PHYSICAL REVIEW B VOLUME 23, NUMBER 7 1 APRIL 1981

Roton second sound and roton scattering

Humphrey J. Maris and Richard W. Cline*
Department of Physics, Brown University, Providence, Rhode Island 02912

(Received 20 October 1980)

The theory of roton second sound in superfluid helium-4 is considered. We first derive the hydrodynamic equations which determine the motion of a pure roton gas. The collisions between rotons are assumed to conserve energy, momentum, and the number of rotons. The hydrodynamic equations without dissipative terms have the same form as the corresponding equations for an ordinary fluid. When lowest-order dissipative effects are included, it is found that there are some extra dissipative processes in addition to heat conduction and viscosity. We next consider modifications of the hydrodynamic equations which occur when the effects of collisions between rotons and phonons are included, and when there are some roton-roton collisions in which the roton number changes. It is found that at the temperatures and pressures where roton second sound has so far been observed, the wave motion is best described as isothermal second sound. We show that current experimental data indicate that in nearly all collisions between rotons ($>99.97\%$) the total number of rotons is conserved.

I. INTRODUCTION

Second sound in superfluid helium was first predicted by Landau[1] and by Tisza.[2] Landau showed that the velocity c_2 of second sound is given by the formula

$$c_2 = \left(\frac{\rho_s}{\rho_n} \frac{TS^2}{\rho C} \right)^{1/2} , \quad (1)$$

where S, C, ρ_s, ρ_n, and ρ are the entropy, specific heat, superfluid density, normal fluid density, and total density, respectively. At low temperatures ($T \lesssim 1.2$ K) the normal fluid density is much smaller than the total density and so Eq. (1) can be simplified to read

$$c_2 = \left(\frac{TS^2}{\rho_n C} \right)^{1/2} . \quad (2)$$

At these temperatures second sound is closely analogous to an ordinary sound wave in a gas.[3] The "atoms" of the gas are the elementary excitations of helium (i.e., phonons and rotons). This gas has a bulk modulus $B_{S\mu}$ given by

$$B_{S\mu} = \frac{TS^2}{C} . \quad (3)$$

Thus, the velocity can be expressed as

$$c_2 = (B_{S\mu}/\rho_n)^{1/2} . \quad (4)$$

The modulus $B_{S\mu}$ is the modulus that determines pressure changes when the entropy and chemical potential of the gas are held constant. For an ordinary gas the same formula holds for the velocity of sound except that instead of $B_{S\mu}$, the bulk modulus B_{SN} at constant entropy and particle number is used. The difference comes about because collisions between rotons and phonons do not usually conserve the total number of excitations, and so the chemical potential of the roton-phonon gas is always zero.

The theory of the interactions amongst the elementary excitations in helium was first considered by Landau and Khalatnikov.[4] They concluded that at low temperatures the roton and phonon components of the excitation gas interacted only weakly with each other, and that the number of collisions between excitations in which rotons were converted into phonons (or vice versa) was small. Consequently Khalatnikov and Chernikova[5,6] suggested that it might be possible for there to be a wave motion similar to second sound, but involving only the roton part of the excitation gas. They predicted that this mode could be observed if high-frequency measurements were made at temperatures below about 1.2 K. The frequency has to be high compared to the rate at which the roton and phonons interact with each other. The first observation of roton second sound (RSS) was made by Dynes, Narayanamurti, and Andres.[7-9] Instead of propagating a continuous wave of RSS they generated short pulses. With this method, and for a propagation distance of 0.234 cm, they were able to detect RSS at temperatures between 0.4 and 0.7 K. The pulse had a velocity that was in rough agreement with the predictions of the Khalatnikov-Chernikova theory. In addition to the RSS signal the detector received also a higher velocity pulse due to phonons. This provided some support for the view that the interaction between phonons and rotons was weak.

Further observations of RSS have been made by Castaing and Libchaber,[10–12] and by Cline and Maris.[13] These experiments were primarily designed to give more accurate results for the velocity of RSS, but also provided some information about the attenuation. The theory has been extended by Maris,[14] Weiss and Polder,[15,16] Khalatnikov,[17] and Castaing and Libchaber.[11] Although the basic picture of RSS is certainly correct many important points have still not been resolved. The measured velocity of RSS, at least under the range of experimental conditions investigated so far, is somewhat larger than the velocity predicted by Khalatnikov and Chernikova. Possible reasons for this have been suggested in the later theories mentioned above, but no definite conclusion has been reached. In addition, the attenuation and dispersion of RSS is fairly large and this has not yet been satisfactorily related to theory. In this paper we consider these problems. In the next section we derive hydrodynamic equations for a roton gas. These hold when there is no interaction between the rotons and the phonons. These equations include the lowest order dissipative corrections due to viscosity, etc. Section III compares and contrasts these results with the corresponding equations of conventional hydrodynamics (e.g., the Navier-Stokes equation). In Sec. IV we relate the dissipative coefficients to the roton-roton scattering time, and in Sec. V calculate the velocity and attenuation of a wave propagating in a roton gas. Section VI describes how the hydrodynamic equations are modified when various sorts of interactions between rotons and phonons can occur.

II. HYDRODYNAMICS OF A PURE ROTON GAS

We derive here the hydrodynamics of a roton gas under the following conditions.

(i) The number density n_R of the rotons is sufficiently small that the superfluid density ρ_s is approximately equal to the total density ρ.

(ii) The collisions between the rotons conserve the total energy, momentum, and number of the rotons.

(iii) The collision rate is large on the time scale of motions we consider, and the roton mean free path is short compared to the distances over which the various hydrodynamic quantities vary.

Because of (i) we can assume in the present context that the superfluid velocity $\vec{v}_s$ is zero, and that the superfluid density is constant. Hence, the excitations move in a static background superfluid, and the distribution function n_p for the rotons is just determined by the Boltzmann equation

$$\frac{\partial n_p}{\partial t}=\left(\frac{\partial n_p}{\partial t}\right)_{\text{coll}}-\vec{v}_p\cdot\vec{\nabla}n_p \ . \tag{5}$$

The first term on the right-hand side (RHS) is the collision term, and $\vec{v}_p$ is the roton group velocity. Thus, this equation is of exactly the same form as the Boltzmann equation for an ordinary gas of atoms. In addition, conditions (ii) and (iii) are the same as those that hold for an ordinary gas when a hydrodynamic description is a good approximation. Thus, the hydrodynamics of a roton gas can differ from conventional hydrodynamics only because the energy-momentum relation (dispersion relation) for rotons is not $p^2/2m$. We shall see that this difference does not change the equations of nondissipative hydrodynamics, but does introduce some extra, and physically interesting, lowest order dissipative effects.

For small amplitude disturbances we can linearize Eq. (5) to get

$$\frac{\partial n_p}{\partial t}=\int C(\vec{p},\vec{p}\,')\,\delta n_{p'}\,d\tau_{p'}-\vec{v}_p\cdot\vec{\nabla}n_p \ , \tag{6}$$

where

$$n_p=\bar{n}_p+\delta n_p \ , \tag{7}$$

$\bar{n}_p$ is the equilibrium distribution function, and δn_p is small. Our method of solution follows closely that used in Ref. 18. We introduce a new function $\tilde{C}$ related to C by

$$\bar{n}_p^{1/2}(\bar{n}_p+1)^{1/2}\tilde{C}(\vec{p},\vec{p}\,')=C(\vec{p},\vec{p}\,')\,\bar{n}_{p'}^{1/2}(\bar{n}_{p'}+1)^{1/2} \ . \tag{8}$$

$\tilde{C}$ is then symmetric with respect to interchange of the variables $\vec{p}$ and $\vec{p}\,'$. We define eigenfunctions $\psi_i(\vec{p})$ and eigenvalues λ_i by

$$\int \tilde{C}(\vec{p},\vec{p}\,')\,\psi_i(\vec{p}\,')\,d\tau_{p'}=-\lambda_i\psi_i(\vec{p}) \ . \tag{9}$$

Because of the properties of $\tilde{C}$ the eigenvalues are non-negative and the eigenfunctions can be chosen to be orthonormal:

$$\int \psi_i(\vec{p})\,\psi_j(\vec{p})\,d\tau_p=\delta_{ij} \ . \tag{10}$$

Then if we expand δn_p as

$$\delta n_p=\bar{n}_p^{1/2}(\bar{n}_p+1)^{1/2}\sum_i A_i\psi_i(\vec{p}) \ , \tag{11}$$

the Boltzmann equation (5) leads to the following equation for the expansion coefficients A_i:

$$\dot{A}_i=-\lambda_iA_i-\sum_j\langle i|v_{p\alpha}|j\rangle\frac{\partial A_j}{\partial X_\alpha} \ . \tag{12}$$

$\langle i|v_{p\alpha}|j\rangle$ is the matrix element of the α component of $\vec{v}_p$ between eigenfunctions ψ_i and ψ_j, and a summation over the repeated Greek indices is implied. X_α is the α component of the position vector $\vec{X}$. Consider now any quantity $f(\vec{p})$ which is conserved in collisions between rotons. The total amount of this quantity per unit volume is

$$F=\int f(\vec{p})\,n_p\,d\tau_p \ . \tag{13}$$

The rate at which F changes as a result of collisions must be zero. Therefore, from Eq. (6)

$$\int d\tau_p \int d\tau_{p'} f(\vec{p})C(\vec{p},\vec{p}\,')\delta n_{p'} = 0 \ . \tag{14}$$

But this must be true for any distribution $\delta n_{p'}$, and so

$$\int d\tau_p f(\vec{p})C(\vec{p},\vec{p}\,') = 0 \ . \tag{15}$$

If we combine this with Eq. (8) it follows that

$$f(\vec{p})\bar{n}_p^{-1/2}(\bar{n}_p+1)^{1/2} \tag{16}$$

is an eigenfunction of $\tilde{C}$ with eigenvalue zero. Thus there are as many zero-eigenvalue eigenfunctions as there are conserved quantities, i.e., five in all. In normalized form we can write these eigenfunctions as follows:

$$\psi_{1S} = a_{000}^{-1/2}\bar{n}_p^{-1/2}(\bar{n}_p+1)^{1/2} \ , \tag{17}$$

$$\psi_{2S} = a_{000}^{1/2}(a_{200}a_{000}-a_{100}^2)^{-1/2} \times (\epsilon_p-\bar{\epsilon})\bar{n}_p^{-1/2}(\bar{n}_p+1)^{1/2} \ , \tag{18}$$

$$\psi_{1P\alpha} = (3/a_{020})^{1/2}p_\alpha\bar{n}_p^{-1/2}(\bar{n}_p+1)^{1/2} \ , \tag{19}$$

where $\alpha = 1,2,3$ and

$$a_{ijk} \equiv \int \epsilon^i p^j v_p^k \bar{n}_p(\bar{n}_p+1)\,d\tau_p \ . \tag{20}$$

$\bar{\epsilon}$ is the average energy of a roton, given by

$$\bar{\epsilon} = \int \epsilon_p\bar{n}_p(\bar{n}_p+1)\,d\tau_p \Big/ \int \bar{n}_p(\bar{n}_p+1)\,d\tau_p = a_{100}/a_{000} \ . \tag{21}$$

Since the system is rotationally invariant, the angular parts of all eigenfunctions must be spherical harmonics. This is reflected in the notation used in Eqs. (17)–(19).

For an eigenfunction i with a nonzero eigenvalue we can write Eq. (12) as

$$A_i = -\lambda_i^{-1}\dot{A}_i - \lambda_i^{-1}\sum_j \langle i|v_{p\alpha}|j\rangle \frac{\partial A_j}{\partial X_\alpha} \ . \tag{22}$$

The eigenvalues are of the order of magnitude of the collision rate. Thus for slowly varying disturbances we can assume that the magnitude of $\lambda_i^{-1}\dot{A}_i$ is much smaller than the magnitude of A_i. Similarly the terms in the sum over j for the RHS of Eq. (22) which are from eigenfunctions with nonzero eigenvalues must also be small. Hence to lowest order in λ^{-1}

$$A_i \approx -\lambda_i^{-1}\sum_j{}'' \langle i|v_{p\alpha}|j\rangle \frac{\partial A_j}{\partial X_\alpha} \ , \tag{23}$$

where the sum is now restricted to include just the eigenfunctions of zero eigenvlaue. $\langle i|v_{p\alpha}|j\rangle$ is nonzero only if i and j are eigenfunctions with spherical harmonics having l's differing by 1. Hence, we get the following results:

$$A_{nS} = -\lambda_{nS}^{-1}\langle nS|v_{p\alpha}|1P\beta\rangle \frac{\partial A_{1P\beta}}{\partial X_\alpha} \ , \tag{24}$$

$$A_{nP} = -\lambda_{nP}^{-1}\left(\langle nP|v_{p\alpha}|1S\rangle \frac{\partial A_{1S}}{\partial x_\alpha} + \langle nP|v_{p\alpha}|2S\rangle \frac{\partial A_{2S}}{\partial X_\alpha}\right) \ , \tag{25}$$

$$A_{nD} = -\lambda_{nD}^{-1}\langle nD|v_{p\alpha}|1P\beta\rangle \frac{\partial A_{1P\beta}}{\partial X_\alpha} \ , \tag{26}$$

where nS, nP, nD refer to the nth eigenfunction with $l = 0, 1$, and 2, respectively. We now consider Eq. (12) when i is a zero-eigenvalue eigenfunction, and use Eqs. (24)–(26) on the right-hand side. This gives a set of differential equations for the time rate of change of A_{1S}, A_{2S}, and $A_{1P\alpha}$ in terms of first and second spatial derivatives of these five quantities. In the Appendix we show that these quantities can be related to macroscopic variables as follows:

$$A_{1S} = a_{000}^{-1/2}\delta n \ , \tag{27}$$

$$A_{2S} = (C_n\beta/T)^{1/2}\delta T \ , \tag{28}$$

$$A_{1P\alpha} = (\beta\rho_n)^{1/2}v_{n\alpha} \ , \tag{29}$$

where δn and δT are the local change in total number density and temperature, respectively, $\vec{v}_n$ is the drift velocity of the roton gas, $\beta = 1/kT$, ρ_n is the normal fluid density due to the rotons, and C_n is the specific heat of the rotons at constant number density. Note, therefore, that C_n is not the usual roton specific heat, which is a derivative of the entropy at constant chemical potential. The differential equations for A_{1S}, A_{2S}, and $A_{1P\alpha}$ then lead to the hydrodynamic equations

$$\frac{\partial n}{\partial t} = -n_R\,\mathrm{div}\,\vec{v}_n + D\nabla^2 n + \nu_2\nabla^2 T \ , \tag{30}$$

$$C_n\frac{\partial T}{\partial t} = -\alpha T B_T\,\mathrm{div}\,\vec{v}_n + \kappa\nabla^2 T + \nu_1\nabla^2 n \ , \tag{31}$$

$$\rho_n\frac{\partial \vec{v}_n}{\partial t} = -\frac{B_T}{n_R}\nabla n - \alpha B_T\nabla T + \zeta\nabla\,\mathrm{div}\,\vec{v}_n + \eta\left(\tfrac{1}{3}\,\mathrm{grad}\,\mathrm{div} + \nabla^2\right)\vec{v}_n \ , \tag{32}$$

where n_R is the equilibrium number of rotons per unit volume, B_T is the isothermal bulk modulus of the roton gas, and α is the thermal expansion coefficient. Relations between these quantities and the a_{ijk} coefficients are given in the Appendix. The second-order terms on the RHS's of these equations represent dissipative terms. Formal expressions for the dissipative coefficients are

$$D=\tfrac{1}{3}\sum_{nP}{}'\frac{[1S|v_p|nP]^2}{\lambda_{nP}}\ , \tag{33}$$

$$\kappa=\tfrac{1}{3}C_n\sum_{nP}{}'\frac{[2S|v_p|nP]^2}{\lambda_{nP}}\ , \tag{34}$$

$$\nu_1=\tfrac{1}{3}\left[\frac{C_n T B_T}{n_R^2}\right]^{1/2}\sum_{nP}{}'[1S|v_p|nP]\frac{[2S|v_p|nP]}{\lambda_{nP}}\ , \tag{35}$$

$$\nu_2=\nu_1 n_R^2/TB_T\ , \tag{36}$$

$$\zeta=\tfrac{1}{3}\rho_n\sum_{nS}{}'\frac{[1P|v_p|nS]^2}{\lambda_{nS}}\ , \tag{37}$$

$$\eta=\tfrac{1}{5}\rho_n\sum_{nD}{}'\frac{[1P|v_p|nD]^2}{\lambda_{nD}}\ . \tag{38}$$

The sums are now over eigenfunctions with the indicated angular momentum, excluding those with eigenvalue zero. By the quantity $[1S|v_p|nP]$, for example, we mean the "radial integral"

$$h^{-3}\int_0^\infty S_1(p)v_pP_n(p)p^2dp\ ,$$

where S_1 and P_n are the radial parts of ψ_{1S} and the nth P state, respectively. In the sums only one member of each set of degenerate eigenfunctions with different magnetic quantum number m is to be counted.

III. COMPARISON WITH CLASSICAL FLUIDS

Consider first Eqs. (30)–(32) for very slowly varying disturbances. If we retain only the lowest-order spatial gradients on the RHS of these equations we can write Eqs. (30) and (32) as

$$\frac{\partial n}{\partial t}=-n_R\,\mathrm{div}\vec{v}_n\ , \tag{39}$$

$$\rho_n\frac{\partial \vec{v}_n}{\partial t}=-\nabla P\ , \tag{40}$$

where P is the pressure. The combination of Eq. (39) with the simplified version of Eq. (31) can be reduced to

$$\frac{\partial S}{\partial t}=-S\,\mathrm{div}\vec{v}_n\ , \tag{41}$$

where S is the entropy per unit volume. Equations (39)–(41) are identical with the equations of motion for small amplitude disturbances in an ordinary fluid when dissipative effects are neglected

Consider now the more general equations (30)–(32). The extra terms in these equations represent various dissipative processes. By comparison with the equations of conventional hydrodynamics[19] including dissipation one can see immediately that κ is the thermal conductivity, and ζ and η are the bulk and shear viscosities. The extra terms are thus those that involve D, ν_1, and ν_2. It follows from Eq. (30) that the particle current $\vec{j}$ must be

$$\vec{j}=n_R\vec{v}_n-D\nabla n-\nu_2\nabla T\ . \tag{42}$$

In ordinary hydrodynamics

$$\vec{j}=n\vec{v} \tag{43}$$

even when dissipative effects occur. In fact, one may consider that the velocity $\vec{v}$ is defined by this equation, and so it is a truism to say that no extra terms occur. On the other hand, $\vec{v}_n$ is defined in a different and specific way. Formally, $\vec{v}_n$ is fixed by Eq. (29). This means we can consider $\vec{v}_n$ to be defined as the following moment of the local distribution function

$$\vec{v}_n=\frac{1}{\rho_n}\int\vec{p}\,n_p\,d\tau_p\ . \tag{44}$$

For an ordinary fluid $\rho_n=nm$ and $\vec{p}=m\vec{v}_p$ where m is the mass of the atoms, and then Eq. (44) gives $\vec{v}_n$ equal to $\vec{j}/n$, in agreement with Eq. (43).

The extra terms $-D\nabla n$ and $-\nu_2\nabla T$ in Eq. (42) represent dissipative contributions to the current. D has the physical meaning of a diffusion coefficient. The remaining extra term $\nu_1\nabla^2 n$ in Eq. (31) implies that there is a dissipative contribution $-\nu_1\nabla n$ to the heat flux. Note that ν_1 and ν_2 are connected by an Onsager-type relation [see Eq. (36)].

One can verify that for an ordinary gas these extra terms dissappear. If $\vec{v}_p=\vec{p}/m$ then it follows that

$$[1S|v_p=m^{-1}[1S|p\ . \tag{45}$$

But the momentum dependence of $[1S|p$ is exactly the same as that of the function $[1P|$ [see Eqs. (17) and (19)]. Hence for $n\neq 1$ the matrix element $[1S|v_p|nP]$ is proportional to

$$[1P|nP]\ ,$$

which vanishes because of orthogonality. Thus, the coefficients D, ν_1, and ν_2 are all zero.

IV. ESTIMATE OF DISSIPATIVE COEFFICIENTS

We can derive approximate expressions for D, κ, ζ, η, ν_1, and ν_2 by the following method which is roughly equivalent to the relaxation time approximation. The eigenvalues λ_i of the collision matrix $\tilde{C}$ determine the rate at which a component of the distribution function which is proportional to a given eigenfunction $\psi_i(\vec{p})$ decays because of collisions amongst the rotons. The simplest approximation is to set all of the nonzero λ_i equal to the rate τ_{RR}^{-1} at which roton-roton collisions occur. In this approxi-

TABLE I. Roton-roton collision time τ_{RR} in seconds as a function of temperature at pressures of 0 and 25 bars according to Landau and Khalatnikov (LK) and Donnelly and Roberts (DR).

	$P=0$		$P=25$ bars	
T (K)	LK	DR	LK	DR
0.45	3.6×10^{-5}	2.4×10^{-5}	1.3×10^{-6}	8.0×10^{-7}
0.5	4.9×10^{-6}	3.3×10^{-6}	2.5×10^{-7}	1.5×10^{-7}
0.55	9.5×10^{-7}	6.6×10^{-7}	6.5×10^{-8}	4.0×10^{-8}
0.6	2.4×10^{-7}	1.7×10^{-7}	2.1×10^{-8}	1.3×10^{-8}
0.65	7.6×10^{-8}	5.5×10^{-8}	7.9×10^{-9}	5.1×10^{-9}
0.7	2.8×10^{-8}	2.0×10^{-8}	3.5×10^{-9}	2.2×10^{-9}
0.75	1.2×10^{-8}	8.7×10^{-9}	1.7×10^{-9}	1.1×10^{-9}
0.8	5.5×10^{-9}	4.1×10^{-9}	8.9×10^{-10}	5.9×10^{-10}

mation the set of all eigenfunctions with nonzero eigenvalues is degenerate, and hence can be chosen arbitrarily subject only to the conditions that they be normalized and orthogonal to each other and to the five eigenfunctions which have $\lambda=0$. These are given by Eqs. (17)–(19). For rotons we can write the dispersion relation as

$$\epsilon_p=\Delta+(p-p_0)^2/2\mu \ . \tag{46}$$

At the temperatures of interest (<1.2 K) Δ and $p_0^2/2\mu$ are much larger than kT. Then all of the rotons are near to the roton minimum p_0. The special eigenfunctions Eq. (17)–(19) all contain a factor

$$\bar{n}_p^{1/2}(\bar{n}_p+1)^{1/2}\cong e^{-\beta\Delta/2}e^{-\beta(p-p_0)^2/4\mu} \ .$$

This suggests that these radial parts of these functions be approximated by harmonic oscillator wave functions in the variable

$$q\equiv(p-p_0)/(2\mu kT)^{1/2} \ . \tag{47}$$

The radial parts of ψ_{1S} and $\psi_{1P\alpha}$ vary with q like the $n=0$ harmonic oscillator wave function and ψ_{2S} is proportional to the $n=2$ state. Then all of the other eigenfunctions which have nonzero λ_i can be written as harmonic oscillator wave functions in the variable q multiplied by an angular part consisting of a spherical harmonic. In this representation it is easy to calculate the matrix elements involved in the sums involved in the transport coefficients. The results are

$$D=kT\tau_{RR}/3\mu \ , \tag{48}$$

$$\kappa=\tfrac{5}{6}n_Rk^2T\tau_{RR}/\mu \ , \tag{49}$$

$$\nu_1=k^2T^2\tau_{RR}/3\mu \ , \tag{50}$$

$$\nu_2=n_Rk\tau_{RR}/3\mu \ , \tag{51}$$

$$\zeta=n_Rp_0^2\tau_{RR}/9\mu \ , \tag{52}$$

$$\eta=n_Rp_0^2\tau_{RR}/15\mu \ , \tag{53}$$

where n_R is the number of rotons per unit volume. If the roton dispersion is approximated by the formula (47), then

$$n_R=(4\pi/h^3)\,p_0^2\,(2\pi\mu kT)^{1/2}e^{-\Delta/kT} \ . \tag{54}$$

Equations (48)–(54) are correct to lowest order in the quantities kT/Δ and $\mu kT/p_0^2$.

The roton-roton scattering time has been estimated by Landau and Khalatnikov.[20] They used Eq. (53), which they derived by a different method, and argued that above 1 K the total shear viscosity of the normal fluid was dominated by the roton contribution. They then used the experimental value of η to estimate τ_{RR} in the temperature range above 1 K. In addition, they carried out a microscopic calculation of τ_{RR} based on a very simplified model for roton-roton interactions. This calculation gave $\tau_{RR}n_R$ independent of temperature. If this is assumed to be true, one obtains the values of τ_{RR} at low temperatures shown in Table I.[21] An alternative way to calculate τ_{RR} is to use the theory of Roberts and Donnelly.[22] This assumes a more specific form of the roton-roton interactions, and gives

$$\tau_{RR}n_R=[2^{3/2}/\pi^{1/3}\Gamma(\tfrac{1}{3})]\mu^{1/2}\rho^{2/3}p_0^{-4/3}(kT)^{1/6} \ . \tag{55}$$

The values of τ_{RR} obtained in this way are fairly close to the estimates based on the Landau-Khalatnikov calculation (see Table I).

V. WAVE PROPAGATION IN PURE ROTON GAS

Consider now the propagation of a longitudinal wave of frequency Ω and wave vector $\vec{K}\parallel z$ through the roton gas. K must be sufficiently small so that the dissipative terms make a relatively small contribution. There exist two solutions that correspond to roton second sound waves which propagate in the + and $-z$ direction. Their frequencies are

$$\Omega_{\pm} = \pm K c_S - \tfrac{1}{2} i K^2 \left[x^2 D + \frac{(\zeta + \frac{4}{3}\eta)}{\rho_n} + \frac{y^2 \kappa}{C_n} + \frac{2 y \nu_1 n}{(\rho_n C_n T)^{1/2}} c_S \right] + O(K^3) \ , \tag{56}$$

where c_T and c_S are the isothermal and isentropic sound velocities, equal to $(B_T/\rho_n)^{1/2}$ and $(B_S/\rho_n)^{1/2}$, respectively, and

$$x = c_T/c_S \ , \tag{57}$$

$$y = (1 - c_T^2/c_S^2)^{1/2} \ . \tag{58}$$

If the roton spectrum is approximated by Eq. (46) and terms of lowest order in kT/Δ and $\mu kT/p_0^2$ are kept, one finds

$$B_T = n_R kT \ , \tag{59}$$

$$B_S = 3 n_R kT \ , \tag{60}$$

$$\alpha = 1/T \ , \tag{61}$$

$$\rho_n = n_R p_0^2 / 3kT \ , \tag{62}$$

$$C_n = n_R k/2 \ , \tag{63}$$

$$c_T = 3^{1/2} kT/p_0 \ , \tag{64}$$

$$c_S = 3kT/p_0 \ . \tag{65}$$

Hence, the velocity of the sound waves is $3kT/p_0$,[23] and the attenuation of a wave of frequency Ω is from Eq. (56):

$$\alpha = \frac{17}{405} \frac{\Omega^2 \tau_{RR} p_0^3}{\mu k^2 T^2} + O(\Omega^4) \ . \tag{66}$$

These waves are conventional isentropic sound waves. They have fluctuations in both number density and temperature associated with them. It is straightforward to show that

$$\frac{\delta n}{n_R} = \frac{\alpha B_T}{C_n} \frac{\delta T}{T} = 2 \frac{\delta T}{T} \ . \tag{67}$$

In addition to these solutions there is a purely damped solution with frequency

$$\Omega_D = -iK^2 \left[y^2 D + \frac{x^2 \kappa}{C_N} - \frac{2 y \nu_1 n_R}{(\rho_n C_n T c_S^2)^{1/2}} \right] + O(K^4)$$

$$= -\frac{i}{3} K^2 \frac{kT \tau_{RR}}{\mu} + O(K^4) \ . \tag{68}$$

This damped solution has temperature and density fluctuations associated with it. Their magnitudes and phases are such that the pressure is constant.

We have investigated the limits of validity of the hydrodynamic equations with first-order dissipation. Higher-order dissipative terms make a negligible con-

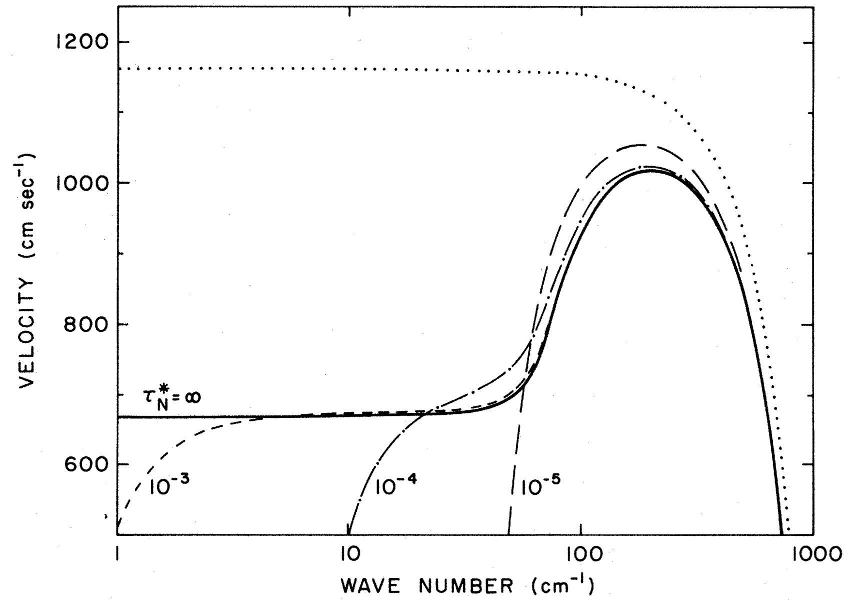

FIG. 1. Velocity of roton second sound at 0.6 K as a function of wave number K. Dotted curve is for no coupling between rotons and phonons ($\tau_{RP} = \infty$). Other curves include roton-phonon coupling and are labeled by the value of the number-nonconserving roton scattering time τ_N^* measured in seconds.

tribution provided that the conditions

$$\Omega \tau_{RR} \ll 1 \quad , \tag{69}$$

$$K \Lambda_{RR} \ll 1 \quad , \tag{70}$$

are satisfied, where Λ_{RR} is the roton mean-free path defined as

$$\Lambda_{RR} \equiv \tau_{RR}(kT/\mu)^{1/2} \quad . \tag{71}$$

Since for sound waves K is of the order of magnitude of $(\Omega p_0/kT)$, the condition (70) becomes

$$\Omega \tau_{RR}(p_0^2/\mu kT)^{1/2} \ll 1 \quad . \tag{72}$$

Since $(p_0^2/\mu kT)^{1/2}$ is about 25 to 0.6 K this condition is more severe than condition (69). Note, however, that the series for Ω, α, etc. [Eqs. (56), (66), and (68)] are expansions in the parameter

$$\gamma \equiv \Omega \tau_{RR} p_0^2/\mu kT \tag{73}$$

or the square of this. Thus, these fail unless $\gamma \ll 1$ and this is a more restrictive condition than Eq. (72). In the regime of frequencies where Eq. (72) still holds but γ is $\gtrsim 1$, one can show from the hydrodynamic equations that Ω and K are related by

$$\tilde{\Omega}^3 - 9\tilde{\Omega}\tilde{K}^2 + (13i/5)\tilde{\Omega}^2\tilde{K}^2 - 3i\tilde{K}^4 - (23/15)\tilde{\Omega}\tilde{K}^4 - i\tilde{K}^6/5 = 0 \quad , \tag{74}$$

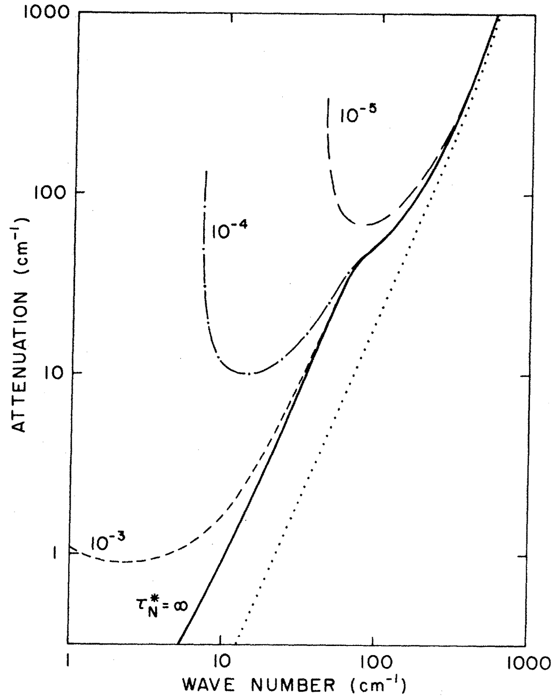

FIG. 2. Attenuation of roton second sound at 0.6 K as a function of wave number K. Dotted curve is for no coupling between rotons and phonons ($\tau_{RP} = \infty$). Other curves include roton-phonon coupling and are labeled by the value of the number nonconserving roton scattering time τ_N^* measured in seconds.

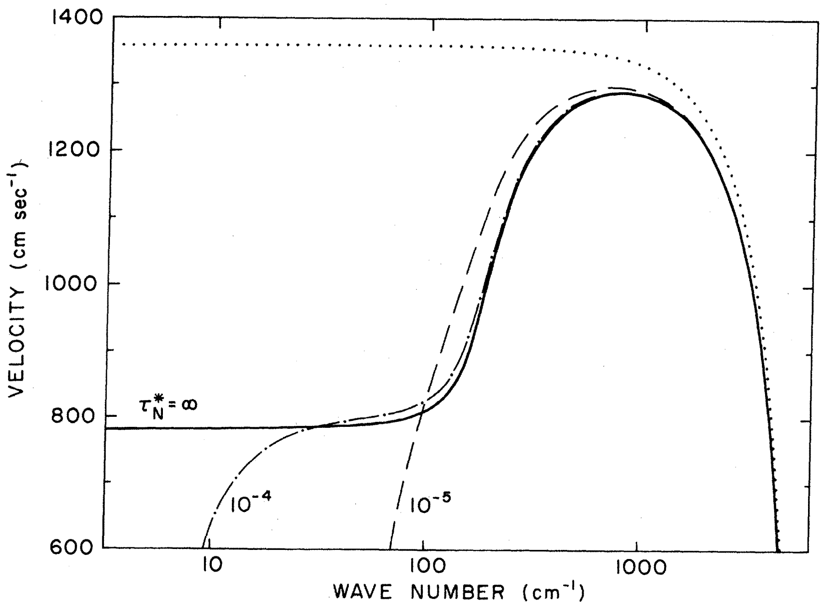

FIG. 3. Velocity of roton second sound at 0.7 K as a function of wave number K. Dotted curve is for no coupling between rotons and phonons ($\tau_{RP} = \infty$). Other curves include roton-phonon coupling and are labeled by the value of the number-nonconserving roton scattering time τ_N^* measured in seconds. The velocity for $\tau_N^* = 10^{-3}$ is essentially the same as the velocity for $\tau_N^* = \infty$.

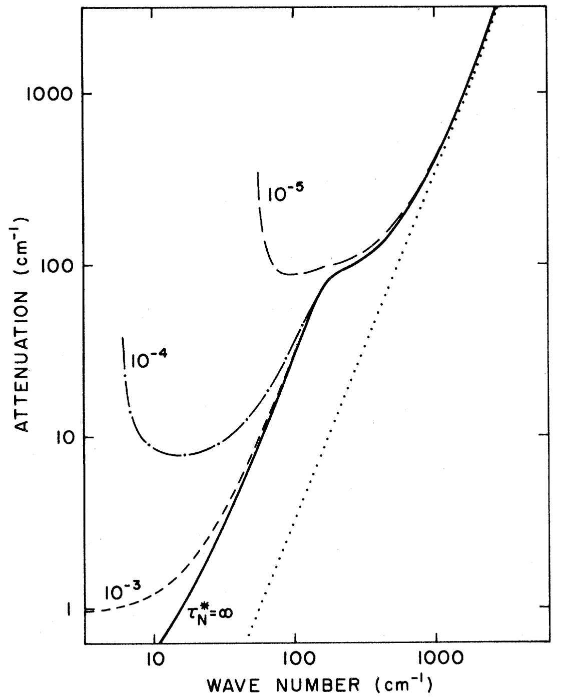

FIG. 4. Attenuation of roton second sound at 0.7 K as a function of wave number K. Dotted curve is for no coupling between the rotons and phonons ($\tau_{RP}=\infty$). Other curves include roton-phonon coupling and are labeled by the value of the number-nonconserving roton scattering time τ_N^* measured in seconds.

where

$$\tilde{\Omega}=\Omega\tau_{RR}p_0^2/\mu kT \ , \tag{75}$$

$$\tilde{K}=K\tau_{RR}p_0/\mu \ . \tag{76}$$

We have solved this cubic equation in Ω numerically. The results for the velocity and the attenuation of the roton second-sound waves are shown by the dotted lines in Figs. 1–4. These calculations used the Landau-Khalatnikov values of the roton-roton collision time. As a rough estimate of the value of K for which the theory is beginning to break down we may choose the point where $K\Lambda_{RR}=0.2$. This corresponds to a K value of 1000 cm^{-1} at 0.6 K and 5500 cm^{-1} at 0.7 K.

VI. INTERACTIONS WITH PHONONS

A. Energy and momentum transfer to phonons

Consider now the effect on roton second sound of energy and momentum transfer to the phonons. For the moment we consider only those interactions between phonons and rotons which conserve the total number of rotons. The theory of roton-phonon interactions was first worked out by Landau and Khalatnikov.[4] The dominant process is believed to be

$$R+P\rightarrow R'+P' \ . \tag{77}$$

Their result for the total cross section for scattering of phonons with momentum p by rotons is

$$\sigma_{PR}=(p_0^2p^4/4\pi\hbar^4\rho^2c^2)(\tfrac{2}{9}+B^2/25+2AB/9+A^2) \ , \tag{78}$$

where

$$A=(\rho^2/p_0c)[(\partial^2\Delta/\partial\rho^2)+(\partial p_0/\partial\rho)^2/\mu] \ , \tag{79}$$

$$B=p_0/\mu c \ , \tag{80}$$

and c is the phonon velocity. We can estimate A and B from the tables of roton parameters of Brooks and Donnelly.[21] Since the group velocity of the rotons is much less than the sound velocity the time τ_{RP} that a roton goes in between collisions with phonons is given by

$$\frac{1}{\tau_{RP}}=c\langle\sigma_{PR}\rangle n_p \ , \tag{81}$$

where c is the sound velocity and n_p is the number of phonons per unit volume, and $\langle\sigma_{PR}\rangle$ is σ_{PR} averaged over phonon momenta. This gives the values of τ_{RP} shown in Table II. One can also calculate the time τ_{PR} a phonon goes in between collisions with rotons (see Table II). τ_{PR} is equal to $\tau_{RP}n_p/n_R$.

The effect of phonon-roton interactions on roton second sound has been considered by Khalatnikov and Chernikova.[5,6] However, they concentrated attention on what happens in helium under zero pressure. Experimentally, roton second sound has so far only been observed at high pressures, and we believe that the physical situation is quite different under these conditions. It is clear from Table II that the possibility of seeing roton second sound is reduced at $P=0$ because of the greatly increased rate of collisions between rotons and phonons.

Consider now the physical situation at a pressure of 25 bars and for temperatures around 0.6 K. The phonon mean free path for collisions with rotons is long (i.e., of the order of centimeters). In addition, at this temperature and pressure the phonon mean free path due to collisions with other phonons is also long. Thus in a typical experiment which uses a helium cell of linear dimensions of the order of 1 cm the phonons are making random transversals of the liquid, going from wall to wall occasionally colliding with a roton. Thus, the phonon gas is effectively clamped by the walls of the container. We therefore have to consider that the drift velocity of the phonon gas is always zero.

TABLE II. Roton-phonon and phonon-roton collision times in seconds as a function of temperature at pressures of 0 and 25 bars.

T (K)	$P=0$		$P=25$ bars	
	τ_{RP}	τ_{PR}	τ_{RP}	τ_{PR}
0.45	1.5×10^{-6}	1.9×10^{-2}	7.8×10^{-5}	9.9×10^{-3}
0.5	7.0×10^{-7}	1.7×10^{-3}	3.7×10^{-5}	1.2×10^{-3}
0.55	3.6×10^{-7}	2.3×10^{-4}	1.9×10^{-5}	2.2×10^{-4}
0.6	2.0×10^{-7}	4.2×10^{-5}	1.0×10^{-5}	4.9×10^{-5}
0.65	1.1×10^{-7}	9.5×10^{-6}	6.0×10^{-6}	1.4×10^{-5}
0.7	6.7×10^{-8}	2.6×10^{-6}	3.5×10^{-6}	4.4×10^{-6}
0.75	4.1×10^{-8}	8.4×10^{-7}	2.2×10^{-6}	1.6×10^{-6}
0.8	2.6×10^{-8}	3.0×10^{-7}	1.4×10^{-6}	6.7×10^{-7}

The roton specific heat at constant number is

$$C_R=C_n=\tfrac{1}{2}n_R k \ . \tag{82}$$

This is much smaller than the phonon specific heat[24] (see Table III). Thus we can regard the phonon system as a heat bath of constant temperature.

Consider now the rate at which energy and momentum is transferred between the rotons and the phonons. The condition of conservation of momemtum is

$$\vec{p}_1+\vec{q}_1=\vec{p}_2+\vec{q}_2 \ , \tag{83}$$

where $\vec{p}$ and $\vec{q}$ refer to rotons and phonons, respectively. The conservation of energy conditions is

$$(p_1-p_0)^2/2\mu+cq_1=(p_2-p_0)^2/2\mu+cq_2 \ . \tag{84}$$

Let $\vec{p}_1=\hat{p}_1(p_0+\pi_1)$ where $\hat{p}_1$ is a unit vector in the direction of $\vec{p}_1$. The order of magnitude of π_1 is $(\mu kT)^{1/2}$, and cq_1 is typically $3kT$. Both of these quantities are much smaller than p_0 and so

$$\begin{aligned} p_2=|\vec{p}_1+\vec{q}_1-\vec{q}_2|&=|\hat{p}_1(p_0+\pi_1)+\vec{q}_1-\vec{q}_2| \\ &\approx p_0+\pi_1+\hat{p}_1\cdot(\vec{q}_1-\vec{q}_2) \ . \end{aligned} \tag{85}$$

Hence to lowest order the energy transferred from the phonon to the roton is

$$\begin{aligned} \delta E&=c(q_1-q_2) \\ &=[\hat{p}_1\cdot(\vec{q}_1-\vec{q}_2)]^2/2\mu+\pi_1\hat{p}_1\cdot(\vec{q}_1-\vec{q}_2)/\mu \ . \end{aligned} \tag{86}$$

Detailed investigation shows that all relative orientations of $\hat{p}_1$ and $\vec{q}_1-\vec{q}_2$ are possible. Averaging over all possible directions of $\vec{q}_1$ and $\vec{q}_2$ gives

$$\langle[\hat{p}_1\cdot(\vec{q}_1-\vec{q}_2)]^2/2\mu\rangle=(q_1^2+q_2^2)/6\mu \ .$$

If the energies of the phonons are both on the order of $3kT$ the typical contribution to δE from this term is $\sim3k^2T^2/\mu c^2$. The order of magnitude of the second term on the right-hand side of Eq. (86) is

TABLE III. Number density, specific heat, and normal fluid density for rotons and phonons at 25 bars.

T (K)	n_R (cm^{-3})	n_P (cm^{-3})	C_R (erg cm^{-3} K^{-1})	C_P (erg cm^{-3} K^{-1})	ρ_{nR} (g cm^{-3})	ρ_{nP} (g cm^{-3})
0.45	4.1×10^{15}	5.1×10^{17}	2.8×10^{-1}	1.9×10^{2}	9.94×10^{-7}	9.5×10^{-8}
0.5	2.1×10^{16}	7.1×10^{17}	1.5×10^{0}	2.6×10^{2}	4.69×10^{-6}	1.5×10^{-7}
0.55	8.3×10^{16}	9.5×10^{17}	5.7×10^{0}	3.5×10^{2}	1.66×10^{-5}	2.1×10^{-7}
0.6	2.6×10^{17}	1.2×10^{18}	1.8×10^{1}	4.4×10^{2}	4.76×10^{-5}	3.1×10^{-7}
0.65	6.8×10^{17}	1.6×10^{18}	4.7×10^{1}	6.0×10^{2}	1.15×10^{-4}	4.2×10^{-7}
0.7	1.6×10^{18}	2.0×10^{18}	1.1×10^{2}	7.5×10^{2}	2.45×10^{-4}	5.7×10^{-7}
0.75	3.2×10^{18}	2.5×10^{18}	2.2×10^{2}	9.3×10^{2}	4.71×10^{-4}	7.6×10^{-7}
0.8	6.1×10^{18}	3.0×10^{18}	4.2×10^{2}	1.1×10^{3}	8.34×10^{-4}	9.9×10^{-7}

$3kT(kT/\mu c^2)^{1/2}$. And the typical kinetic energy of a roton (i.e., the energy above Δ) is $\frac{1}{2}kT$. Thus, viewed *from the roton* the fractional change in energy in the collision is of order

$$\frac{\delta E}{E} \sim 6\frac{kT}{\mu c^2} + 6\left(\frac{kT}{\mu c^2}\right)^{1/2} \quad . \tag{87}$$

At 0.6 K and 25(bars) $kT/\mu c^2$ is 0.07. Thus the change in energy of a roton in one collision is of about the same magnitude as the average kinetic energy of a roton, and so in one collision the roton energy, or temperature, is effectively randomized. Hence we should modify Eq. (31) to become

$$C_n\left(\frac{\partial T}{\partial t} + \frac{\delta T}{\tau_T^*}\right) = -\alpha T B_T \operatorname{div} \vec{v}_n + \kappa \nabla^2 T + \nu_1 \nabla^2 n \quad , \tag{88}$$

where δT is the difference between the local temperature of the roton gas and the ambient temperature, and τ_T^* is the effective "temperature relaxation time" equal to τ_{RP}.

Consider now the momentum transfer to the phonons. In a single collision between a phonon and a roton the direction of the momentum of the phonon is randomized. Hence if we consider a phonon gas of drift velocity $\vec{v}_{nP}$ undergoing collisions with a roton gas of drift velocity $\vec{v}_{nR}$ we would have

$$\frac{\partial \vec{v}_{nP}}{\partial t} = -\frac{(\vec{v}_{nP} - \vec{v}_{nR})}{\tau_{PR}} \quad . \tag{89}$$

This is in the absence of interactions between the phonons and external walls. Then from conservation of momentum

$$\frac{\partial \vec{v}_{nR}}{\partial t} = -\frac{\rho_{nP}}{\rho_{nP}\tau_{PR}}(\vec{v}_{nR} - \vec{v}_{nP}) \quad , \tag{90}$$

where ρ_{nP} and ρ_{nR} are the phonon and roton contributions to the normal fluid density. This can be made more physical if we let

$$\rho_{nP} \equiv n_P m_P^* \quad , \tag{91}$$

$$\rho_{nR} \equiv n_R m_R^* \quad , \tag{92}$$

where m_P^* and m_R^* are effective masses for phonons and rotons. One finds $m_P^* = 3.6kT/c^2$, and $m_R^* = p_0^2/3kT$. Then Eq. (90) becomes for the acceleration of the roton fluid due to interactions with the phonons

$$\frac{\partial \vec{v}_{nR}}{\partial t} = -\frac{m_P^*}{m_R^* \tau_{RP}}(\vec{v}_{nR} - v_{nP}) \quad . \tag{93}$$

As noted above, at 0.6 K and 25 bars the phonon mean free path is long and the phonon gas is effectively clamped in place by the container of the helium sample. Thus, it is a reasonable approximation to take $\vec{v}_{nP} = 0$ in Eq. (93). This then leads to an extra term in the equation for the acceleration of the roton fluid [see Eq. (32)]:

$$\rho_n\left(\frac{\partial \vec{v}_n}{\partial t} + \frac{\vec{v}_n}{\tau_V^*}\right) = -\frac{B_T}{n_R}\nabla n - \alpha B_T \nabla T + \zeta \nabla \operatorname{div} \vec{v}_n + \eta\left(\tfrac{1}{3}\operatorname{grad}\operatorname{div} + \nabla^2\right)\vec{v}_n \quad , \tag{94}$$

where τ_V^* is the velocity relaxation time equal to $m_R^* \tau_{RP}/m_P^*$. At 0.6 K m_P^* and m_R^* have the values $0.33 m_{H_e}$ and $27.7 m_{H_e}$, respectively, where m_{H_e} is the mass of a helium atom. Thus, τ_V^* is longer than τ_T^* by ~ 840.

The effective times for energy and momentum transfer we have calculated here do not differ significantly from the results of Khalatnikov and Chernikova.[5] Castaing,[10] however, obtains a much slower rate for the equilibration of the phonon and roton temperatures, essentially because he uses for the roton specific heat the specific heat at constant chemical potential, rather than at constant number of rotons as we have used. For reasons discussed in Sec. VII, we believe that the majority of roton interactions conserve roton number, and that the specific heat at constant number is therefore the correct one to use. Energy and momentum transfer to the phonons is not considered in the theory of Weiss.[15]

B. Conversion of phonons into rotons

Various mechanisms have been proposed by which rotons can be converted into phonons. These processes relax the roton number density back to the equilibrium value corresponding to the temperature of the phonon gas with a certain time constant τ_N^*. A simultaneous relaxation of the local temperature and drift velocity of the roton gas must also occur, but the times τ_{CT} and τ_{CV} that it takes for these quantities to relax by means of "conversion" processes are not necessarily equal to τ_N^*. The equation for the number density becomes

$$\frac{\partial n}{\partial t} + \frac{n - n_R}{\tau_N^*} = -n_R \operatorname{div} \vec{v}_n + D\nabla^2 n + \nu_2 \nabla^2 T \quad . \tag{95}$$

The relaxation times in Eqs. (88) and (94) are modified to be

$$\frac{1}{\tau_T^*} = \frac{1}{\tau_{CT}} + \frac{1}{\tau_{RP}} \quad , \tag{96}$$

$$\frac{1}{\tau_V^*} = \frac{1}{\tau_{CV}} + \frac{m_P^*}{m_R^* \tau_{RP}} \quad . \tag{97}$$

It is also possible to have a more complicated relaxation structure. For example, the relaxation rate of the number of density can include a term proportional to δT, and this term can have an effect comparable to the "diagonal" relaxation term we have included. Inclusion of terms like this does not seem justified since so little is known about the relaxation mechanism.

Definite values for τ_N^* τ_{CT}, τ_{CV} are not available. We discuss the various theories in the next section. We shall see that the experimental observations that have been made of roton second sound provide an interesting limitation on the rate at which number-nonconserving processes can occur.

C. Wave propagation with coupling to phonons

It is straightforward to find the velocity when the coupling to phonons is considered, but the cubic equation satisfied by the frequency Ω is very complicated. Hence, we have calculated the attenuation and velocity of roton second sound numerically for representative values of the parameters. Figures 1 and 2 show results at 0.6 K and $P = 25$ bars. The Landau-Khalatnikov values for τ_{RR} and τ_{RP} have been used (Tables I and II). The different curves are for different assumed values of the number nonconserving collision time τ_N^*. It has been assumed that τ_{CT} is equal to τ_N^* and that τ_{CV} is infinity. Figures 3 and 4 show equivalent results at 0.7 K. These results are not changed in any dramatic way if τ_{CV} is set equal to τ_N^*, or if τ_{CT} is made infinite.

VII. DISCUSSION

Consider first the numerical results shown in Figs. 1–4 for a pure roton gas, i.e., for when the interactions between the rotons and the phonons are neglected and roton number is conserved. The results are shown up to a K value K_c such that $K_c\Lambda_{RR}$ is 0.2, and this is roughly the limit of the validity of the hydrodynamic theory with lowest order dissipation. A remarkable feature of the results is that within this hydrodynamic regime there is a large variation of the sound velocity. One can understand this as follows. The condition $K\Lambda_{RR} \ll 1$ has the effect that second- and higher-order dissipative terms are small compared to the first-order dissipative terms which we have included. However, it turns out that for rotons the first-order dissipative terms become comparable to the *nondissipative terms* (such as the terms involving the bulk modulus B_T) considerably before $K\Lambda_{RR}$ approaches 1. This happens because the pressure and the bulk moduli of a roton gas are peculiarly small in the following sense. For a gas of particles of number density n_R, momentum p_0, and particle velocity $(kT/\mu)^{1/2}$ one would naively expect the pressure to be of the order of

$$P \sim n_R p_0 (kT/\mu)^{1/2} \ . \qquad (98)$$

In fact, it is easy to show that when $kT \ll \Delta$ the pressure is exactly[25]

$$P = n_R kT \ . \qquad (99)$$

This is smaller than Eq. (98) by a factor $(\mu kT)^{1/2}p_0$. The difference arises because of the peculiar dispersion relation of rotons. In Fig. 5 we plot the energy of a roton as a function of p_z. Rotons that travel in the positive z direction come from the parts of the dispersion curve marked A and B. Thus, rotons which are reflected by a wall perpendicular to the z axis may have z momenta close to $+p_0$ (A rotons) or close to $-p_0$ (B rotons). The A rotons give momentum to the wall and hence push on it. However, the B rotons have the remarkable property that they pull on the wall when they collide with it, and give a negative contribution to the pressure. To lowest order the total pressure produced by A and B rotons vanishes. In higher order there is a finite positive pressure (even for an exactly parabolic roton spectrum) because the density of states in momentum space assures that there are slightly more A rotons than B rotons. This explains why the pressure is given by Eq. (99) and not by Eq. (98), and in turn why the bulk modulus of the roton gas is anomalously small.

Because of the small nondissipative terms there is a range of K values where hydrodynamics with lowest-order dissipation is a valid approximation, but the dissipative terms dominate over the nondissipative terms in controlling the dynamical behavior of the system. Such a regime does not exist for an ordinary gas of particles. It is in this regime that the large variation of sound velocity occurs.

Consider now the effects of coupling to the phonons. This is characterized by three times τ_N^*, τ_T^*, and τ_V^* which describe the relaxation of number, temperature, and drift velocity. Clearly when τ_V^* is such that $\Omega\tau_V^* < 1$ no propagating wave is possible. According to our estimates τ_V^* is about $840\tau_{RP}$ at 0.6

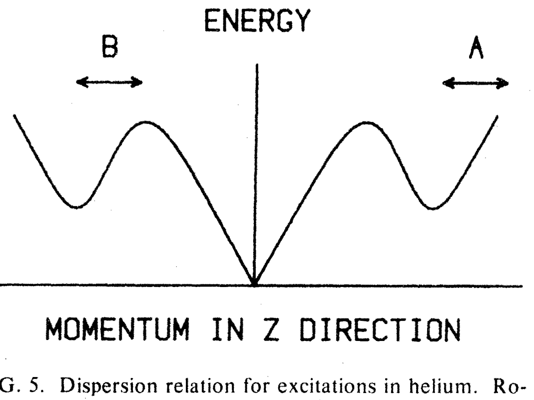

FIG. 5. Dispersion relation for excitations in helium. Rotons from regions A and B move in the positive z direction.

K when number nonconserving processes are neglected. This has the effect that $\Omega\tau_V^*$ becomes comparable to 1 only at frequencies which are lower than those of physical interest. Thus for practical purposes the relaxation of the velocity can be neglected. In fact, the results shown in Figs. 1–4 are unaltered if τ_V^* is set equal to infinity.

The effect of temperature relaxation is to lower the velocity of sound for small K. This is because the period of the wave is sufficiently long for energy exchange with the phonons to occur. In this case when $\Omega\tau_T^* << 1$ the sound velocity becomes the isothermal velocity c_T. This can be seen clearly in Figs. 1 and 3. The existence of isothermal second sound is impossible to understand on the conventional view that second sound is a "temperature wave." However, when second sound is regarded simply as a sound wave in the gas of excitations it is clear that isothermal second sound can exist. Consider now a typical experiment with a propagation distance ~ 1 cm. If a short pulse is used to generate the roton second sound, waves with a wide distribution of wave numbers will be produced. However, because the attenuation increases rapidly with K (see Figs. 2 and 4) the waves which actually reach the detector are predominantly of small K, and hence lie in the isothermal regime. At 0.6 K this isothermal regime extends up to $K \sim 50\ \text{cm}^{-1}$, and the attenuation at this point is $\sim 40\ \text{cm}^{-1}$. Thus, for a path length of ~ 1 cm waves *other* than those in the isothermal range of K are very highly attenuated (by e^{-40}) and so the propagation over distances of this order is primarily isothermal.

Consider now the effect of number relaxation. The principal effect of this is to add a contribution to the attenuation which increases rapidly at low frequencies (Figs. 2 and 4). Below a critical wave number K_c the modes become nonpropagating, i.e., of imaginary frequency. One can understand this as follows. In this regime $\Omega\tau_T^*$ is $<<1$ and so no temperature oscillations can occur. If $\Omega\tau_N^*$ is also $<< 1$, roton number oscillations are also impossible, and hence there is nothing left from which to construct the wave.

For a given value of τ_N^* the attenuation as a function of wave number has a minimum value, which increase as τ_N^* decreases (Figs. 2 and 4). Hence the fact that roton second sound is actually observed experimentally puts a limit on the maximum rate at which number nonconserving processes can occur. This limit is set most stringently by experiments which use a long path length. For example, Cline and Maris[13] have shown that roton second sound can be observed with a propagation distance of 0.493 cm up to at least 0.7 K. At this temperature our calculations give a minimum attenuation of $8\ \text{cm}^{-1}$ if τ_N^* is 10^{-4} sec, and $90\ \text{cm}^{-1}$ if τ_N^* is 10^{-5} sec. Thus, all Fourier components in the pulse are attenuated by a factor of at least $\exp(-4)$ if τ_N^* is 10^{-4} sec and a factor of $\exp(-45)$ if τ_*^N is 10^{-5} sec. Thus it is clearly impossible for a roton pulse to be detected if τ_N^* is as short as 10^{-5} sec, and so a reasonable estimate for the shortest possible value of τ_N^* might be 3×10^{-5} sec. This is a remarkable result since at 0.7 K the roton-roton collision time is 3.5×10^{-9} sec (see Table I). Thus τ_N^* is bigger than τ_{RR} by a factor of at least 10 000, i.e, in nearly all collisions rotons number is conserved.

One can carry out a similar analysis at lower temperatures using the data of Castaing and Libchaber.[11] They used a propagation distance of 1.775 cm at temperatures up to 0.576 K. The same argument as used above implies that at $0.576K\tau_N^*$ must be greater than 10^{-4} sec, and is therefore bigger than τ_{RR} by at least a facor of 3000.

According to Khalatnikov[25] roton-phonon conversion occurs primarily via the process

$$R + R' \rightleftharpoons R'' + P \quad . \tag{100}$$

He originally argued[26] that the anomalously large attenuation observed for first sound in the MHz frequency range at around 1 K was related to this relaxation process. This would make τ_N^* about 10^{-8} sec at 1 K and 25 bars. At lower temperatures it should vary roughly as $\exp(\Delta/kT)$, and so at 0.6 K τ_N^* would be $\sim 10^{-6}$ sec. This is clearly inconsistent with the observations of roton second sound. In the later papers of Khalatnikov and Chernikova[5,6] the ultrasonic attenuation is attributed to the *energy* transfer process between rotons and phonons (as distinct from number transfer), and is thus determined by τ_{RP} of the previous section. If this is the case, the rate of number nonconserving collisions cannot be determined from the ultrasonic data. However, if the dominant process is (100) one would still expect an approximately exponential temperature dependence to the rate. An additional point is that τ_{CV} should be considerably larger than τ_N^* and τ_{CT} because the momentum of the phonons in (100) is approximately Δ/c. This is much smaller than p_0, and so the total momentum of the rotons is only slightly changed as a result of the transition.

Khalatnikov[26] pointed out a long time ago that τ_N^* should be significantly bigger than τ_{RR}. This occurs because two rotons with momenta near to p_0 cannot collide and produce a roton and a phonon, unless the angle between the momenta of the initial rotons has a special value. This follows from kinematic considerations, i.e., conservation of energy and momentum. It is straightforward to show that the process Khalatnikov envisioned should suppress number nonconserving processes by a factor of order p_0c/Δ, which is roughly eight. This is much less than the factor of 10^4 we need, and so there must be some other suppression mechanism.

It is exciting to note that such a mechanism occurs naturally if we accept the Feynman-Cohen[27] picture of a roton as a small vortex ring. According to this view, two rotons at large distances interact primarily via their back-flow velocity fields. This gives a very large total scattering cross section, essentially because the potential is long range (r^{-3}) and comparable to the roton kinetic energy ($\sim \frac{1}{2}kT$) even at a separation of 10 Å. Since the rotons do not have to approach each other very closely for scattering to occur, one would expect that in this type of process the rotons in no sense "lose their identities," and thus their total number should be conserved. For a number-changing collision the rotons have to get much closer together, and so the cross section σ_{NC} for this sort of process is much smaller.

Roberts and Donnelly[22] have used classical mechanics to calculate the trajectories of two rotons which are interacting through their velocity fields. When the simple dipole form was used for the velocity field at all separation distances, they found that some of the classical trajectories of the rotons "fall in," i.e., the two rotons approach each other without limit. Roberts and Donnelly speculated that after the "fall in" process phonons might be produced. They calculated the scattering cross section σ_F for these processes to be

$$\sigma_F = \pi^{1/3}(\tfrac{10}{9})(\tfrac{5}{4})^{2/3}(\mu/\rho)^{2/3} \quad . \tag{101}$$

This gives $\sigma_F = 5.5$ Å^2 at $P = 25$ bars, whereas the total cross section σ_T at 0.6 K was calculated to be 220 Å^2. Thus, this leads to a suppression factor of

$$\sigma_T/\sigma_F \sim 40 \quad . \tag{102}$$

This is really a lower limit to the suppression since it assumes that phonons are always produced every time fall occurs. Roberts and Donnelly found this result for collisions between rotons of zero total momentum. In this special case there are no kinematic restrictions on the production of phonons. In the case of an arbitrary distribution of total momenta of the incoming rotons one would expect that in addition to the Roberts-Donnelly factor the Khalatnikov kinematic suppression factor should still apply, thereby giving a total suppression of ~ 320. This is still much smaller than the factor of 10^4 indicated by the roton second sound experiments, but may well be the explanation of a large part of the effect.

The experiments on roton second sound performed so far have all used pulses. The velocity of the pulse (as measured by the peak position) has been found to be between the isothermal and adiabatic velocities c_T and c_S. Because of the large dispersion and damping of the waves it is very hard to predict from our calculations the velocity that should be observed in pulse experiments. We have performed some numerical calculations on pulse propagation. These are based directly on the hydrodynamic equations for the rotons, together with some assumed initial condition, such as a δ function of number density or temperature increase as the source. These calculations give pulse velocities and shapes which depend very considerably on the initial conditions (which are, of course unknown) and on whether the detector in the experiments is assumed to be sensitive primarily to roton number density or roton temperature. It is certainly possible to explain the details of what is seen if suitable assumptions are made.

In summary, therefore, we have derived hydrodynamic equations for a roton gas and have studied the waves which can propagate in the gas. We conclude that the mode observed in the roton second sound experiments is best described as isothermal number-conserved second sound. The experiments indicate that roton collisions conserve roton number to a high degree of accuracy.

We wish to thank Kurt Weiss for much helpful discussion and correspondence. Jerry Tessendorf helped with the computer calculations of pulse propagation. This work was supported in part by the National Science Foundation through Grant Nos. DMR 77-12249 and DMR 80-11284, and through the Materials Research Laboratory of Brown University.

APPENDIX

Consider a gas of rotons with temperature $T + \delta T$, chemical potential μ, and "drift velocity" $\vec{v}_n$. The distribution function is

$$\begin{aligned} n_p &= [\exp(\epsilon_p - \mu - \vec{p}\cdot\vec{v}_n)/k(T+\delta T) - 1]^{-1} \\ &\approx \bar{n}_p + \beta\bar{n}_p(\bar{n}_p + 1)(\mu + \vec{p}\cdot\vec{v}_n + \epsilon_p\delta T/T) \quad . \end{aligned} \tag{A1}$$

This assumes μ, δT, and $\vec{v}_n$ are small. The α component of the momentum density $\vec{\pi}$ is

$$\pi_\alpha = \int n_p p_\alpha d\tau_p \equiv \rho_n v_{n\alpha} \tag{A2}$$

thereby defining ρ_n. Using Eq. (A1) gives

$$\rho_n = \frac{\beta}{3}\int p^2 \bar{n}_p(\bar{n}_p + 1)d\tau_p = \beta a_{020}/3 \quad . \tag{A3}$$

The pressure P is given by

$$P = \frac{1}{3}\int p v_p n_p d\tau_p \quad . \tag{A4}$$

From this it is straightforward to show that

$$B_T = -V\left(\frac{\partial T}{\partial V}\right)_T = \beta a_{011}^2/9a_{000} \quad . \tag{A5}$$

The number density n_R for μ, δT, and $\vec{v}_n$ equal to zero is

$$n_R = \int \bar{n}_p d\tau_p = \beta\frac{a_{011}}{3} \quad . \tag{A6}$$

If μ and δT are nonzero one finds

$$\left(\frac{\partial n_R}{\partial T}\right)_\mu = \beta \frac{a_{100}}{T} , \tag{A7}$$

$$\left(\frac{\partial n_R}{\partial \mu}\right)_T = \beta a_{00} . \tag{A8}$$

The total energy density

$$U = \int \epsilon_p n_p d\tau_p = U_0 + \beta\mu a_{100} + \beta \delta T a_{200}/T ,$$

where U_0 is the energy density for μ, δT, and $\vec{v}_n$ zero. Thus the specific heat at constant μ is

$$C_\mu = \beta a_{200}/T . \tag{A9}$$

By using Eqs. (A7) and (A8) one can show that the specific heat at constant roton number is

$$C_n = \beta(a_{200}a_{000} - a_{100}^2)/T a_{000} . \tag{A10}$$

The thermal expansion coefficient is

$$\alpha = 3(a_{111}a_{000} - a_{100}a_{011})/T a_{011}^2 . \tag{A11}$$

Thus, the adiabatic bulk modulus is

$$B_S = B_T + B_T^2 \alpha^2 T / C_n$$

$$= (\beta/9) \frac{a_{011}^2 a_{200} - 2a_{111}a_{100} + a_{111}^2 a_{000}}{a_{200}a_{000} - a_{100}^2} . \tag{A12}$$

From Eqs. (A1) and (11) it must be true that

$$\beta \bar{n}_p(\bar{n}_p + 1)(\mu + \vec{p} \cdot \vec{v}_n + \epsilon_p \delta T/T)$$

$$= \bar{n}_p^{1/2}(\bar{n}_p + 1)^{1/2}$$

$$= \left(A_{1S}\psi_{1S} + A_{2S}\psi_{1S} + \sum_{\alpha=1}^{3} A_{1P\alpha}\psi_{1P\alpha} \right) .$$

Therefore

$$A_{1S} = \beta\mu a_{000}^{1/2} + \beta a_{100} \delta T / a_{000}^{1/2} T , \tag{A13}$$

$$A_{2S} = \beta(a_{200}a_{000} - a_{100}^2)^{1/2} \delta T / a_{000}^{1/2} T , \tag{A14}$$

$$A_{1P\alpha} = \beta(a_{020}/3)^{1/2} v_{n\alpha} . \tag{A15}$$

If we use the relation

$$\mu = \left[\delta n - \left(\frac{\partial n}{\partial T}\right)_\mu \delta T \right] \Big/ \left(\frac{\partial n}{\partial \mu}\right)_T$$

and Eqs. (A3), (A7), (A8), and (A10), we obtain Eqs. (27)–(29) in the text.

---

[*] Present address: Department of Physics, Massachusetts Institute of Technology, Cambridge, Mass. 02139.

[1] L. D. Landau, J. Phys. (Moscow) 5, 71 (1941).

[2] L. Tisza, C. R. Acad. Sci. 207, 1035, 1186 (1938).

[3] H. J. Maris, Phys. Rev. Lett. 36, 907 (1976).

[4] L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 19, 637 (1949).

[5] I. M. Khalatnikov and D. M. Chernikova, Zh. Eksp. Teor. Fiz. 49, 1957 (1965) [Sov. Phys. JETP 22, 1336 (1966)].

[6] I. M. Khalatnikov and D. M. Chernikova, Zh. Eksp. Teor. Fiz. 50, 411 (1966) [Sov. Phys. JETP 23, 274 (1966)].

[7] R. C. Dynes, V. Narayanamurti, and K. Andres, Phys Rev. Lett. 30, 1129 (1973).

[8] V. Narayanamurti, R. C. Dynes, and K. Andres, Phys. Rev. B 11, 2500 (1975).

[9] V. Narayanamurti and R. C. Dynes, Phys. Rev. B 13, 2898 (1976).

[10] B. Castaing, Phys. Rev. B 13, 3854 (1976).

[11] B. Castaing and A. Libchaber, J. Low Temp. Phys. 31, 887 (1978).

[12] B. Castaing and A. Libchaber, J. Phys. (Paris) 39, C6-219 (1978).

[13] R. W. Cline and H. J. Maris, in *Phonon Scattering in Condensed Matter*, edited by H. J. Maris (Plenum, New York, 1980), p. 161.

[14] H. J. Maris, Phys. Rev. Lett. 36, 907 (1976).

[15] K. Weiss, Phys. Rev. B 15, 4227 (1977). Equation (32) of this paper contains a misprint and should read: $c_{\to N} \sim c_N[1 + \frac{1}{3} i (d_2/\Omega\tau_0' + \frac{17}{5}\Omega\tau')]$. When $\Omega\tau_0' >> 1$ this is then in agreement with our result (66).

[16] D. Polder and K. Weiss, Phys. Rev. A 17, 1478 (1978).

[17] I. M. Khalatnikov, Phys. Lett. 60A, 205 (1977).

[18] H. J. Maris, Phys. Rev. A 7, 2074 (1973).

[19] L. D. Landau and I. M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1959).

[20] L. D. Landau and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. 19, 709 (1949).

[21] We have taken n_R and the other roton parameters Δ, p_0, μ from J. S. Brooks and R. J. Donnelly, J. Phys. Chem. Ref. Data 6, 51 (1977).

[22] P. H. Roberts and R. J. Donnelly, J. Low Temp. Phys. 15, 1 (1974).

[23] In Ref. 10 it is claimed that nonparabolicity of the roton spectrum can lead to a significant correction to the velocity of roton second sound. As far as we can tell this result is incorrect, and arises from an improper evaluation of the a_{ijk} integrals.

[24] It is probably correct to compare C_R with the specific heat of the phonons also at constant number density. This is the specific heat C_P shown in Table III. The phonon specific heat at constant chemical potential is larger than this by a factor of 4.

[25] In fact, this follows from Eq. (A4) by integration by parts. The only assumption is that Maxwell-Boltzmann statistics may be used. This requires $kT << \Delta$.

[26] I. M. Khalatnikov, Zh. Eskp. Teor. Fiz. 20, 243 (1950).

[27] R. P. Feynman and M. Cohen, Phys. Rev. 102, 1189 (1956).