution. Thus, we obtain

$$(\sigma_K')^2=b^{2y-d}\sigma_K^2 \tag{2.24}$$

and the scaling relation (2.7) is again obtained.

In physical terms, for the bond diluted case we observe the following. Initially, the system flows to a distribution dominated by the two largest values of the coupling. Under iteration the *weight* of the smaller of these increases, while their *separation* decreases. Eventually one maps into a region in which the separation is small enough so that the recursion relations can be expanded about the most probable coupling, and the scaling relation (2.7) is obtained. This picture makes physical sense; if a small fraction of the bonds is missing, one should expect, after one step of renormalization, that missing renormalized bonds ($K'=0$) occur with very low probability. Forcing the renormalized distribution into a two δ-function form with one coupling fixed to have the value zero,[6,15] will yield a procedure that violates the scaling relation. For scaling to hold, the distribution has to be such that allows the "competition" between increasing weight and decreasing difference between the dominant couplings.

III. POSITION-SPACE RENORMALIZATION METHOD

In the previous section we saw how scaling of randomness near the pure system can be exactly reproduced by simple position-space renormalizations. In this section we describe such a renormalization transformation. We generalize the Migdal-Kadanoff recursion relations[16] for inhomogeneous systems[6,17] to random Potts models.[4] The important part of the method is a simple parametrization of the renormalized bond distribution such that the transformation satisfies the Harris criterion. We start from the Hamiltonian of Potts spins $S_i=1,\ldots,q$ on a square or triangular lattice

$$-\beta H=\sum_{\langle ij\rangle}K_{ij}\delta_{S_iS_j}\ , \tag{3.1}$$

H contains only nearest-neighbor bonds K_{ij} (in units of the temperature) which are randomly distributed with probability distribution $P(K_{ij})$.

The version of the Migdal recursion relation[16] we use is illustrated in Fig. 1. First, every other bond is shifted. For instance, this gives new bonds

$$K_{12}=K_1+K_2,\quad K_{34}=K_3+K_4\ . \tag{3.2}$$

Then those Potts spins which are connected to only two neighbors are summed over, giving new nearest-neighbor couplings for the remaining Potts spins. This decimation is conveniently, expressed by the ratio of the two eigenvalues of the transfer matrix,

$$T(S_i,S_j)=\exp(K_{ij}\delta_{S_iS_j})\ ,$$

e.g.,

$$t(K)=\frac{e^K-1}{e^K+q-1}\ , \tag{3.3}$$

where q is the number of Potts states. Thus the new

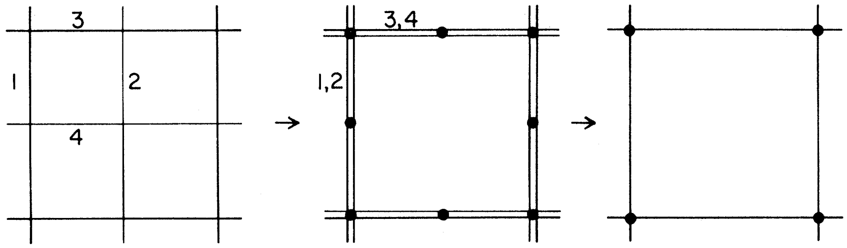

FIG. 1. Illustration of the renormalization procedure: In the first step every second bond is moved and added to the nonshifted bonds. In the second step the summation over the spins on the edges is performed, giving the renormalized interactions for the remaining spins.

coupling K' is given by

$$t(K')=t(K_1+K_2)t(K_3+K_4) \ . \tag{3.4a}$$

Finally, after rescaling with a factor $b=2$, we obtain again the random Potts model as defined by Eq. (3.1), but with a renormalized distribution of couplings K'_{ij},

$$P'(L)=\int \prod dK_i\, P(K_i)\, \delta(L-K'(K_1,K_2,K_3,K_4)) \ . \tag{3.4b}$$

One sees that this method leads to the same equations for the square and triangular lattices.[13] However, the pure Ising critical temperature K_c and the percolation concentration p_c, which are easily obtained from Eq. (3.4b) are closer to the exactly known values of the triangular lattice than to those of the square lattice.

It is easy to see that the recursion relation (3.4a) satisfies the conditions of Sec. II which are sufficient to reproduce the Harris criterion. Thus, following the full distribution of the couplings numerically we would find the correct scaling behavior near the pure system. However, calculations with full distributions consume much computer time, and statistical errors make the determination of second derivatives such as the specific heat very difficult, if not impossible. Also the discussion of fixed point distributions and their eigenvalues needs additional approximations. We employ therefore a simple parametrization of the renormalized coupling distribution.

Let us start with two bond strengths K_1 and K_2 distributed randomly with probability $1-p$ and p, respectively. Equation (3.4a) gives renormalized couplings K' with different weights p_i which are easily calculated. For instance, the new coupling may take the value K'_2 obtained from

$$t(K'_2)=[t(2K_2)t(K_1+K_2)] \ ,$$

with probability $p'_2=4p^3(1-p)$, etc. (for details see Ref. 6). We approximate the new distribution $P'(t')$ of $t'=t(K')$ by

$$P(t')=(1-p')\delta(t'-t'_1)+p'\delta(t'-t'_2) \ , \tag{3.5}$$

where t'_1,t'_2,p' are determined by setting the first three moments of $P(t')$ to be equal to those of the "full" distribution[6] $P'(t')$. This leads to the recursion relations

$$P'=\frac{1}{2}\left(1-\frac{B}{(4A^3+B^2)^{1/2}}\right) \ , \tag{3.6a}$$

$$t'_{1,2}=\langle t'\rangle+\frac{1}{2A}[B\pm(4A^3+B^2)^{1/2}] \ , \tag{3.6b}$$

with

$$A=\langle t'^2\rangle-\langle t'\rangle^2 \ , \tag{3.6c}$$

$$B=\langle t'^3\rangle-3\langle t'^2\rangle\langle t\rangle+2\langle t'\rangle^3 \ , \tag{3.6d}$$

where $\langle\cdots\rangle$ means an average over the renormalized full distribution $P'(t')$.

This parametrization satisfies the Harris criterion. In linear order in $1-p$, $P'(t')$ is just a two δ function distribution which is reproduced exactly by the distribution of Eq. (3.5). Therefore our approximation contains the competition between increasing weight and decreasing difference between couplings which is responsible for the scaling of randomness (see Sec. II). After a sufficient number of iterations the bond differences are small, and the second moment scales like b^{2y-d}. Since we transfer the exact second moment to the distribution $P(t')$ of Eq. (3.5), our procedure has the correct scaling behavior in this case, too. Furthermore, on the percolation line ($K_1=O_1$, $K_2=\infty$), the Migdal recursion relation (3.4) gives only zero and infinite bonds with renormalized weight p'. Thus in this case we obtain the same results as in Ref. 6, namely, a percolation fixed point with a critical exponent which describes the formation of geometrical clusters at the critical concentration. The numerical results based on this procedure for various random Potts models are discussed in Sec. V, to which the reader may proceed directly.

IV. DUALITY, GEOMETRICAL ARGUMENTS, AND POTTS MODELS WITH COMPETING INTERACTIONS

In this section we apply a duality transformation to the random-bond Potts model. On this basis an exact critical line is derived. Second, some heuristic geometrical arguments are presented, that relate percolative and thermal critical properties. In particular, an exponent ξ, that governs a purely geometrical property, is estimated on the basis of thermal considerations. Finally, we consider Potts models with competing positive and negative bonds, randomly distributed.

A. Duality

Consider the Potts model on a square lattice, with positive couplings K_1, K_2 distributed at random with probabilities $1-p$ and p, respectively. The dual K_i^* of a coupling K_i is given by[18]

$$(e^{K_i^*}-1)(e^{K_i}-1)=q \quad . \tag{4.1}$$

Replacing each coupling by its dual we get (on the dual lattice) a system with couplings K_1^* and K_2^*, distributed at random, with probabilities $1-p$ and p. The free energies of the original system and its dual have the same singular part, and therefore one can write

$$F_s(K_1,K_2,p)=F_s(K_2^*,K_1^*,1-p) \quad . \tag{4.2}$$

This relation can be used to derive an exact critical line for the random-bond Potts model at the percolation concentration $p=\frac{1}{2}$. Such a line has been derived for the Ising model by Fisch.[19] First, consider the case $K_1=0$, $K_2=\infty$, that corresponds to the bond diluted system at $T=0$, and, therefore, to the geometrical bond percolation problem. Using Eq. (4.2), one gets

$$F_s(0,\infty,p)=F_s(0,\infty,1-p) \quad ,$$

so that if there is a single singularity as a function of p, it has to occur at $p=1-p$, e.g., $p=p_c=\frac{1}{2}$. This exact result for the percolation concentration is well known.[20] Consider now a system with $p=\frac{1}{2}$, and $K_2>K_1>0$. This system is self-dual at $K_1^*=K_2$ (which implies also $K_1=K_2^*$); therefore, the critical line at $p=\frac{1}{2}$ is given by the implicit equation

$$(e^{K_1^c}-1)(e^{K_2^c}-1)=q \quad . \tag{4.3}$$

In Sec. V we compare this exact result with phase boundaries obtained by an approximate calculation.

The duality relations (4.1)–(4.2) can be used to obtain information about the critical lines of a Potts model with infinite and finite bonds distributed at random. Such a system is the dual of the dilute bond case, for which results concerning the critical line near the pure and the percolation limits are known.[2,4,21(a)] Using these results, we obtain for the case of randomly distributed finite bonds $K_2=\beta J_2$ (with probability p) and infinite bonds $K_1\rightarrow\infty$ (with probability $1-p$)

$$\frac{T_c(p)}{J_2}\sim\frac{q-1}{q\ln q}\,\frac{p_c}{p-p_c} \quad \text{for } p\rightarrow p_c \quad , \tag{4.4a}$$

$$\frac{1}{T_c(1)}\left.\frac{dT_c}{dp}\right|_{p=1}=-\frac{2\sqrt{q}}{(1+\sqrt{q}\,)\ln(1+\sqrt{q}\,)} \quad . \tag{4.4b}$$

One should note that in such a random system the clusters connected by infinite bonds can be represented by a single Potts spin: these, in turn, are connected by couplings of finite strength, creating some kind of random lattice of Potts spins.

B. Geometric picture of transitions near percolation

Consider again a bond-diluted Potts model, this time on an arbitrary lattice in d dimensions with nearest-neighbor interactions. Just above the percolation concentration we have many finite clusters and one infinite cluster. We expect that summation over the Potts spins in the finite clusters does not change the singular part of the free energy. The same holds for summation over chains of spins which are connected to the infinite cluster at one end only. Thus for a discussion of thermal critical properties it is sufficient to consider only the backbone of the infinite cluster.[21(b)] Near p_0 this backbone consists of large loops of chains of spins which are connected to each other by nodes of mainly three bonds. According to scaling theories,[22(a)] the average shortest distance and the average distance along the chains between these nodes diverge at p_c as powers of $p-p_c$, with critical exponents ν and ξ, respectively. We can now easily sum over the spins in the chains, i.e., spins with two neighbors only. This gives a new interaction K' between spins at two adjacent nodes. For the Potts model K' can be expressed in terms of K as

$$t(K')=t^n(K) \quad , \tag{4.5}$$

where n is the number of spins in the chain and

$$t(K)=\frac{e^K-1}{e^K+q-1} \tag{4.6}$$

is the ratio of the two eigenvalues of the transfer matrix.

Since n is different for different chains, K' is different, too. Thus the diluted model has the same singular free energy as a *random-bond* model for spins at the nodes of the infinite cluster's backbone. These nodes form a lattice with random structure.

Equations (4.5) and (4.6) also give us an estimate for the critical temperature. To a first approximation we may take into account fluctuations of the chain lengths n by replacing n in Eq. (4.5) by some effective value n_0, which diverges near p_c like[15,22(b)]

$$n_0 = A\left(\frac{p-p_c}{p_c}\right)^{-\xi} . \tag{4.7}$$

Thus, the random-bond system of the node spins is approximated by an average coupling K_0 given by

$$t(K_0) = t^{n_0}(K) . \tag{4.8}$$

The node lattice will be critical at some coupling K_{0c}, and Eq. (4.8) can be inverted to obtain the critical coupling K_c of the original random system, e.g.,

$$t(K_c) = [t(K_{0c})]^{A^{-1}[(p-p_c)/p_c]^{\xi}} \tag{4.9}$$

The lattice structure of the node spins is not expected to vary drastically as a function of p near p_c. The important part of the variation is taken into account via the divergence of the distance between nodes, Eq. (4.7). Therefore we can assume that near p_c the critical coupling of the node spins is constant, to leading order in $p-p_c$. Since K_c is large for p near p_c Eq. (4.9) can be expanded to yield, to leading order in $p-p_c$,

$$e^{-K_c(p)} \simeq A_q\left(\frac{p-p_c}{p_c}\right)^{\xi} \tag{4.10a}$$

with

$$A_q = \frac{|\ln t(K_{0c})|}{Aq} . \tag{4.10b}$$

It should be noted that comparison of Eq. (4.10a) with exact bounds derived by Bergstresser[23] immediately yields $\xi = 1$, thereby yielding a purely geometrical exponent on the basis of thermal and geometrical considerations.[22(b)]

Equations (4.10a) and (4.10b) give us a relation between a thermal phase transition and a pure geometrical property characterized by the coefficient A and exponent ξ. The same arguments hold for the bond dilute XY model.[24] There the low-temperature variable is K^{-1} instead of e^{-K}.

Equation (4.10a) has also been derived by scaling arguments.[25] Our position-space approach has the advantage that we do not have to assume a certain number of length scales and that we can give an estimate for the coefficient A. We mentioned above that near p_c the node lattice has a coordination number very close to 3. Thus it should be a good approximation to replace K_{0c} by the critical temperature of the Potts model on a honeycomb lattice,[26] given by

$$|\ln t(K_{0c})| = \ln(1+q/B) \tag{4.11a}$$

with

$$B = 2\sqrt{q}\,\cos[\tfrac{1}{3}\tan^{-1}(4/q-1)^{1/2}] . \tag{4.11b}$$

Then, if we know A_q we can calculate A by Eqs. (4.10). If our approximation makes sense, A should be independent of the number q of Potts states, since A is a geometrical property only. From the replica method A_q is given by[27]

$$A_q = \frac{\ln q}{q-1} . \tag{4.12}$$

Using Eqs. (4.10–4.12) we find

$$A = \frac{\ln(1+q/B)(q-1)}{q\ln q} , \tag{4.13}$$

with B given by Eq. (4.11b). Since A_q is believed to be lattice independent,[4] A should not depend on the lattice either. Solving Eq. (4.13) we find for $q=2$, 3, and 4 the values $A=0.396$, 0.382, and, 0.375 respectively. For larger q, A increases to the value 0.5 for $q=\infty$. Thus A is relatively insensitive to q which gives us confidence in our estimate, which yields for n_0, the average number of bonds that connect adjacent nodes of the infinite cluster's backbone

$$n_0 \sim 0.4\frac{p_c}{p-p_c} .$$

We have shown that both duality and decimation map different random systems onto each other. This supports the idea that if there is a sharp phase transition, we have a large universality class of random systems. We want to mention that there are random systems which belong to other types of universality classes, like spin-glasses[28] and random field models.[29]

C. Frustration in Potts Models

We now briefly discuss effects of competing interactions in random Potts models. Ising models with randomly distributed positive and negative bonds have many new and interesting properties. These models describe spin-glasses very well.[28]

For the moment let us consider only bonds with equal strengths and random sign. For Ising models it has been shown that the concept relevant to physical properties is "frustration"[30,31] described as follows. Consider a closed loop of bonds. If it is *not* possible to arrange all spins in such a way that the energy associated with *each* bond is lowest, then the loop is called frustrated. A measure of the frustration is the product of bond signs around the loop. It is negative if the loop is frustrated.

When a pure ferromagnet is diluted with frustrated plaquettes, the transition temperature decreases. Domany[32] has given a physical picture to estimate the concentration where the ferromagnetism disappears completely. He argues that ferromagnetism can occur only if the *unfrustrated* plaquettes percolate. Consider the square lattice. Then there are two types of frustrated plaquettes, shown in Fig. 2, which occur with probabilities $4p^3(1-p)$ and $4p(1-p)^3$, respectively (p is the concentration of positive bonds). Neglecting small correlations,[12] the unfrustrated plaquettes percolate when $p > p_0$ given by

$$1-4p_0^3(1-p_0)+4p_0(1-p_0)^3 \cong 0.59 \ . \qquad (4.14)$$

Here $p_c = 0.59$ is the concentration for site percolation.[11] Equation (4.14) gives $p_0 = 0.826$ which agrees very well with Monte Carlo estimates[22(a)] $0.8 \leq p_0 \leq 0.85$.

Now consider Potts models with general q. *First*, note that the gauge invariance of the Ising model, which leaves the frustration invariant,[30] is lost for $q \neq 2$. This means, for $q \neq 2$, that a change of the spin variable at one site cannot be compensated by a change in the sign of bonds surrounding it. Thus the ferromagnetic-antiferromagnetic symmetry in the square lattice holds only for $q = 2$, the Ising model.

Second, consider decimation. Summation over one spin with two neighbors leads to a new coupling K' for the neighbors given by

$$t(K') = t^2(K) \qquad (4.15)$$

with $t(K)$ given by Eq. (4.6).

So $t(K)$ is a measure of the effect of interactions. Therefore let us compare the ratio $t(-K)/t(K)$ in order to study the importance of negative couplings with respect to positive ones. One obtains

$$\frac{t(-K)}{t(K)} = \frac{2-q+(q-1)e^{-K}-e^{K}}{2-q+(q-1)e^{K}-e^{-K}} \ . \qquad (4.16)$$

At low temperatures, i.e., for $\pm K \to \pm\infty$, this ratio approaches $-(q-1)^{-1}$. Thus for large q, at low temperature, negative bonds are renormalized to zero bonds. At high temperatures, in linear order in K, the ratio is -1 independent of q. Thus we expect, that the slope $dT_c(p)/dp$ at $p = 1$ is only slightly sensitive to q, in agreement with Ref. 4, whereas the low temperature $T_c(p)$ is shifted to lower concentrations with increasing q.

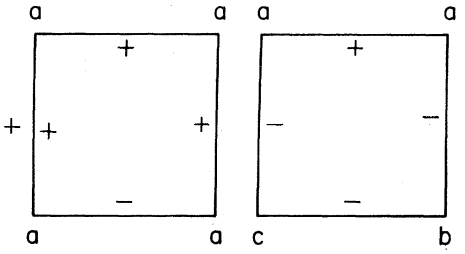

FIG. 2. Two ground-state configurations of Potts spins, the signs denote the signs of the interaction $K\delta_{s_i s_j}$, the Potts variable is denoted by a,b,c. In the Ising case (no state c is allowed), both squares are frustrated. In the three- or more-state Potts model, however, only the left square is frustrated.

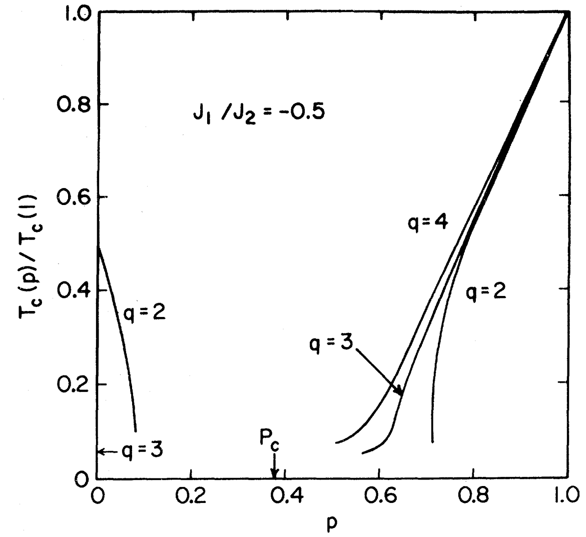

FIG. 3. Critical temperature as a function of concentration p of J_2 bonds for different number q of Potts states. Since J_1 is negative, there is an antiferromagnetic phase for small p and q.

Third, note that frustration is strongly reduced when going from $q = 2$ to larger q. From Fig. 2 it is obvious that any loop which contains more than one negative bond is not frustrated. For instance this leads to a phase transition in the $q = 3$ state antiferromagnetic Potts model on a triangular lattice.[33] Only when one single bond is negative, no spin alignment that satisfies all bonds is possible. Using Domany's percolation arguments to derive p_0 for $q > 2$, we see that only the plaquette in Fig. 2(a) is frustrated. Using the same argument that led to Eq. (4.14), one obtains $p_0 = 0.7$ for the square lattice. Thus for $q > 2$ the ferromagnetic phase at low temperatures is extended to much lower concentrations, in agreement with the decimation arguments given above.

In Sec. V we also see that the position-space renormalization shows this effect; namely, that for low temperatures the ferromagnetic phase extends up to higher negative bond concentration, as the number of Potts components increases (see Fig. 3).

V. NUMERICAL RESULTS

This section presents the results of the Migdal-Kadanoff renormalization transformation for the bond-random Potts model in two dimensions. We use the parametrization and the recursion relations described in Sec. III.

A. Random Fixed Point

We consider a q-component Potts model with J_1 and J_2, distributed at random with probabilities $1-p$ and p, respectively. As mentioned in Sec. II, the recursion relations given by Eqs. (3.6) are conveniently expressed in terms of the variables $t_1 = t(J_1/k_B T)$, $t_2 = t(J_2/k_B T)$, and p, with t given by Eq. (3.3). Note that $t=0$ for zero bonds and $t=1$ for infinite bonds ($T=0$). Figure 4 shows the space (t_1,t_2,p). We can always choose $t_1 \leq t_2$. For $t_1 = t_2, p$ is not defined, and for $p=0(1)$, $t_2(t_1)$ is not defined. Therefore the pure critical system, given by

$$t(K_c) = t^2(2K_c) \tag{5.1}$$

corresponds to the entire dotted line in Fig. 4. Furthermore, we know the percolation fixed point from Ref. 6, located at $p=0.38$. In our three-dimensional space it is threefold unstable, with all three scaling powers given by $y=0.611$, independent of the number q of Potts states. This shows that the crossover exponent from percolation to thermal critical behavior is unity, independent of the direction in coupling space.

Between the pure line and the percolation fixed point we have a critical surface which separates the ordered ferromagnetic phase from the paramagnetic one. Details of the phase diagram will be discussed below. We have already shown in Sec. II that our renormalization procedure satisfies the Harris scaling of the randomness. Indeed, for negative specific-heat exponent α_{pure} of the pure system ($q < 6.8$ in our approximation) we find only the pure critical fixed point, located at $J_1 = J_2$ and $p=0.48$. In this case the whole critical surface flows under the renormalization iteration into this fixed point. So the pure line is not a line of fixed points. Our recursion relations give a uniquely defined pure fixed point in the limit of vanishing randomness.

Increasing the number q of Potts states beyond $q=6.8$ we obtain a positive value for α_{pure}. In this case we find a new random fixed point spliting off from the pure one. Now the whole critical surface of the random system flows into this random fixed point. Its location changes with increasing q as shown by the arrows in Fig. 4. When α_{pure} goes to unity ($q \to \infty$ in our approximation[13]) the random fixed point approaches the pure system again but not at the pure fixed point.

The random fixed point has one relevant eigenvalue which gives a new specific-heat exponent α_{random} for all random systems crossing the critical surface. This means that the random system has a divergent specific heat. In Table I the properties of this new random fixed point are shown for α_{pure} values which correspond to Ising, three- and four-state Potts models.[34] Figure 4 shows the difference $\alpha_{\text{pure}} - \alpha_{\text{random}}$ as a function of α_{pure}. We expect the largest effect for the three-state Potts model. In this case $\alpha_{\text{pure}} = \frac{1}{3}$ is reduced to $\alpha_{\text{random}} = 0.24$. When α_{pure} goes to one, the system approaches a first-order transition.[13] In this case we see from Fig. 5 that the random system also approaches a first-order transition.

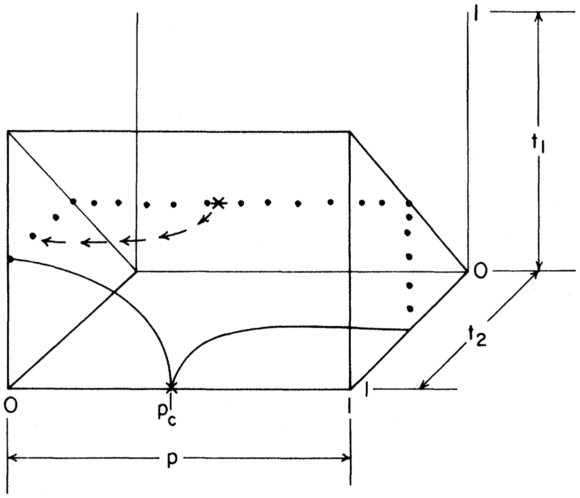

FIG. 4. Parameter space of our renormalization equation (3.6). $t_{1,2}$ are essentially bond strengths, p is the concentration of t_2. The pure critical system is described by the dotted line. The other two lines sketch the critical temperatures of the usual (zero-bond) percolation ($t_1=0$) and the infinite bond percolation ($t_2=1$). The whole critical surface spans between these three lines. For $\alpha_{\text{pure}} > 0$ the critical surface flows to a new random fixed point. With increasing α, this fixed point splits off the pure fixed point and moves along the dashed line.

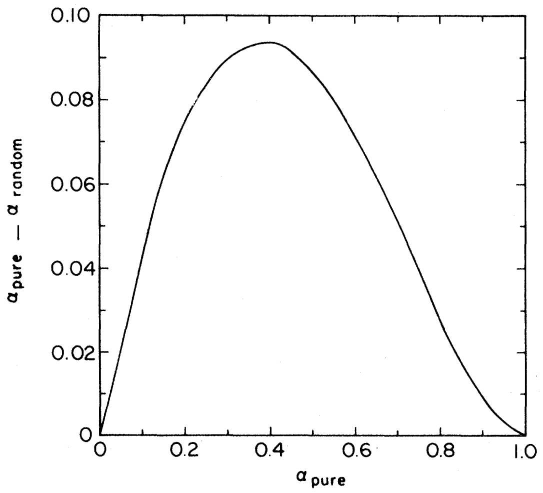

FIG. 5. Difference between the random and pure specific-heat exponents as a function of the pure one.

TABLE I. For $\alpha_{\text{pure}} > 0$, the critical behavior of the random-bond Potts model is governed by a random fixed point. The fixed-point bond distribution in the form of Eq. (3.3), is parametrized in terms of J_1/J_2 (where $J_i = k_B T K_i$), p (the concentration of J_2 bonds), and $T_c/T_{c\,\text{pure}}$. α_{pure} and α_{random} are the respective specific-heat exponents.

α_{pure}	α_{random}	J_1/J_2	p	$T_c/T_{c\,\text{pure}}$ $(p=1)$
0	0	1	0.49	1
$\frac{1}{3}$	0.24	0.5	0.15	0.57
$\frac{2}{3}$	0.60	0.48	0.04	0.50
1	1	0.51	0	1

We find that both α_{pure} and α_{random} increase as a function of the number q of Potts states. So, keeping the bond configuration (for instance the cluster geometry for diluted systems) fixed, the divergence (versus finiteness) of the specific heat is the same for both pure and random systems. This seems to be plausible. However, it is in contrast to scaling arguments of Ma.[35] These arguments allow a negative α_{random} only, i.e., only a nondivergent specific heat. Also, expansions near four dimensions[36,37] and a recent application of the scaling field approach in three dimensions[38] give a negative α for the random Ising model. At the moment we cannot explain this contradiction. It would be useful to investigate this problem by other means, for instance Monte Carlo methods, series expansions, and experiments on adsorbates on surfaces which belong to the three-state Potts universality class.[39]

B. Phase Diagrams

Figures 6 and 7 show cuts through the critical surface, now in units more appropriate for experiments, namely, the critical temperature in units of T_c(pure) (only J_2 present) as a function of J_2 bond concentration p and the ratio of the two bond strengths J_1/J_2.

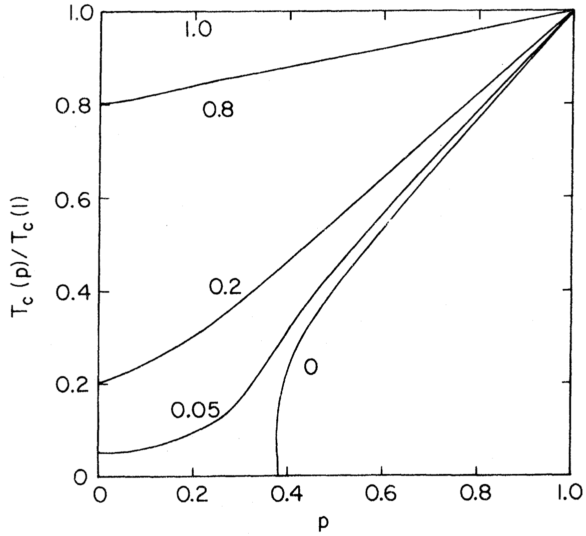

FIG. 6. Critical temperature as a function of concentration p of J_2 bonds for different bond ratios $J_1/J_2 (q=2)$.

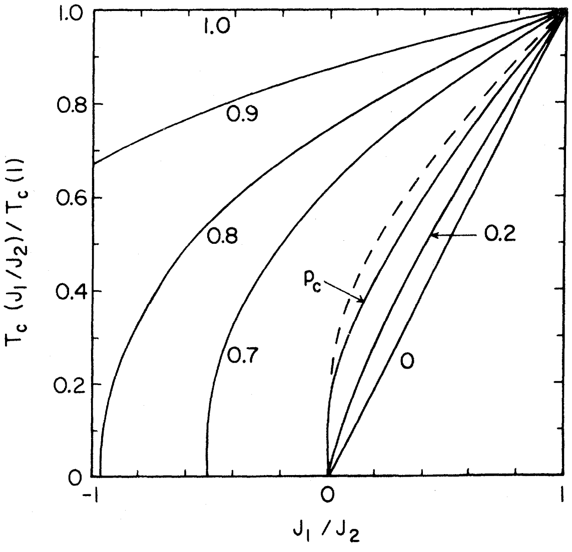

FIG. 7. Critical temperature as a function of bond ratio J_1/J_2 for different concentrations p of J_2 bonds. The dashed line gives the exact critical line at p_c for the square lattice $(q=2)$.

For $J_1 = 0$, i.e., the diluted system, $T_c(p)$ is very similar to the corresponding results of Ref. 6. We can also discuss $T_c(p)$ for nonzero J_1 with concentration $1-p$. For $p=0$ one has trivially

$$T_c(0) = \frac{J_1}{J_2} T_c(1) \ . \tag{5.2}$$

$T_c(p)$ for different ratios J_1/J_2 is shown in Fig. 5. Just below p_c, T_c is very sensitive to a small coupling ratio. This is also seen in Fig. 7 which shows T_c as a function of coupling ratio for different concentrations p of J_2. We used $q=2$, but in general the phase diagram is rather insensitive to the number q of Potts states.

For $0 \leq p \leq p_c$ the J_2 bonds do not percolate. In this case one has a ferromagnetic transition only for positive J_1. However, for $p \geq p_c$ we have also shown the phase diagram for a ferromagnet with a fraction of negative bonds. Note that the Migdal recursion relations which we use here do not have a spin-glass fixed point[17] in two dimensions. For $J_1/J_2 = -1$ we see from Fig. 6 that ferromagnetism is suppressed below $p_0 \lesssim 0.87$. It is interesting that our simple approximation yields a p_0 which agrees with values observed for the square lattice using several other methods (see Ref. 32 for a discussion of p_0). For the square lattice at p_c we know the exact phase boundary from duality, Eq. (4.3). Figure 7 shows that our results at p_c agrees well with the exact result.

Figure 3 shows the effect of negative bonds as discussed in Sec. III. For $q=2$ we have the symmetry $(J_1, J_2) \leftrightarrow (-J_1, -J_2)$ leading to an antiferromagnetic phase at small p. For $p=0$, i.e., the pure antiferromagnet, we obtain a transition for $q < 3.03$. This shows that entropy effects may lead to an ordering of one sublattice with the other one being disordered. However, the antiferromagnetic transition for $q=3$ is destroyed by a very small concentration of positive bonds. The ferromagnetic transition shows the effects discussed in Sec. IV; for small negative bond concentration we see only a small change in $T_c(p)$ with q. However, for small $T_c(p)$ and $q > 2$ the space boundary turns towards larger negative bond concentrations than that of the Ising case.

C. Specific Heat

In Sec. V A we have seen that for positive pure specific-heat exponent α_{pure} the random system has new critical behavior. However one would like to know whether the new random α can be observed in experiments. Therefore we calculated the specific heat with the renormalization transformation Eq. (3.6). The free energy, averaged over the bond configurations, is given by[6]

$$\bar{f}(\{K\}) = \sum_{n=0}^{\infty} b^{-(n+1)d} \bar{g}(t_1^{(n)}, t_2^{(n)}, p^{(n)}) \ . \tag{5.3}$$

$\bar{g}$ is an average over the constants g given by the decimation

$$g = 2\ln(e^{K_1+K_2} + e^{K_3+K_4} + q - 2) \ , \tag{5.4}$$

to be averaged with the bond distribution Eq. (3.4), which gives six different terms for $\bar{g}$. The specific heat is given by

$$C = \beta^2 \frac{\partial^2 \bar{f}}{\partial \beta^2} \ . \tag{5.5}$$

Figure 8 shows the specific heat of the *pure* system with $\alpha_{\text{pure}} = \frac{1}{3}$ which should correspond to the three-state Potts model.[34] From the simple log-log plot the exponent can be observed below $t = (T - T_c)/T_c \simeq 10^{-3}$. However, fitting a straight line between $10^{-3} < t < 10^{-2}$ gives an effective exponent $\alpha \simeq 0.42$. Indeed, this exponent has been observed in recent computer experiments[40] of the pure three-state Potts model for $t \gtrsim 10^{-2}$.

Figure 8 also shows the specific heat for the random (positive) bond Potts model near the random fixed point. One should note the slow crossover to the true (random) critical behavior: only for $t \lesssim 10^{-4}$ can the random exponent $\alpha = 0.24$ be observed. Between $10^{-3} < t < 10^{-2}$ one finds an effective exponent of 0.37. In the critical diluted system, also shown in Fig. 8, the renormalization flow needs many steps to approach the fixed point. This gives an additional analytic contribution to the specific

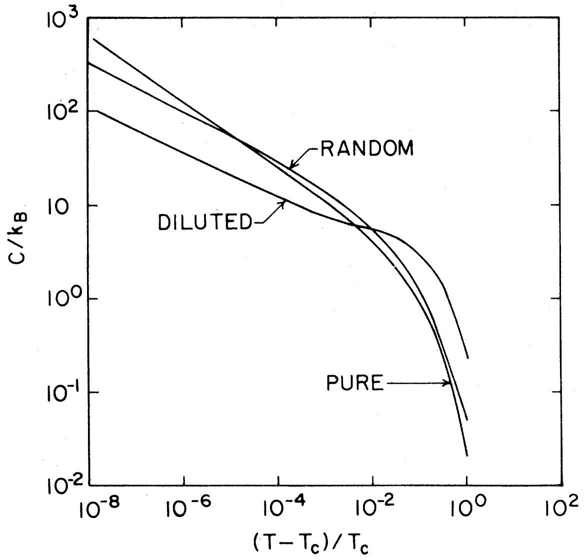

FIG. 8. Specific heat of a random Potts model. The pure system has an exponent $\alpha = \frac{1}{3}$, which corresponds to the pure three-state Potts model. Furthermore, the specific heat of the system at the random fixed point and of the diluted system ($p = 0.6$) are shown.

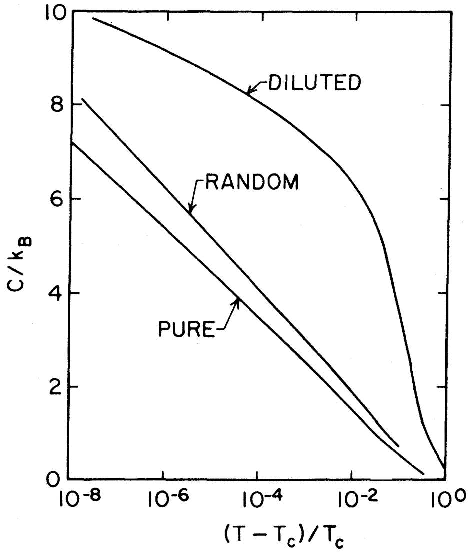

FIG. 9. Specific heat of a random Potts model, which corresponds to a random Ising model (logarithmic divergence of the pure system). The figure shows the pure system, a diluted one ($p=0.6$) and a system near the pure fixed point ($J_1/J_2=0.9$, $p=0.5$).

heat. Thus analyzing the log-log plot between $10^{-3} < t < 10^{-2}$ an effective exponent $\alpha = 0.15$ is found which is smaller than the asymptotic value $\alpha = 0.24$. We note that the critical amplitude is smaller for the diluted system than for the one at the random fixed point. Figure 9 shows the specific heat for the $\alpha_{\text{pure}} = \alpha_{\text{random}} = 0$ system as a function of $\log_{10} t$. Introducing randomness at the pure fixed point increases the amplitude of the logarithmic divergence. We could not find any corrections to critical behavior due to the marginal randomness. In the dilute system the logarithmic dependence of C disappears for reasonable values of t. However, C still increases rapidly as t goes to zero. Only very close to the percolation concentrations p_c we see a maximum of C above T_c. For larger concentrations the contribution of clustering effects is only a broad satellite on top of the critical contribution with no peak of its own. This differs from the results in Ref. 6. There the approximate recursion relation gave a negative α for the Ising model which underestimated the critical contribution to the specific heat. Here we study a system with the exact exponent $\alpha = 0$.

VI. SUMMARY

This paper discusses critical properties of quenched random Potts models. In particular, we consider the two-dimensional random bond model on a regular lattice with two bond strengths K_1 and K_2 distributed randomly with probabilities $1-p$ and p, respectively.

By duality transformation and geometrical arguments different random Potts models are mapped onto each other. This supports the idea of a large universality class of random systems. We derive an exact critical line at the percolation concentration for the square lattice. For the diluted system ($K_1 = 0$) we show how the geometrical properties of the infinite percolating cluster determine the thermal phase diagram. Using known results of phase diagrams, we obtain estimates of the exponent and amplitude of the scaling law of the chain length in the infinite cluster.

Negative bonds in a ferromagnet create so-called "frustration" which reduces the transition temperature and leads, at least in higher dimensions, to new "spin-glass" phases. We show that negative bonds in Potts ferromagnets with more than three states per site create much less frustration than in Ising models. This leads to the fact that in Potts models with a fraction p of negative bonds, $T_c(p)$ decreases strongly for small p, but then turns back and decreases weakly for larger p, in contrast to Ising models.

Critical properties of random systems near the pure system are described by the Harris criterion. This means, that a measure of randomness scales like α/ν, the pure specific heat and correlation exponent, respectively. As a consequence, the random critical behavior is different from the pure one for $\alpha \geqslant 0$ only. We derive this result with the replica method. Furthermore we show how the Harris criterion can be seen in approximate position-space renormalizations. We see that it also holds for the bond-diluted system, where the difference in bond strengths (zero and finite) is large. Following the renormalization flow we see that the Harris criterion arises from a competition between increasing weight of randomness and decreasing difference in bond strengths.

This picture helps us to find a simple three parameter position-space renormalization which gives the exact scaling of randomness near the pure system. Since the equation contains the number of Potts states as a continuous parameter q we can observe what happens when α goes from negative values to positive ones. We see that for $\alpha > 0$ a new random fixed point splits off from the pure one. The whole critical surface (except the pure systems) flows to this random fixed point. This means that the random system has a new specific-heat exponent α_{random}. We find that α_{random} is smaller than α_{pure} but positive. Both α_{random} and α_{pure} increase with increasing q, which is in contradiction with scaling arguments of Ma. Also, if one assumes a single correlation length near T_c, α_{random} should be negative according to Harris. Whether a system at a random fixed point, i.e., a random-bond distribution at every length scale, can be described by a single correlation length, remains doubtful. Thus we hope that our results stimulate other investigations of random three- or four-state Potts models using different methods.

By our renormalization method we also obtain nonuniversal properties. Thus we calculate phase diagrams and specific heat of random Potts models for different K_1, K_2, and p values. We found that the true critical behavior that characterizes the random system emerges only very close to T_c, because of slow crossover. For models with $\alpha_{\text{pure}} \geqslant 0$ the specific heat is dominated by the critical singularity, as opposed to Ref. 6, where a narrow peak superimposed on a large analytic background was found.

Note added in proof. The possibility of a positive specific-heat exponent α in a random system has been discussed by D. J. Bergman, A. Aharony, and Y. Imry, J. Magn. Magn. Mater. 7, 217 (1978). The value of the crossover exponent ϕ quoted by Harris and Lubensky in Ref. 11 is not correct. The correct value agrees with the exact one [Eq. (2.17)] $\phi = \alpha/\nu$ [A. Aharony (unpublished)]. We thank A. Aharony for bringing these points to our attention.

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[1] For recent reviews see J. M. Kosterlitz and D. J. Thouless, Prog. Low. Temp. Phys. B 8, 371 (1978); M. N. Barber, Phys. Rep. 59, 375 (1980). A large sample of recent experimental work can be found in J. Phys. (Paris) C 4, 1 (1977).

[2] E. Domany and E. K. Riedel, J. Appl. Phys. 49, 1315 (1978); E. Domany, M. Schick, J. S. Walker, and R. B. Griffiths, Phys. Rev. B 18, 2209 (1978); E. Domany and M. Schick *ibid.* 20, 3828 (1979); M. J. Tejwani, O. Ferreira, and O. E. Vilches, Phys. Rev. Lett. 44, 152 (1980).

[3] For recent reviews see, S. Kirkpatrick, *Les Houches Summer School in Physics, 1978* (North-Holland, Amsterdam, 1978); M. Wortis, C. Jayaprakash, and E. K. Riedel, J. Appl. Phys. 49, 1335 (1978).

[4] B. W. Southern and M. F. Thorpe, J. Phys. C 12, 5351 (1979); J. M. Yeomans and R. B. Stinchcombe (unpublished).

[5] D. Stauffer, Z. Phys. B 22, 161 (1975); H. E. Stanley, J. J. Birgenau, P. J. Reynolds, and J. F. Nicoll, J. Phys. C 9, L553 (1976).

[6] C. Jayaprakash, E. K. Riedel, and M. Wortis, Phys. Rev. B 18, 2244 (1978).

[7] A. B. Harris, J. Phys. C 7, 1671 (1974).

[8] V. J. Emery, Phys. Rev. B 11, 239 (1975).

[9] E.g., L. P. Kadanoff and F. J. Wegner, Phys. Rev. B 4, 3989 (1971).

[10] F. J. Wegner and E. K. Riedel, Phys. Rev. B 7, 248 (1973).

[11] T. C. Lubensky, Phys. Rev. B 11, 3573 (1975); G. Grinstein and A. H. Luther, *ibid.* 13, 1329 (1976); A. B. Harris and T. C. Lubensky, Phys. Rev. Lett. 33, 1540 (1974).

[12] While this manuscript was in final stages of preparation, we received unpublished work of J. M. Yeomans and R. B. Stinchcombe, who studied the critical properties of random three- and four-state Potts models.

[13] A. N. Berker and S. Ostlund, J. Phys. C 12, 4961 (1979).

[14] E.g., A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6, p. 357.

[15] T. Tatsumi and K. Kawasaki, Prog. Theor. Phys. 55, 614 (1976); M. Plischke and D. Zobin, J. Phys. C 10, 4571 (1977); J. M. Yeomans and R. B. Stinchcombe, *ibid.* 12, 347 (1979).

[16] A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975) [Sov. Phys. JETP 42, 743 (1975)]; L. P. Kadanoff, Ann. Phys. 100, 359 (1976).

[17] S. Kirkpatrick, Phys. Rev. B 15, 1533 (1977); C. Jayaprakash, J. Chalupa, and M. Wortis, *ibid.* 15, 1495 (1977).

[18] See, for example, F. Y. Wu and Y. K. Wang, J. Math. Phys. 17, 439 (1976).

[19] R. Fisch, J. Stat. Phys. 18, 111 (1978).

[20] J. W. Essam, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1972), Vol. 2.

[21] (a) H. Nishimori, J. Phys. C 12, L64 (1979), has conjectured an exact equation for $T_c(p)$ for the bond-diluted Ising model; however, A. Aharony and M. J. Stephen have disproved the conjecture J. Phys. C 13, L407 (1980). (b) This may be incorrect for magnetic exponents, see G. Shlifer, W. Klein, P. J. Reynolds, and H. E. Stanley, J. Phys. A 12, L169 (1979).

[22] (a) D. Stauffer, Phys. Rev. 54, 1 (1979). (b) A. Coniglio Phys. Rev. Lett. 46, 250 (1981).

[23] T. K. Bergstresser, J. Phys. C 10, 3831 (1977).

[24] W. Kinzel, E. K. Riedel, and S. Solla (unpublished).

[25] T. C. Lubensky, Phys. Rev. B 15, 311 (1977).

[26] D. Kim and R. I. Joseph, J. Phys. C 7, L167 (1974).

[27] E. Domany, J. Phys. C 11, L337 (1978).

[28] See, for example, K. Binder, in Ordering in Strongly Fluctuating Condensed-Matter Systems, NATO Advanced Study Institute, Geilo 1979 (Plenum, New York, in press).

[29] Y. Imry and S. K. Ma, Phys. Rev. Lett. 35, 1399 (1975); G. Grinstein, *ibid.* 37, 944 (1976); I. Morgenstern, K. Binder, and R. M. Hornreich, Phys. Rev. B 23, 287 (1981).

[30] G. Toulouse, Commun. Phys. 2, 115 (1977).

[31] W. Kinzel and K. H. Fischer, J. Phys. C 11, 2115 (1978).

[32] E. Domany, J. Phys. C 12, L119 (1979).

[33] M. Schick and R. B. Griffiths, J. Phys. A 10, 2123 (1977).

[34] M. P. M. den Nijs, J. Phys A (in press); B. Nienhuis, A. N. Berker, E. K. Riedel, and M. Schick, Phys. Rev. Lett. 43, 737 (1979).

[35] S. K. Ma, *Modern Theory of Critical Phenomena,* (Benjamin, New York, 1976), Chap. X.

[36] D. E. Knmelnitzkii, Zh. Eksp. Teor. Fiz. 68, 1960 (1975), [Sov. Phys. JETP 41, 981 (1975)].

[37] A. Aharony, J. Magn. Magn. Mater. 7, 198 (1978).

[38] K. E. Newman and E. K. Riedel (unpublished).

[39] S. Alexander, Phys. Lett. A 54, 353 (1975).

[40] K. Binder (unpublished).