Static susceptibilities of the hydrodynamic order parameter variables of ^{3}He-*B*

M. Dörfle
Fachbereich Physik Universität Essen, 4300 Essen, West Germany
(Received 18 August 1980)

The coefficients of the $1/k^2$ divergence of the static susceptibilities of the hydrodynamic order-parameter variables in the superfluid B phase of ^{3}He are calculated by means of a microscopic theory. Their Bardeen-Cooper-Schrieffer value as well as the Landau corrections are written down explicitly. It is shown that only F_1^a, F_3^a, and the effective mass contribute to the static susceptibilities of the hydrodynamic order parameter in spin space. The results are compared with those of other work, and the experimental implications are discussed.

I. INTRODUCTION

The superfluid phases of ^{3}He provide an interesting testing ground for many ideas which are of importance in various fields of physics. There is no other system with so many broken symmetries in both real and spin space, due to the 18 degrees of freedom of the triplet order parameter.

From the beginning special attention has been drawn to the magnetic behavior of superfluid ^{3}He, which has turned out as an appropriate tool to identify the different phases.[1,2] Several authors have calculated the dynamical spin-correlation function in order to explain the NMR results. The various Goldstone modes have been determined by means of Green's-functions technique[3-8] or by Boltzmann technique[9] in the collisionless as well as in the hydrodynamic limit. The price to be paid, as far as the Green's-functions technique is concerned, is the restriction to a weak-coupling model which, in some cases, includes the paramagnon enhancement. The phenomenlogical theories,[10-12] on the other hand, give more general results for they exploit the symmetries of the order parameter as the only input. They give also explicit expressions for sound and spin waves but, in contrast to the pure microscopic calculations, a set of phenomenological parameters has to be defined which cannot be determined within the scope of these theories. The transport parameters describing the sound attenuation and the spin-wave damping can be connected with the dynamical correlation functions of the hydrodynamic variables via a set of Kubo relations.[13] They can be used as a starting point for microscopic approaches to the transport parameters.[14]

A second type of input parameters are the static susceptibilities. The conserved variables—density, momentum density, energy density, and magnetization density—give rise to static susceptibilities, which are found in the normal fluid and the superfluid phase likewise. Another group of static susceptibilities belongs to the hydrodynamic order-parameter variables. As hydrodynamic we characterize those order-parameter variables, which refer to the broken symmetries of the superfluid phase. These variables, together with the conserved ones, form a set of slow variables from which the hydrodynamic theory is constructed.

In the hydrodynamic regime, we may separate our problem into two parts which may be solved step by step. First, the symmetries of the system are exploited in order to establish a linearized hydrodynamics and to evaluate the sound and spin waves. In a second step, we are concerned with the static susceptibilities and the transport parameters which enter the hydrodynamic modes. The static susceptibilities for $\vec{k} \to 0$, however, are much easier to calculate than the dynamical ones for finite $\vec{k}$ and ω, necessary in the pure microscopic approach.

This paper is devoted to the static susceptibilities of the hydrodynamic order-parameter variables of superfluid ^{3}He-*B*. They enter the gradient part of the superfluid free energy.

Brinkman and Smith[15] specialized the Ginzburg-Landau free energy, calculated by Ambegaokar, de Gennes, and Rainer,[16] to the B phase in the weak-coupling limit. Cross[17] evaluated a generalized Ginzburg-Landau function for A and B phase valid in the whole temperature region. Landau corrections have been taken into account in terms of the first Landau parameters F_1^s and F_1^a.

In the present paper we start from the linear-response functions and specialize them in the static limit. The results obtained are more general than those of Refs. 15 and 17 in taking into account the dependence on all Landau parameters exactly. Complete agreement with both works is obtained if we specialize our results by taking the higher-order Landau parameters equal to zero.

We show that only two Landau parameters, F_1^a and F_3^a, enter the static susceptibility of the hydrodynamic order-parameter variable in spin space. Below T_c, we find that new temperature-dependent Landau parameters can be defined, which give rise to very simple results.

In Sec. II we introduce the general procedure. We generalize a concept, developed by Leggett[18] for the neutral s-wave superconductor, to the case of triplet pairing. In Sec. III the static susceptibilities are calculated in the weak-coupling limit. We show that the $1/k^2$ dependence of the static susceptibilities is a consequence of the gap equation. Section IV is devoted to the influence of the Landau parameters on the static susceptibilites. Section V contains a discussion of the results and the comparision with other work.

II. LINEAR RESPONSE FUNCTIONS

The hydrodynamics of the superfluid phases of ^{3}He is characterized by two types of variables: the conserved ones and those due to broken symmetries. In the various phases of superfluid ^{3}He different symmetries are broken, depending on the vacuum states. The order parameter of the B phase has the group structure U(1) ⊗ SO(3) and can be parameterized by a phase factor times a three-dimensional rotation matrix n_{ij}. In the A phase the group structure is characterized by $S^2 \otimes SO(3)/Z_2$ thus giving rise to the bivector factorization of the order parameter. Throughout this paper we restrict ourselves to the B phase. Four additional hydrodynamic order-parameter variables enter the hydrodynamics: ϕ and the three angles characterizing the rotation matrix. Their static susceptibilities must diverge as $1/k^2$ for $k \to 0$. The physical background is rather simple: the Hamiltonian is invariant against phase transformation and transformations of n_{ij}. Thus, it costs no energy to change globally the phase or the rotation matrix. Introducing a small inhomogeneity of ϕ or n_{ij} into the system the restoring forces which bring the system back to its homogeneous state, become infinitly small as the system appproaches $k \to 0$. In other terms, the inverse susceptibilities must behave at least like k^2 for $k \to 0$. It is this feature we want to calculate by microscopic means.

The superfluid phase of ^{3}He is described generally by a complex order-parameter matrix $\hat{A}_{ij}$. (Operators are indicated by the symbol $\hat{A}$, their expectation values are described by the same symbol without $\hat{}$, equilibrium values by a superscript ^{0}.)

$$\hat{A}_{ij} = \int d^3x F^*(\vec{x}) \hat{\psi}_\alpha(\vec{r} - \tfrac{1}{2}\vec{x}) \times \hat{\psi}_\beta(\vec{r} + \tfrac{1}{2}\vec{x}) \frac{x_j}{|\vec{x}|} (\sigma_i \sigma_2)_{\alpha\beta} \ . \tag{2.1}$$

Sums over repeated indices are always implied if not indicated otherwise. The unit vector $\vec{x}/|\vec{x}|$ selects the p-wave part of the anomalous expectation value, $(\vec{\sigma}\sigma_2)$ gives the projection onto the triplet spin axes. Thus, A_{ij} describes a pairing of a triplet system which has the symmetry of a p-wave state. The function $F^*(\vec{x})$ is chosen in such a way to make sure that always

$$A_{ij} A_{ij}^* = 1 \ . \tag{2.2}$$

In the B phase A_{ij} is proportional to a rotation matrix n_{ij}:

$$A_{ij} = \frac{1}{\sqrt{3}} e^{i\phi} n_{ij} \ . \tag{2.3}$$

The changes of $\delta A_{ij}(\vec{x}) = A_{ij}(\vec{x}) - A_{ij}^0$ from the equilibrium value can be formulated in terms of the expectation values of the phase deviation $\delta\phi$ and the deviation from the angles $\delta\theta_i$:

$$\delta A_{ij}(\vec{x}) = i A_{ij}^0 \delta\phi(\vec{x}) + \epsilon_{ikl} \delta\theta_k(\vec{x}) A_{lj}^0 \ . \tag{2.4}$$

The four variables $\delta\hat{\phi}$ and $\delta\hat{\vec{\theta}}$ are the microscopic operators we look for. They can be expressed in terms of the operator $\hat{A}_{ij}$:

$$\begin{aligned} \delta\hat{\phi}(\vec{x}) &= \frac{1}{2i} [A_{jl}^{0*} \hat{A}_{jl}(\vec{x}) - A_{jl}^0 \hat{A}_{jl}^\dagger(\vec{x})] \ , \\ \delta\hat{\theta}_i(\vec{x}) &= \tfrac{3}{4} \epsilon_{ijk} [A_{jl}^{0*} \hat{A}_{kl}(\vec{x}) + A_{jl}^0 \hat{A}_{kl}^\dagger(\vec{x})] \ . \end{aligned} \tag{2.5}$$

Due to the high symmetry of the B phase, $\delta\phi$ is the only additional hydrodynamic variable in real space. Therefore, the real-space part of the B phase dynamics is equivalent to the neutral singlet superconductor, whereas the spin-space part offers a very rich and complicated structure as we will see below.

It is our aim to calculate microscopically the static susceptibilities of $\delta\phi$ and $\delta\vec{\theta}$. If we exploit the Bogoljubov inequality to show the $1/k^2$ divergence we make explicit use of various conservation laws. To show, for example, that $\chi_{\delta\phi,\delta\phi}(\vec{k}) \sim 1/k^2$ we must exploit the number conservation law. Thus it is only natural that we have to take care that any approximation scheme we pursue ensures the conservation laws. The situation were different, if we calculated the transport coefficients. They are determined by microscopic scattering processes which occur on a very short time scale. Thus, a crude Hartree-Fock approximation may give reasonable expressions, whereas this procedure fails completely in the case of the static susceptibilities. This problem is well known since the early sixties, when Kadanoff and Baym[19,20] presented a systematic approximation scheme, which ensures the conservation laws on each stage.

It was applied by Leggett[18] in order to calculate microscopically the static and dynamic susceptibilities of the conserved variables of the singlet superconductor. The calculations presented here make use of this work although there are some modifications due to the triplet pairing. In order to establish the notations we give a short summary of the method, but stress only those points where special problems of the triplet state are involved.

The static susceptibility of any microscopic variable can be calculated from the correlation function for imaginary times

$$\langle\langle A(1)A(2)\rangle\rangle = \langle A(1)A(2)\rangle - \langle A(1)\rangle\langle A(2)\rangle \; . \tag{2.6}$$

The time ordering is always implied. Taking the Fourier transform of space and time we get the static susceptibilities after continuing the discrete imaginary frequencies to the real axis. The static susceptibilities of $\delta\phi$ and $\delta\vec{\theta}$ can be built up from the static susceptibilities of the order parameter $\hat{A}_{ij}$:

$$\langle\delta\phi(1)\delta\phi(2)\rangle = -\tfrac{1}{12} n_{ik}^{0} n_{jl}^{0} \times \langle\langle(\hat{A}_{ik}^{\dagger} - \hat{A}_{ik})_1(\hat{A}_{jl}^{\dagger} - \hat{A}_{jl})_2\rangle\rangle \; ,$$

$$\langle\delta\theta_q(1)\delta\theta_p(2)\rangle = \tfrac{3}{16}\epsilon_{qim}\epsilon_{pjn} n_{mk}^{0} n_{nl}^{0} \times \langle\langle(\hat{A}_{ik}^{\dagger} + \hat{A}_{ik})_1(\hat{A}_{jl}^{\dagger} + \hat{A}_{jl})_2\rangle\rangle \; . \tag{2.7}$$

The time-ordered expectation values $\langle\langle\hat{A}_{ik}\hat{A}_{jl}\rangle\rangle$ are related with the linear response function $\mathfrak{L}$ for imaginary times:

$$\mathfrak{L}(12,1'2') = G_2(12,1'2') - G(1-1')G(2-2') \; , \tag{2.8}$$

where G_2 denotes the two-particle and G the one-particle Green's functions:

$$\langle\langle\hat{A}_{ik}^{(\dagger)}\hat{A}_{jl}^{(\dagger)}\rangle\rangle(k\omega_n) = \int\frac{d^3q}{(2\pi)^3}\int\frac{d^3q'}{(2\pi)^3}\gamma(q)\hat{q}_k(\sigma_i\sigma_2)_{\alpha\beta}\frac{1}{\beta^2}\sum_{\omega_l,\omega_{l'}}\mathfrak{L}^a_{\alpha\alpha',\beta\beta'}(\vec{q}\,\omega_l,\vec{q}\,'\omega_{l'};\vec{k}\omega_n)\gamma(q')\hat{q}_l'(\sigma_j\sigma_2)_{\alpha'\beta'} \; , \tag{2.9}$$

where

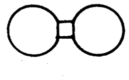
$$i\gamma(q)\hat{q}_k = \int d^3x\, F(\vec{x})\frac{x}{|\vec{x}|_k}e^{i\vec{q}\cdot\vec{x}}$$

and $\mathfrak{L}^a$ is that part of $\mathfrak{L}$ which consists of anomalous propagators. The symbol ($\dagger$) on the left-hand side of Eq. (2.9) is meant to imply that both $\hat{A}_{ik}$ and $\hat{A}_{ik}^{\dagger}$ may be inserted. All static susceptibilities of the order parameter A_{ij} are composed of particle-particle $\langle\langle\hat{A}_{ik}\hat{A}_{jl}^{\dagger}\rangle\rangle$ or hole-hole propagators $\langle\langle\hat{A}_{ik}^{\dagger}\hat{A}_{jl}\rangle\rangle$ or of propagators which transpose a particle-particle fluctuation into the hole-hole channel $\langle\langle\hat{A}_{ik}^{\dagger}\hat{A}_{jl}^{\dagger}\rangle\rangle$ and vice versa $\langle\langle\hat{A}_{ik}\hat{A}_{jl}\rangle\rangle$. The four rows and four columns of the particle-hole matrix $\mathfrak{L}$ are ordered in the following way: *hh*, *pp*, *hp*, and *ph* (*p* denotes particle, *h* hole). Thus the left upper 2×2 submatrix is the anomalous part $\mathfrak{L}^a$ of $\mathfrak{L}$. For details we refer to Ref. 18.

The response function $\mathfrak{L}$ can be obtained from the Bethe-Salpeter equation:

$$\mathfrak{L} = (12,1'2') = L(12,1'2') + L(13,1'3') \times\frac{\delta^2\Phi}{\delta G(3,4)\delta G(3',4')}\mathfrak{L}(42,4'2') \; , \tag{2.10}$$

where $L(12,1'2') = G(1-2')G(2-1')$. The conservation laws are ensured writing the vertex as a second functional derivative of a functional without external lines. The approximations are to be carried out on Φ. We take into account only the leading order in T_c/T_F, i.e., we have

$\Phi =$  ,

where the square denotes the interaction vertex and the lines symbolize one-particle Green's functions. Thus, the BCS interaction as well as the effects of the mean field described by the Landau parameters are taken into account. We have not included strong-coupling terms in the sense of Serene and Rainer,[21,22] which are of order $(T_c/T_F)^3$.

The leading contributions to the correlation functions are due to quasiparticle excitations near the Fermi surface. Therefore, we separate the one-particle Green's functions into the quasiparticle part, which is proportional to the Green's functions of free particles, and an incoherent part which is included in the interaction vertex renormalizing it. After these approximations, we arrive at

$$\mathfrak{L} = L_k(1-\Gamma^{\omega}L_k)^{-1} \; , \tag{2.11}$$

where we have omitted an additional incoherent part of $\mathfrak{L}$, which remains finite for $\vec{k}\to0$ and therefore cannot contribute to the divergent behavior of the static susceptibilities of the hydrodynamic order-parameter variables. L_k is the singular quasiparticle part of $L = GG$. (Since only L_k will be used in the following, we will drop the index k.) Γ^{ω} denotes the quasiparticle irreducible vertex part introduced by Landau. We neglect its dependence on the quasiparticle energy and the frequencies because it is assumed to be a smooth function of energy and momentum near the Fermi surface.

Restricting ourselves to the unitary phases of superfluid ^{3}He, the one-particle Green's functions read

$$G_{\alpha\beta}(\vec{q}\omega_n) = a\frac{i\omega_n+\xi}{\omega_n^2+\xi^2+|\Delta|^2}\delta_{\alpha\beta} \equiv G\,\delta_{\alpha\beta} \; , \tag{2.12}$$

$$F_{\alpha\beta}(\vec{q}\omega_n) = a\frac{\Delta_0}{\omega_n^2+\xi^2+|\Delta|^2}id_j(\sigma_j\sigma_2)_{\alpha\beta} \equiv F i d_j(\hat{q})(\sigma_j\sigma_2)_{\alpha\beta} \; , \tag{2.13}$$

where ω_n are the discrete Matsubara frequencies, $\vec{q}$ the wave vector, $\xi^2 = q^2/2m^* - \mu$ the excitation energy, Δ_0 the maximum of the gap:

$$|\Delta|^2 = \Delta_0^2 |\vec{d}|^2$$

and

$$d_i = A_{ij}^0 \hat{q}_j \ . \tag{2.14}$$

Thus, we allow for the A and B phases as well as the more academic planar and polar phases. According to our model Γ^ω is diagonal in the particle-hole space. The scattering in the anomalous channel is described by the BCS interaction, the scattering in the particle-hole channel by the usual quasiparticle scattering vertex:

(BCS)

$$\Gamma^{\rm BCS} = \Gamma_1(\hat{q}\cdot\hat{q}')\delta_{\alpha\beta}\delta_{\gamma\delta} + \Gamma_2(\hat{q}\cdot\hat{q}')\sigma^m_{\alpha\beta}\sigma^m_{\gamma\delta} \ , \tag{2.15a}$$

(Landau)

$$\Gamma^{\rm L} = \Gamma^s(\hat{q}\cdot\hat{q}')\delta_{\alpha\beta}\delta_{\gamma\delta} + \Gamma^T(\hat{q}\cdot\hat{q}')\sigma^m_{\alpha\beta}\sigma^m_{\gamma\delta} \ . \tag{2.15b}$$

The response functions are to be calculated in the static limit but with finite external wave vector $\vec{k}$. Since the vertex does not depend on the quasiparticle energy ξ and the frequencies ω_n we perform the integration and summation over the internal variables at once

$$L^{GG}(\hat{q},\vec{k}) = a^2 v(0)\int d\xi \sum_{\omega_n} G_{\alpha\beta}(\vec{q}+\tfrac{1}{2}\vec{k},\ \omega_n) \times G_{\gamma\delta}(\vec{q}-\tfrac{1}{2}\vec{k},\ \omega_n)$$

$$\equiv GG\,\delta_{\alpha\beta}\delta_{\gamma\delta} \ . \tag{2.16}$$

$v(0)$ is the density of states at the Fermi surface. A minus sign on either of the Green's functions indicates that we use the variables

$$G^- = G(-(\vec{q}+\tfrac{1}{2}\vec{k}),-\omega_n) \ .$$

In order to calculate the leading order in $\vec{k}$ of the static susceptibilities we have to expand the propagators L in terms of $\vec{k}$. The results are in the anomalous channel

$$G^-G + FF|\vec{d}|^2 = a^2 v(0)\int d\xi \frac{1}{\beta}\sum_{\omega_n}\frac{1}{\omega_n^2+\xi^2+|\Delta|^2} - \frac{1}{2}\frac{(\vec{v}_F\cdot\vec{k})^2}{|\Delta|^2}\lambda(\hat{q},T) \tag{2.17}$$

[in Sec. III it will become clear that we must keep the $(\vec{v}_F\cdot\vec{k})^2$ term] and

$$FF = \lambda(\hat{q},T) + \frac{1}{6}(\vec{v}_F\cdot\vec{k})^2 \frac{\partial}{\partial|\Delta|^2}\frac{1}{|\vec{d}|^2}\lambda(\hat{q},T) \ . \tag{2.18}$$

The coupling of normal and anomalous channels reads

$$GF = \frac{1}{2}\frac{(\vec{v}_F\cdot\vec{k})}{\Delta_0|\vec{d}|^2}\lambda(\hat{q},T) \tag{2.19}$$

and in the normal channel we get

$$GG + |\vec{d}|^2 FF = -a^2 v(0)\,Y(\hat{q},T) \ , \tag{2.20}$$

where $\lambda(\hat{q},T) = \frac{1}{2}a^2 v(0)[1 - Y(\hat{q},T)]$ and $Y(\hat{q},T)$ is the anisotropic Yoshida function:

$$Y(\hat{q},T) = \int_0^\infty d\xi\, \tfrac{1}{2}\beta\, {\rm sech}^2 \tfrac{1}{2}\beta[\xi^2 + |\Delta(\hat{q})|^2]^{1/2} \ . \tag{2.21}$$

Up to the anisotropy, these expressions are the same as those given by Leggett.[18] All other products of Green's functions can be expressed by Eqs. (2.17)–(2.21).

The system of integral equations (2.11) can be simplified a good deal evaluating the spin sums. We expand (2.11) in a ladder diagram by expanding the right-hand side in a geometric series. Equation (2.9) shows that both sides of each ladder have to be multiplied with $(\vec{\sigma}\sigma_2)$. Thus L^a, the anomalous submatrix of L is converted to

$$(\sigma_i\sigma_2)_{\alpha\beta} L^a_{\alpha\gamma,\beta\delta}(\sigma_j\sigma_2)_{\gamma\delta} \rightarrow L^a_{ij} \ . \tag{2.22}$$

We start on the left-hand side of an arbitrary diagram. By means of simple algebra, one can show

$$(\sigma_i\sigma_2)_{\alpha\beta} G_{\alpha\gamma'} G_{\beta\gamma}(\Gamma_1\delta_{\gamma\delta'}\delta_{\gamma'\delta} + \Gamma_2\sigma^m_{\delta'\gamma}\sigma^m_{\delta\gamma'}) = \delta_{ij} GG(\Gamma_1+\Gamma_2)(\sigma_j\sigma_2)_{\delta\delta'} \ , \tag{2.23}$$

and

$$(\sigma_i\sigma_2)_{\alpha\beta} F_{\alpha\gamma'} F_{\beta\gamma}(\Gamma_1\delta_{\gamma\delta'}\delta_{\gamma'\delta} + \Gamma_2\sigma^m_{\delta'\gamma}\sigma^m_{\delta\gamma'}) = (2d_id_j - \vec{d}^2\delta_{ij}) FF(\Gamma_1+\Gamma_2)(\sigma_j\sigma_2)_{\delta\delta'} \ . \tag{2.24}$$

In order to determine the static susceptibilites of the order-parameter variables, we start always in the particle-particle or hole-hole channel. Passing step by step through the whole diagram—a single step being shown in Eqs. (2.23) and (2.24)—all Pauli matrices are eliminated and we arrive at

$$\mathcal{L}^a_{ij} = L^a_{ik}(1 - \Gamma^\omega L^a)^{-1}_{kj} \ . \tag{2.25}$$

This is still a system of integral equations due to the dependence of Γ^ω on the solid angles.

If we restrict ourselves to the fluctuations in the anomalous channels, then we arrive at the BCS approximation of the static susceptibilities. If we want to take into account the strong-coupling corrections due to the Fermi-liquid parameters, we have to admit the coupling to the normal scattering channels which are described by the remaining parts of L. The evaluation of spin sums in expressions like (2.22) but containing the remaining parts of L does not bring about further difficulties.

In order to deal with all fluctuations possible in the normal channels (i.e., with singlet and triplet symmetry) it is convenient to extend the matrices in the particle-hole space. This is done by extending the two-dimensional normal channel into a four-dimensional normal channel which contains singlet and triplet parts separately. The vertex matrix Γ^{ω} remains diagonal, but L is composed of various parts. L^a denotes the anomalous propagator

$$\underline{L}^a(\hat{q},\vec{k})=\begin{pmatrix} -G^-G\delta_{rp} & (2d_r^*d_p^*-\vec{d}^{*2}\delta_{rp})FF \\ (2d_rd_p-\vec{d}^2\delta_{rp})FF & -GG^-\delta_{rp} \end{pmatrix} . \tag{2.26}$$

The indices r,p mark the spin axes. In the extended normal channels we take into account singlet and triplet fluctuations separately which do not couple within the normal channel:

(singlet channel)

$$\underline{L}^s(\hat{q},\vec{k})=\begin{pmatrix} G^-G^- & -FF|\vec{d}|^2 \\ -FF|\vec{d}|^2 & GG \end{pmatrix} , \tag{2.27}$$

(triplet channel)

$$\underline{L}^T(\hat{q},\vec{k})=\begin{pmatrix} G^-G^-\delta_{rp} & (2d_rd_p^*-\delta_{rp}|\vec{d}|^2)FF \\ (2d_r^*d_p-\delta_{rp}|\vec{d}|^2)FF & GG\delta_{rp} \end{pmatrix} . \tag{2.28}$$

The normal channels are coupled to the anomalous channels by $\underline{L}^{a,s}$ and $\underline{L}^{a,T}$:

(coupling anomalous singlet)

$$\underline{L}^{a,s}(\hat{q},\vec{k})=\begin{pmatrix} iG^-Fd_p & iFG^-d_p^* \\ iFGd_p & iGFd_p^* \end{pmatrix} , \tag{2.29}$$

(coupling anomalous triplet)

$$\underline{L}^{a,T}(\hat{q},\vec{k})=\begin{pmatrix} -\epsilon_{lpm}d_pG^-F & \epsilon_{lpm}d_p^*FG^- \\ -\epsilon_{lpm}d_pFG & \epsilon_{lpm}d_p^*GF \end{pmatrix} . \tag{2.30}$$

We take care of all possible couplings constructing the extended particle-hole matrix $\underline{L}$ from these contributions:

$$\underline{L}(\hat{q},\vec{k})=\begin{pmatrix} \underline{L}^a & (\underline{L}^{a,s})^{\dagger} & (\underline{L}^{a,T})^{\dagger} \\ \underline{L}^{a,s} & \underline{L}^s & 0 \\ \underline{L}^{a,T} & 0 & \underline{L}^T \end{pmatrix} , \tag{2.31}$$

$$\Gamma^{\omega}(\hat{q}\cdot\hat{q}')=\begin{pmatrix} \Gamma^{\phi} & 0 & 0 \\ 0 & \Gamma^s & 0 \\ 0 & 0 & \Gamma^T \end{pmatrix} , \tag{2.32}$$

where $\Gamma^{\phi}=\Gamma_1+\Gamma_2$. All elements of L are still functions of $\vec{k}$ and the internal $\hat{q}$. The dressed response function $\mathcal{L}$ is calculated from Eq. (2.11) applying usual matrix algebra and integrating over all internal unit vectors.

III. BCS APPROXIMATION

The expressions (2.26)–(2.32) are valid for all unitary phases. In this paper, these equations will be applied to the isotropic B phase. We neglect the small symmetry-breaking magnetic dipole-dipole energy which fixes one of the three parameters due to the rotation matrix n_{ij}. Without loss of generality, we can assume $n_{ij}^0=\delta_{ij}$, i.e., the axes of spin space and real space are parallel to each other. Thus, $d_i=A_{ik}^0\hat{q}_k$ becomes identical with the unit wave vector

$$d_i=d_i^*=\hat{q}_i$$

and the hydrodynamic order parameter variables take a much simpler form

$$\delta\hat{\phi}=\frac{i}{2\sqrt{3}}(\hat{A}_{ii}^{\dagger}-\hat{A}_{ii}) , \tag{3.1}$$

$$\delta\hat{\theta}_i=\frac{\sqrt{3}}{4}\epsilon_{ijk}(\hat{A}_{jk}^{\dagger}-\hat{A}_{jk}) . \tag{3.2}$$

Furthermore, the gap Δ and the Yoshida function $Y(\hat{q},T)$ become isotropic.

The polar and the planar phase are two other real unitary phases. In their case, the equilibrium order parameter A_{ij}^0 is proportional to $\delta_{ij}-l_il_j$ and l_il_j in the planar and the polar state, respectively. Although the path of thoughts is quite similar to that of the isotropic phase we will put them apart for they are only of academic interest.

The calculations can be simplified a good deal fixing $\vec{k}$ parallel to the $\hat{z}$ axis

$$(\vec{v}_F\cdot\vec{k})=v_Fk\hat{q}_3 . \tag{3.3}$$

In this section, we restrict ourselves to the BCS interaction omitting the corrections due to the coupling to the normal channels. The static susceptibilities will diverge proportional to $1/k^2$— this feature follows from the broken symmetries already contained in the weak-coupling model—whereas the superfluid density will take the BCS value. Hence, we have to deal with the reduced set of integral equations (2.25) and (2.26). In this model, only the fluctuations $\underline{L}^a$ in the anomalous channel and the attractive BCS pair interaction $\Gamma^\phi = \Gamma_1 + \Gamma_2$ are taken into account. Γ^ϕ is parameterized by

$$\Gamma^\phi(\hat{q}\cdot\hat{q}') = \Gamma^\phi \hat{q}_i \hat{q}_i' \tag{3.4}$$

exploiting that the pairs are condensed in a p-wave state. The constant Γ^ϕ describes the strength of the attractive pair interaction. It is purely microscopic and must not occur in any microscopic expression. The value of Δ is determined from the self-consistent gap equation

$$1 = -\tfrac{1}{3}\Gamma^\phi a\, v(0) \int d\xi \frac{1}{\beta} \sum_{\omega_n} \frac{1}{\omega_n^2 + \xi^2 + \Delta^2} , \tag{3.5}$$

where we have already exploited that Δ is isotropic. Inserting the Green's functions it is straightforward to show that

$$1 = -\tfrac{1}{3}\Gamma^\phi (GG^- + FF)|_{k=0} \tag{3.6}$$

is an alternative form of the gap equation. The integral equations (2.25) can be transformed into a set of algebraic equations: let us write down a small part of any ladder. It reads

$$\cdots \int \frac{d\Omega}{4\pi} \Gamma^\phi(\tilde{q}\cdot q) \underline{L}^a(q, \vec{k}) \times \int \frac{d\Omega'}{4\pi} \Gamma^\phi(q\cdot q') \underline{L}^a(q', \vec{k}) \times \int \frac{d\Omega''}{4\pi} \Gamma^\phi(q'\cdot q'') \underline{L}^a(q'', \vec{k}) \cdots . \tag{3.7}$$

The interaction vertex Γ^ϕ depends only on the angle between two quasiparticles excited on the Fermi surface. Taking into account the p-wave model (3.4), the integrals are separable

$$\cdots \Gamma^\phi \tilde{q}_i \left[\int \frac{d\Omega}{4\pi} \hat{q}_i \underline{L}^a(\hat{q}, \vec{k}) \hat{q}_r \right] \times \Gamma^\phi \left[\int \frac{d\Omega}{4\pi} \hat{q}_r \underline{L}^a(\hat{q}, \vec{k}) \hat{q}_j \right] \cdots . \tag{3.8}$$

Comparing the solid angle integrals with Eqs. (2.9), (2.16), and (2.22), we recognize that they are proportional to the bare equilibrium propagators,

$$\int \frac{d\Omega}{4\pi} \hat{q}_k \underline{L}^a_{ij}(\hat{q}, \vec{k}) \hat{q}_l \sim \begin{bmatrix} \langle\langle \hat{A}_{ik} \hat{A}_{jl}^\dagger \rangle\rangle_0 & \langle\langle \hat{A}_{ik}^\dagger \hat{A}_{jl}^\dagger \rangle\rangle_0 \\ \langle\langle \hat{A}_{ik} \hat{A}_{jl} \rangle\rangle_0 & \langle\langle \hat{A}_{ik}^\dagger \hat{A}_{jl} \rangle\rangle_0 \end{bmatrix} . \tag{3.9}$$

The calculations are further simplified passing from the variables $\hat{A}_{ik}, \hat{A}_{ik}^\dagger$ to $\hat{A}_{ik}^\dagger + \hat{A}_{ik}$ and $\hat{A}_{ik}^\dagger - \hat{A}_{ik}$. Equations (3.1) and (3.2) reveal that these variables are the most appropriate choice in order to calculate the static correlation functions of $\delta\phi$ and $\delta\vec{\theta}$. Rotating the matrix (3.9), we get

$$\begin{bmatrix} \underline{L}^{11} & \underline{L}^{12} \\ \underline{L}^{21} & \underline{L}^{22} \end{bmatrix} = \begin{bmatrix} \langle\langle (\hat{A}_{ik}^\dagger + \hat{A}_{ik})(\hat{A}_{jl}^\dagger + \hat{A}_{jl}) \rangle\rangle_0 & \langle\langle (\hat{A}_{ik}^\dagger + \hat{A}_{ik})(\hat{A}_{jl}^\dagger - \hat{A}_{jl}) \rangle\rangle_0 \\ \langle\langle (\hat{A}_{ik}^\dagger - \hat{A}_{ik})(\hat{A}_{jl}^\dagger + \hat{A}_{jl}) \rangle\rangle_0 & \langle\langle (\hat{A}_{ik}^\dagger - \hat{A}_{ik})(\hat{A}_{jl}^\dagger - \hat{A}_{jl}) \rangle\rangle_0 \end{bmatrix} . \tag{3.10}$$

The upper indices refer always to the particle-hole space. Each element in this space is a matrix with two spin- and two real-space indices:

$$\underline{L}^{11} = \int \frac{d\Omega}{4\pi} \hat{q}_k \hat{q}_l [-\delta_{ij} G^- G + FF(2\hat{q}_i \hat{q}_j - \delta_{ij})] = L^{11}_{ik,jl} . \tag{3.11}$$

$$\underline{L}^{22} = \int \frac{d\Omega}{4\pi} \hat{q}_k \hat{q}_l [-\delta_{ij} G^- G - FF(2\hat{q}_i \hat{q}_j - \delta_{ij})] = L^{22}_{ik,jl} , \tag{3.12}$$

$$\underline{L}^{12} = \underline{L}^{21} = 0 .$$

$\underline{L}^{22}$ and $\underline{L}^{22}$ are decoupled and can be dealt with independently. We note, that the fluctuations of the real-space variable $\delta\phi$ occur only in the channel $\underline{L}^{22}$, whereas the spin-space fluctuations are restricted to $\underline{L}^{11}$. Within the BCS approximation, fluctuations of real-space variables are decoupled from fluctuations of spin-space variables, since $\underline{L}^{12} = \underline{L}^{21} = 0$.

The static susceptibility of the phase ϕ is obtained by a linear combination of matrix elements of $\underline{L}^{22}$:

$$\chi_{\delta\phi,\delta\phi}(\vec{k}) = \frac{v(0) a^2 \gamma^2}{12\Gamma^\phi} \mathcal{L}^{22}_{ii,kk} , \tag{3.13}$$

where

$$\mathcal{L}^{22}_{ik,jl} = L^{22}_{ik,rs} (1 - \Gamma^\phi L^{22})^{-1}_{rs,jl} . \tag{3.14}$$

If we start from the left-hand side of any ladder, corresponding to Eq. (3.14) two unit vectors in (3.12) are fixed in the same direction: $\hat{q}_i \hat{q}_k \to (\hat{q}_i)^2$. A finite $\vec{k}$ brings in a factor $(\hat{q}_3)^2$. Thus, L^{22} takes nonvanishing values only if the other indices are also fixed parallel to each other. Then, Eq. (3.14) is reduced to

$$\mathcal{L}^{22}_{i,k} = L^{22}_{i,r}(1-\Gamma^{\phi}L^{22})^{-1}_{r,k} \tag{3.15}$$

and

$$\chi_{\delta\phi,\delta\phi}(\vec{k}) = \frac{v(0)a^2\gamma^2}{12\Gamma^{\phi}} \sum_{i,k=1}^{3} \mathcal{L}^{22}_{i,k} , \tag{3.16}$$

where we have dropped one superfluous index by writing ii,kk as i,k.

$$L^{22}_{i,k} = \int \frac{d\Omega}{4\pi} [-\hat{q}_i\hat{q}_k\delta_{ik}G^{-}G - FF\hat{q}_i\hat{q}_k(2\hat{q}_i\hat{q}_k - \delta_{ik})] . \tag{3.17}$$

Summation over repeated indices is not implied here. It is convenient to write Eq. (3.16) in terms of a projector P

$$\underline{P} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} , \tag{3.18}$$

$$\chi_{\delta\phi,\delta\phi}(\vec{k}) = \frac{v(0)a^2\gamma^2}{4\Gamma^{\phi}} Tr_3(\underline{P}\underline{\mathcal{L}}^{22}) . \tag{3.19}$$

We transform the trace by a rotation in such a way that the projector is reduced to one nonvanishing element. This is done by the orthogonal matrix T

$$\underline{T} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & -2 & 1 \end{pmatrix}$$

and

$$\underline{TPT}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.20}$$

In order to perform the trace, we have only to pick up the upper left element of the transformed matrix.

We continue investigating the limit $k=0$. The functions $G^{-}G$ and FF do not depend on $\hat{q}$, and we get

$$L^{22}_{i,k}(k=0) = -(G^{-}G+FF)|_{k=0}\tfrac{1}{3}\delta_{ik} - FF|_{k=0}\tfrac{2}{5}(\underline{P}-\delta_{ik}) . \tag{3.21}$$

It is useful to reformulate Eq. (3.15)

$$\underline{TL}^{22}\underline{T}^{-1}(\underline{1}-\Gamma^{\phi}\underline{TL}^{22}\underline{T}^{-1})^{-1} = \frac{1}{\Gamma^{\phi}}(\underline{1}-\Gamma^{\phi}\underline{TL}^{22}\underline{T}^{-1})^{-1} - \frac{1}{\Gamma^{\phi}}\underline{1} . \tag{3.22}$$

The term $(1/\Gamma^{\phi})\underline{1}$ belongs to the incoherent part of the static susceptibility and can be neglected because the $1/k^2$ divergence is only due to the inverse of $\underline{1}-\Gamma^{\phi}\underline{L}^{22}$. We obtain

$$\Xi = \underline{1}-\Gamma^{\phi}\underline{TL}^{22}\underline{T}^{-1}|_{k=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Gamma^{\phi}\frac{2}{5}FF & 0 \\ 0 & 0 & -\Gamma^{\phi}\frac{2}{5}FF \end{pmatrix} , \tag{3.23}$$

where we have made use of the gap equation (3.6). Obviously, Ξ becomes singular in the limit $\vec{k}\to 0$. The singularity belongs to that linear combination of A_{ik} which can be identified as the macroscopic variable $\delta\phi$. The other linearly independent variables get a finite quasiparticle contribution to their static susceptibilities.

Inspecting the next $\vec{k}$ order of Ξ, we observe that each matrix element will be proportional to k^2. To show that the regular part of Ξ does not contribute to the singular one in order k^2 we exploit the following matrix identity: If we look only for a part of an inverted matrix, we may restrict ourselves to the inversion of a smaller but renormalized matrix:

$$\begin{pmatrix} A & B_1 \\ B_2 & C \end{pmatrix}^{-1} = \begin{pmatrix} (A-B_1C^{-1}B_2)^{-1} & -(A-B_1C^{-1}B_2)^{-1}B_1C^{-1} \\ -C^{-1}B_2(A-B_1C^{-1}B_2)^{-1} & C^{-1}+C^{-1}B_2(A-B_1C^{-1}B_2)^{-1}B_1C^{-1} \end{pmatrix} , \tag{3.24}$$

where A and C are quadratic and the inverse of C must exist. B_1 and B_2 need not have a quadratic form. Thus, if we are only interested in a small part of an inverse, we first take into account the coupling to the other parts and then invert the smaller submatrix

$$A \to A - B_1C^{-1}B_2 . \tag{3.25}$$

C is taken to be the regular part of Ξ. B_1 and B_2 are either zero or proportional to k^2. Therefore, the regular part of Ξ contributes only to the order k^4 which has to be omitted. We arrive at

$$\chi_{\delta\phi,\delta\phi}(\vec{k}) = \frac{\gamma^2a^2v(0)}{4\Gamma^{\phi}}\left[-\Gamma^{\phi}\int\frac{d\Omega}{4\pi}\left(-\frac{1}{2}\frac{(\vec{v}_F\cdot\vec{k})^2}{|\Delta|^2}\lambda\right)\right]^{-1} = \frac{\gamma^2}{(\Gamma^{\phi})^2\Delta_0^2}\frac{m^2}{\rho_s k^2} , \tag{3.26}$$

where

$$\rho_s = \rho\cdot\rho_s^0, \quad \rho_s^0 = 1 - Y(T) .$$

ρ is the density of ^{3}He. Making use of the gap equation and the definition of the order parameter A_{ik}, we can eliminate all microscopic constants and get finally

$$\chi_{\delta\rho,\delta\rho}(\vec{k}) = m^2/\rho_s k^2 \quad , \tag{3.27}$$

which is the expected result.

Next we turn to the BCS approximated static susceptibilities of the hydrodynamic order parameter $\delta\vec{\theta}$ in spin space. Our starting point is Eq. (3.11). Each ladder is multiplied from the left and the right by the permutation tensor ϵ_{pik} selecting fluctuations which belong to perpendicular axes. We fix $p=3$, thus $i,k \in \{1,2\}$, i.e., there is only a coupling between

$$L^{11}_{12,12};\ L^{11}_{21,12};\ L^{11}_{21,21};L^{11}_{12,21}$$

because each internal unit vector $\hat{q}_j$ must be paired. The fluctuations indicated here contribute to the longitudinal part of the order-parameter susceptibility. In order to get those fluctuations due to the transversal part, we have to choose $p=1$ or $p=2$.

We are left with a (2×2) matrix equation. Moreover, the cross susceptibilities $\chi_{\delta\theta_1,\delta\theta_2}$ etc., vanish completely on account of the proper choice of the axes. The system is decoupled in three independent problems.

$$\chi_{\delta\theta_3,\delta\theta_3}(\vec{k}) = \frac{3v(0)a^2\gamma^2}{16\Gamma^\phi}\epsilon_{3ik}L^{11}_{ik,jl}\epsilon_{3jl}$$
$$= \frac{3}{8}\frac{v(0)a^2\gamma^2}{\Gamma^\phi}Tr_2[\underline{P}\underline{L}^{11}(1-\Gamma^\phi\underline{L}^{11})^{-1}] \tag{3.28}$$

where

$$P = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$$

and the reduced $\underline{L}^{11}$ is defined by

$$\underline{L}^{11} = \begin{pmatrix} L^{11}_{12,12} & L^{11}_{12,21}\\ L^{11}_{21,12} & L^{11}_{21,21}\end{pmatrix} \quad . \tag{3.29}$$

We rotate the matrices under the trace with the transformation T

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

and arrive at

$$\chi_{\delta\theta_3,\delta\theta_3}(\vec{k}) = \frac{3}{8}\frac{\gamma^2 v(0)a^2}{\Gamma^\phi}\frac{1}{1-\Gamma^\phi(L^{11}_{12,12}-L^{11}_{21,21})} \quad . \tag{3.30}$$

Inserting (3.11) and (3.6) we get

$$\chi_{\delta\theta_3,\delta\theta_3}(\vec{k}) = 10\frac{m^2}{\rho_s k^2} \quad . \tag{3.31}$$

The transversal susceptibility can be calculated along the same path. $L^{11}_{23,23}$, however, is not identical to $L^{11}_{32,32}$ with respect to the k dependent terms. Thus, the transformed matrix equation becomes diagonal only if $k=0$. On the other hand, the regular part couples only in the order k^4 according to the matrix identity (3.24) and can be neglected

$$\chi_{\delta\theta_1,\delta\theta_1}(\vec{k}) = \chi_{\delta\theta_2,\delta\theta_2}(\vec{k}) = 5\frac{m^2}{\rho_s k^2} \quad . \tag{3.32}$$

Within the BCS approximation, the three hydrodynamic order-parameter susceptibilities, present in the isotropic phase, maintain fixed ratios with respect to each other for all temperatures:

$$\chi_{\delta\phi,\delta\phi}:\chi_{\delta\theta_1,\delta\theta_1}:\chi_{\delta\theta_3,\delta\theta_3} = 1:5:10 \quad . \tag{3.33}$$

This feature is an artefact of the approximation and will disappear if we take Landau corrections into account.

IV. LANDAU CORRECTIONS

Up to this point, the $1/k^2$ divergence of $\chi_{\delta\phi,\delta\phi}$ and $\chi_{\delta\theta,\delta\theta}$ has been established. It was shown that the different characters of the order parameter variables are fixed by the structure of the gap equation. However, the results of Sec. III correspond to a weak-coupling model and do not describe the actual behavior of superfluid ^{3}He. The large values of the Landau parameters indicate, that the quasiparticle interaction is very strong and must not be neglected. Thus, the coupling of fluctuations in the anomalous channels with the normal channels has to be included in a substantial theory.

We return to Eqs. (2.25)–(2.32) which contain all information about fluctuations in the normal and anomalous channels as well as their coupling to each other. However, we are only interested in the anomalous part of $\mathfrak{L}$ which describes the static and dynamical response to fluctuations of the order-parameter variables. According to our generalized RPA approximation (2.11), $\mathfrak{L}$ is calculated from

$$\underline{\mathfrak{L}} = \underline{\Gamma}^{\omega-1}(\underline{1}-\underline{\Gamma}^\omega\underline{L})^{-1} \quad , \tag{4.1}$$

where an additional incoherent part is omitted. The crucial object is the inverse of $(\underline{1}-\underline{\Gamma}^\omega\underline{L})$. The anomalous part can be extracted applying the matrix identity (3.24). This procedure can be understood as a renormalization of the bare L^a, where all possible propagations of fluctuations in the normal channels have been taken into account:

$$\underline{1}-\underline{\Gamma}^\omega\underline{L} = \begin{pmatrix}\underline{1}-\Gamma^\phi\underline{L}^a & -\Gamma^\phi(\underline{L}^{s,a})^\dagger & -\Gamma^\phi(\underline{L}^{T,a})^\dagger\\ -\Gamma^s\underline{L}^{s,a} & \underline{1}-\Gamma^s\underline{L}^s & 0\\ -\Gamma^T\underline{L}^{T,a} & 0 & \underline{1}-\Gamma^T\underline{L}^T\end{pmatrix} \tag{4.2}$$

we take the choice

$$A = \underline{1} - \Gamma^{\phi}\underline{L}^{a} \ ,$$

$$B_1 = B_2^{\dagger} = (-\Gamma^{s}\underline{L}^{s,a},\ -\Gamma^{T}\underline{L}^{T,a}) \ ,$$

and

$$C = \begin{pmatrix} \underline{1} - \Gamma^{s}\underline{L}^{s} & 0 \\ 0 & \underline{1} - \Gamma^{T}\underline{L}^{T} \end{pmatrix} \ . \tag{4.3}$$

Thus, we may substitute

$$\underline{1} - \Gamma^{\phi}\underline{L}^{a} \rightarrow \underline{1} - \Gamma^{\phi}[\underline{L}^{a} + (\underline{L}^{s,a})^{\dagger}(\underline{1} - \Gamma^{s}\underline{L}^{s})^{-1}\underline{L}^{s,a} + (\underline{L}^{T,a})^{\dagger}(\underline{1} - \Gamma^{T}\underline{L}^{T})^{-1}\underline{L}^{T,a}] \ . \tag{4.4}$$

The p-wave assumption makes it possible to rewrite the BCS set of integral equations into a set of linear algebraic equations. This procedure cannot be applied within the normal channels if we want to keep all Landau parameters. Thus, we are confronted with a mixture of both algebraic and integral forms. In a first step we have to solve the integral equations and then may advance to tackle the algebraic problem.

To begin with, we proceed as we have done in the previous chapter algebraizing the equations. We get

$$\underline{L}_{kl} = \hat{q}_k \underline{L}^{a} \hat{q}_l + \hat{q}_k (\underline{L}^{s,a})^{\dagger}(\underline{1} - \Gamma^{s}\underline{L}^{s})^{-1}\underline{L}^{s,a}\hat{q}_l + \hat{q}_k (\underline{L}^{T,a})^{\dagger}(\underline{1} - \Gamma^{T}\underline{L}^{T})^{-1}\underline{L}^{T,a}\hat{q}_l \ , \tag{4.5}$$

where integration with respect to $\hat{q}$ is implied by the notation. $\underline{L}_{k,l}$ describes fluctuations of the order-parameter variables $\hat{A}_{ik}, \hat{A}_{ik}^{\dagger}$ including Fermi-liquid corrections. This expression generalizes Eq. (3.9). Again, it is good policy to pass from $\hat{A}_{ik}$ and $\hat{A}_{ik}^{\dagger}$ to $\hat{A}_{ik}^{\dagger} + \hat{A}_{ik}$ and $\hat{A}_{ik}^{\dagger} - \hat{A}_{ik}$ by means of the particle-hole transformation:

$$\underline{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \ .$$

Exploiting the symmetry properties of GF, $G^{-}F$, etc. (cf. Leggett[18]) the spin and the real space decouple completely, analogous to the case of pure BCS interaction, and we arrive at renormalized $\underline{L}^{11}$ and $\underline{L}^{22}$ [cf. Eqs. (3.11) and (3.12)]

$$L_{ik,jl}^{11} = \int \frac{d\Omega}{4\pi} \hat{q}_k \hat{q}_l [-\delta_{ij} G^{-} G + FF(2\hat{q}_i \hat{q}_j - \delta_{ij})] + 4 \int \frac{d\Omega}{4\pi} \hat{q}_k \epsilon_{ipr} \hat{q}_p GF \int \frac{d\Omega'}{4\pi} \hat{q}_l' \epsilon_{sqj} \hat{q}_q' GF' \Gamma_{rs}^{R}(\hat{q};\hat{q}') \ , \tag{4.6}$$

where

$$\{\delta_{rp} - \Gamma^{T}[GG\delta_{rp} - FF(2\hat{q}_r \hat{q}_p - \delta_{rp})]\}^{-1}\Gamma^{T} = \Gamma_{rp}^{R} \tag{4.7}$$

and

$$L_{ik,jl}^{22} = \int \frac{d\Omega}{4\pi} \hat{q}_k \hat{q}_l [-\delta_{ij} G^{-} G - FF(2\hat{q}_i \hat{q}_j - \delta_{ij})] + 4 \int \frac{d\Omega}{4\pi} \hat{q}_i \hat{q}_k GF \int \frac{d\Omega'}{4\pi} \hat{q}_j' \hat{q}_l' GF' \Gamma^{R}(\hat{q};\hat{q}') \tag{4.8}$$

where

$$[1 - \Gamma^{s}(GG + FF)]^{-1}\Gamma^{s} = \Gamma^{R} \ . \tag{4.9}$$

L^{22} contains the fluctuations of real space and L^{11} those of spin space.

Due to the isotropy of the order parameter the real space is much simpler to deal with than the spin space. Thus, we start again with the static susceptibility of $\delta\phi$. Each GF contributes a factor $\vec{k}$ to the expression (4.8), i.e., the additional term is at least proportional k^2. We are exclusively interested in the lowest k order, i.e., in k^2, therefore, only the $k = 0$ part of Γ^{R} contributes. In that case, the solution of (4.9) is rather simple:

$$\Gamma^{R}(\hat{q};\hat{q}') + \rho_n^0(T) \int \frac{d\overline{\Omega}}{4\pi} \Gamma^{s}(\hat{q}\cdot\overline{q})\Gamma^{R}(\overline{q};\hat{q}') = \Gamma^{s}(\hat{q}\cdot\hat{q}') \ . \tag{4.10}$$

The isotropy of $\rho_n^0(T)$—the normal-fluid density—and Γ^{s} induce that $\Gamma^{R}(\hat{q};\hat{q}')$ is also isotropic. $\Gamma^{s}(\hat{q}\cdot\hat{q}')$ is expanded in terms of Legendre polynomials

$$\Gamma^{s}(\hat{q}\cdot\hat{q}') = \sum_{l=0}^{\infty} F_l^{s} P_l(\hat{q}\cdot\hat{q}') \ , \tag{4.11}$$

where F_l^{s} are the symmetric Landau parameters and we obtain

$$\Gamma^{R}(\hat{q}\cdot\hat{q}') = \sum_{l=0}^{\infty} \frac{F_l^{s}}{1 + [1/(2l+1)]F_l^{s}\rho_n^0} P_l(\hat{q}\cdot\hat{q}') \ . \tag{4.12}$$

The static susceptibility of $\delta\phi$ is constructed of order-parameter fluctuations along parallel spin- and real-space axes [cf. Eq. (3.16)]. We pointed out, that the fluctuations on parallel axes never couple to fluctuations on perpendicular axes within the BCS model. This remains also true if we take the coupling to the fluctuations in the normal channels into account: We start from the left with a parallel fluctuation (i.e., $i=k;GF\sim\hat{q}_3k$). The Landau correction to $\underline{L}^{22}$ takes the form

$$\sim k^2\int\frac{d\Omega}{4\pi}\hat{q}_i^2\hat{q}_3\int\frac{d\Omega'}{4\pi}\hat{q}_j'\hat{q}_l'\hat{q}_3'\Gamma^R(\hat{q}\cdot\hat{q}') \ .$$

The solid angle integral on the left selects the P_1 and P_3 part of the renormalized Landau vertex Γ^R. Making use of the addition theorem for spherical harmonics, we observe at once that this expression does only contribute if $j=l\in\{1,2,3\}$.

We pass to the smaller matrix (3.15) and perform the transformations indicated there. All its elements are either constant or proportional to k^2 and its regular part couples only to the order k^4 if we look for the inverse. Therefore, we may put aside these terms altogether and arrive at

$$\chi_{\delta\phi,\delta\phi}(\vec{k})=\frac{\gamma^2a^2v(0)}{4(\Gamma^\phi)^2}\left[\int\frac{d\Omega}{4\pi}\frac{1}{2}\frac{(\vec{v}_F\vec{k})^2}{\Delta^2}\lambda+4\sum_{i,k}\int\frac{d\Omega}{4\pi}\hat{q}_i^2\frac{(\vec{v}\cdot\vec{k})}{\Delta}\lambda\int\frac{d\Omega'}{4\pi}\hat{q}_k'^2\frac{(\vec{k}\,\vec{v}_F')}{\Delta}\lambda'\Gamma^R(\hat{q}\cdot\hat{q}')\right]^{-1} \ . \tag{4.13}$$

Exploiting the sum over the unit vectors, the equation is further simplified

$$\chi_{\delta\phi,\delta\phi}(\vec{k})=\frac{m^*m}{\rho}\left[\rho_s^0k^2+k^2(\rho_s^0)^2\int\frac{d\Omega}{4\pi}\hat{q}_3\int\frac{d\Omega'}{4\pi}\hat{q}_3'\Gamma^R(\hat{q}\cdot\hat{q}')\right]^{-1} \ . \tag{4.14}$$

The remaining $\hat{q}_3$ selects the P_1 part of Γ^R:

$$\chi_{\delta\phi,\delta\phi}(\vec{k})=\frac{m^2}{\rho_sk^2}(1+\tfrac{1}{3}F_1^s\rho_n^0) \ . \tag{4.15}$$

This result is valid for all temperatures and includes all Landau parameters. It is identical to the expression, we get in the neutral singlet superconductor.

If it were only for the static susceptibility of the phase, the result would not justify the efforts we have undertaken as this expression can be achieved by simple sum-rule arguments.

The matter is different, if we turn to the susceptibilities of $\delta\vec{\theta}$. The k^2 dependence of the Landau term makes sure that we can select exactly those couplings to the normal channel, which are inherent in the anomalous channel. All additional contributions are of the order k^4 and can be neglected. Thus, Eq. (3.30) remains valid if we replace the BCS approximation $L_{12,12}^{11}$, etc., by their renormalized counterparts. The susceptibilities on different axes can be summarized by

$$\chi_{\delta\theta_t,\delta\theta_t}(\vec{k})=\frac{3v(0)a^2\gamma^2}{16\Gamma^\phi}[1-\Gamma^\phi(\epsilon_{tik}L_{ik,jl}^{11}\epsilon_{tjl})]^{-1} \ . \tag{4.16}$$

The index t is not to be summed over. The BCS part of $\underline{L}^{11}$ has already been calculated in Sec. III and we will restrict our attention to the Landau corrections. The strong-coupling contribution to $\chi_{\delta\theta_t,\delta\theta_t}$ takes the form [compare Eqs. (4.6) and (4.7)]

$$4\int\frac{d\Omega}{4\pi}\hat{q}_k\hat{q}_p\epsilon_{tik}\epsilon_{ipr}GF\int\frac{d\Omega'}{4\pi}\hat{q}_l'\hat{q}_q'\epsilon_{sqj}\epsilon_{tjl}GF'\Gamma_{rs}^R(\hat{q};\hat{q}') \ , \tag{4.17}$$

where $\Gamma_{rs}^R(\hat{q};\hat{q}')$ is determined by a system of nine coupled integral equations:

$$\Gamma_{st}^R(\hat{q};\hat{q}')+\int\frac{d\overline{\Omega}}{4\pi}\Gamma^T(\hat{q}\cdot\overline{q})(A\delta_{sw}+B\overline{q}_s\overline{q}_w)\Gamma_{wt}^R(\overline{q};\hat{q}')=\delta_{st}\Gamma^T(\hat{q}\cdot\hat{q}') \ . \tag{4.18}$$

$A=\rho_n^0$ is the normal-fluid density and $B=\rho_s^0$ the superfluid density. Both functions are isotropic and do not change the symmetry of the equation. Expression (4.17) is simplified evaluating the sums over the permutation tensors:

$$4\left[\int\frac{d\Omega}{4\pi}GF\int\frac{d\Omega'}{4\pi}GF'\Gamma_{tt}^R(\hat{q};\hat{q}')+\int\frac{d\Omega}{4\pi}\hat{q}_t\hat{q}_rGF\int\frac{d\Omega'}{4\pi}\hat{q}_s'\hat{q}_t'GF'\Gamma_{rs}^R(\hat{q};\hat{q}')\right.$$
$$\left.-\int\frac{d\Omega}{4\pi}GF\int\frac{d\Omega'}{4\pi}\Gamma_{ts}^R(\hat{q};\hat{q}')\hat{q}_s'\hat{q}_t'GF'-\int\frac{d\Omega}{4\pi}\hat{q}_t\hat{q}_rGF\int\frac{d\Omega'}{4\pi}GF'\Gamma_{rt}^R(\hat{q};\hat{q}')\right] \ . \tag{4.19}$$

Three different forms of the renormalized interaction vertex Γ^R_{rs} appear: a diagonal part Γ^R_{tt}, $\hat{q}_r\Gamma^R_{rt}(\hat{q};\hat{q}')$, and $\Gamma^R_{tr}(\hat{q};\hat{q}')\hat{q}_r'$. Again, we taken the $\vec{k}$ vector parallel to the z axis. Thus GF contributes a factor

$$GF=-\frac{1}{4}\frac{v_F}{\Delta}\rho_s^0 k\hat{q}_3 \ . \tag{4.20}$$

It will prove to be convenient to include one superfluid density ρ_s^0 into the interaction vertex Γ^R_{rs}:

$$\Gamma^R_{rs}\rho_s^0 \to \Gamma_{rs} \ .$$

Γ^T on the right of Eq. (4.18) is now substituted by $\Gamma^T\cdot\rho_s^0$. Next, we define an operator

$$1+A\int\frac{d\overline{\Omega}}{4\pi}\Gamma^T(\hat{q}\cdot\bar{q})$$

and apply its inverse from the left to Eq. (4.18). This gives rise to a new integral equation for Γ_{rs}

$$\Gamma_{st}(\hat{q};\hat{q}')+\int\frac{d\overline{\Omega}}{4\pi}\tilde{\Gamma}(T;\hat{q}\cdot\bar{q})\bar{q}_s\bar{q}_\omega\Gamma_{\omega t}(\bar{q};\hat{q}') = \delta_{st}\tilde{\Gamma}(T;\hat{q}\cdot\hat{q}') \ , \tag{4.21}$$

where $\tilde{\Gamma}$ is determined by

$$\tilde{\Gamma}(\hat{q}\cdot\hat{q}')+\rho_n^0\int\frac{d\overline{\Omega}}{4\pi}\Gamma^T(\hat{q}\cdot\bar{q})\tilde{\Gamma}(\bar{q}\cdot\hat{q}') = \rho_s^0\Gamma^T(\hat{q}\cdot\hat{q}') \ . \tag{4.22}$$

Γ^T is, as usual, expanded in terms of Legendre polynomials

$$\Gamma^T(\hat{q}\cdot\hat{q}')=\sum_{l=0}^{\infty}F_l^a P_l(\hat{q}\cdot\hat{q}') \ , \tag{4.23}$$

where F_l^a are the antisymmetric Landau parameters. Equation (4.22) can be solved immediately

$$\tilde{\Gamma}(T;\hat{q}\cdot\hat{q}')=\sum_{l=0}^{\infty}\frac{F_l^a\rho_s^0}{1+[1/(2l+1)]F_l^a\rho_n^0}P_l(\hat{q}\cdot\hat{q}') \equiv \sum_{l=0}^{\infty}\tilde{F}_l P_l(\hat{q}\cdot\hat{q}') \ . \tag{4.24}$$

Thus, all temperature dependence of Eq. (4.18) has been included in a scaling of the Landau parameters. Equation (4.21) reveals some useful symmetry properties of Γ_{rs}

$$\Gamma_{rs}(\hat{q};\hat{q}')=\Gamma_{sr}(\hat{q}';\hat{q}) \tag{4.25}$$

and

$$\Gamma_{rs}(\hat{q};\hat{q}')=\Gamma_{rs}(-\hat{q};-\hat{q}') \ .$$

We noted above that, besides a diagonal term, only the expressions $\hat{q}_r\Gamma_{rs}$ and $\Gamma_{sr}\hat{q}_r'$ are necessary to evaluate the Landau corrections. We can avoid the utmost cumbersome handling of nine coupled integral equations (4.21) defining a vector $G_s(\hat{q};\hat{q}')$

$$G_s(\hat{q};\hat{q}')\equiv\Gamma_{sr}(\hat{q};\hat{q}')\hat{q}_r'=\hat{q}_r'\Gamma_{rs}(\hat{q}';\hat{q}) \tag{4.26}$$

and thus

$$\hat{q}_r\Gamma_{rs}(\hat{q};\hat{q}')=G_s(\hat{q}';\hat{q}) \ . \tag{4.27}$$

The renormalization equation (4.21) does not depend on $\vec{k}$. The only vector, from which the index s can be constructed, is the unit vector $\hat{q}_s$. Thus, G_s is built up of two isotropic functions $\psi(\hat{q}\cdot\hat{q}')$ and $\phi(\hat{q}\cdot\hat{q}')$

$$G_s(\hat{q};\hat{q}')=\psi(\hat{q}\cdot\hat{q}')\hat{q}_s'+\phi(\hat{q}\cdot\hat{q}')\hat{q}_s \ , \tag{4.28}$$

which can be expanded in terms of Legendre polynomials

$$\psi(\hat{q}\cdot\hat{q}')=\sum_l\psi_l P_l(\hat{q}\cdot\hat{q}') \ ,$$
$$\phi(\hat{q}\cdot\hat{q}')=\sum_l\phi_l P_l(\hat{q}\cdot\hat{q}') \ . \tag{4.29}$$

G_s is determined from Eq. (4.21) multiplying it from the right with $\hat{q}_t'$

$$G_s(\hat{q};\hat{q}')+\int\frac{d\overline{\Omega}}{4\pi}\tilde{\Gamma}(\hat{q}\cdot\bar{q})\bar{q}_s\bar{q}_w G_w(\bar{q};\hat{q}') = \hat{q}_s'\tilde{\Gamma}(\hat{q}\cdot\hat{q}') \ . \tag{4.30}$$

If G_s is known, we can apply once more Eq. (4.21) in order to calculate Γ_{st}. Making use of the symmetry properties defined above, we arrive at

$$\Gamma_{st}(\hat{q};\hat{q}')=\tilde{\Gamma}(\hat{q}\cdot\hat{q}')\delta_{st} - \int\frac{d\overline{\Omega}}{4\pi}\tilde{\Gamma}(\hat{q}\cdot\bar{q})\bar{q}_s G_t(\hat{q}';\bar{q}) \ . \tag{4.31}$$

Before solving Eqs. (4.30) and (4.31), let us turn once more to the static susceptibilities $\chi_{\delta\theta_t,\delta\theta_t}$. Their BCS value is very simple and has been calculated in the last chapter. In order to clarify the influence of the Landau parameters we define two temperature-dependent functions which indicate the deviation of the longitudinal ($t=3$) and the transversal ($t=1,2$) static susceptibility from their BCS value

$$\chi_l^{-1}(\vec{k})=\chi_l^{-1}(\vec{k})|_{\rm BCS}R_l \ ,$$
$$\chi_t^{-1}(\vec{k})=\chi_t^{-1}(\vec{k})|_{\rm BCS}R_t \ . \tag{4.32}$$

We insert the expansion of G_s in terms of ϕ_l and ψ_l into Eq. (4.19) and this into (4.16) and obtain exploiting the BCS results of Sec. III

$$R_l=1+\tfrac{1}{30}(25C_l-19\psi_1-25\phi_0-4\phi_2+\tfrac{12}{7}\psi_3) \tag{4.33}$$

and

$$R_t=1+\tfrac{1}{60}(25C_t-8\psi_1+\tfrac{9}{7}\psi_3-3\phi_2) \ . \tag{4.34}$$

The constants C_l and C_t are due to the first term in Eq. (4.19):

$$\begin{Bmatrix} C_l \\ C_t \end{Bmatrix} = 9\int \frac{d\Omega}{4\pi}\hat{q}_3 \int \frac{d\Omega'}{4\pi}\hat{q}_3' \begin{Bmatrix} \Gamma_{33}(\hat{q};\hat{q}') \\ \Gamma_{22}(\hat{q};\hat{q}') \end{Bmatrix} . \quad (4.35)$$

We can take another point of view. $\hat{q}_3$ is identical to $P_1(\cos\theta)$, so the double integral selects the C_{11}^{00} part of an expansion of Γ_{rs} in terms of spherical harmonics:

$$\Gamma_{rs}(\hat{q};\hat{q}') = (C_{ll'}^{mm'})_{rs} P_l^m(\cos\theta) P_{l'}^{m'}(\cos\theta')\cos m\phi \cos m'\phi' + (D_{ll'}^{mm'})_{rs} P_l^m(\cos\theta) P_{l'}^{m'}(\cos\theta') \times \sin m\phi \sin m'\phi' . \quad (4.36)$$

Thus, we have only to look for terms proportional to $P_1(\cos\theta)P_1(\cos\theta')$ in Eq. (4.31) in order to extract the relevant C_l and C_t:

$$C_l = \tilde{F}_1 - \tfrac{1}{75}\tilde{F}_1(15\psi_1 + 25\phi_0 + 4\phi_2) , \quad (4.37)$$

$$C_t = \tilde{F}_1 - \tfrac{1}{75}\tilde{F}_1(5\psi_1 + 3\phi_2) . \quad (4.38)$$

The four constants ψ_1, ψ_3, ϕ_0, and ϕ_2 are determined from Eq. (4.30). To determine $\psi(\hat{q}\cdot\hat{q}')$ and $\phi(\hat{q}\cdot\hat{q}')$ we multiply (4.30) with $\hat{q}_s'$ and $\hat{q}_s$, respectively. We get two coupled integral equations, which can be transformed into algebraic ones applying the expansions (4.29) and the addition theorem for spherical harmonics:

$$\phi_l = \frac{a_{l-1}'\tilde{F}_{l-1} + a_{l+1}''\tilde{F}_{l+1}}{1 + [1/(2l+1)](a_{l-1}'\tilde{F}_{l-1} + a_{l+1}''\tilde{F}_{l+1})} - a_{l-1}'\psi_{l-1} - a_{l+1}''\psi_{l+1} , \quad (4.39)$$

$$\frac{\tilde{F}_l}{2l+1}a_{l-2}'a_{l-1}'\psi_{l-2} + \left[1 + \frac{\tilde{F}_l}{2l+1}(a_{l-1}'a_l'' + a_l'a_{l+1}'')\right]\psi_l + \frac{\tilde{F}_l}{2l+1}a_{l+1}''a_{l+2}''\psi_{l+2} + \left[1 + \frac{\tilde{F}_l}{2l+1}\right](a_{l-1}'\phi_{l-1} + a_{l+1}''\phi_{l+1}) = \tilde{F}_l , \quad (4.40)$$

where $a_l' = (l+1)/(2l+1)$ and $a_l'' = l/(2l+1)$. The system of equations is decoupled for even and odd l, respectively. According to the Landau parameters which enter into our problem, we are only concerned with odd l's. Inserting ϕ_l into Eq. (4.40), the system is further reduced

$$-a_{l-2}'a_{l-1}'\psi_{l-2} + (1 - a_{l-1}'a_l'' - a_l'a_{l+1}'')\psi_l - a_{l+1}''a_{l+2}''\psi_{l+2} = \tilde{F}_l - a_{l-1}'\frac{a_{l-2}'\tilde{F}_{l-2} + a_l''\tilde{F}_l}{1 + [1/(2l+1)](a_{l-2}'\tilde{F}_{l-2} + a_l''\tilde{F}_l)} - a_{l+1}''\frac{a_l'\tilde{F}_l + a_{l+2}''\tilde{F}_{l+2}}{1 + [1/(2l+3)](a_l'\tilde{F}_l + a_{l+2}''\tilde{F}_{l+2})} . \quad (4.41)$$

In order to solve the system, we start with the first equation ($l=1$) and subtract it from the equation due to ($l=3$). The new equation is then subtracted from that due to $l=5$ and so on. We arrive at

$$a_l'a_{l+1}'\psi_l - a_{l+1}''a_{l+2}''\psi_{l+2} = K_l \text{ for } l = 1,3,5,\ldots , \quad (4.42)$$

where

$$K_l = a_l'\tilde{F}_l - \left[1 + \frac{\tilde{F}_l}{2l+1}\right]a_{l+1}''\frac{a_l'\tilde{F}_l + a_{l+2}''\tilde{F}_{l+2}}{1 + [1/(2l+3)](a_l'\tilde{F}_l + a_{l+2}''\tilde{F}_{l+2})} . \quad (4.43)$$

Apparently, the general solution of (4.42) is

$$\psi_1 = \frac{1}{a_1'a_2'}K_1 + \frac{a_2''a_3''}{a_1'a_2'a_3'a_4'}K_3 + \frac{a_2''a_3''a_4''a_5''}{a_1'a_2'a_3'a_4'a_5'a_6'}K_5 + \cdots ,$$

$$\psi_3 = \frac{1}{a_3'a_4'}K_3 + \frac{a_4''a_5''}{a_3'a_4'a_5'a_6'}K_5 + \frac{a_4''a_5''a_6''a_7''}{a_3'a_4'a_5'a_6'a_7'a_8'}K_7 + \cdots , \quad (4.44)$$

$\psi_5 = \cdots$ etc.

Inserting ψ_l into Eq. (4.39) we obtain ϕ_l and from those, G_s and Γ_{st} are reconstructed, respectively. However, we need not carry out this program, and, the complete solution of ψ_1, ψ_3, ϕ_0, and ϕ_2 is not required.

To demonstrate this proposition, we substitute ϕ_0 and ϕ_2 in R_l and R_t by ψ_1 and ψ_3 according to Eq. (4.39). It is a matter of simple algebra to convince oneself, that ψ_1 and ψ_3 enter the problem only in the linear combination

$$\psi_1 - \tfrac{3}{7}\psi_3 . \quad (4.45)$$

This, however, is the same expression which occurs in the first equation ($l=1$) of the algebraic system (4.42):

$$\psi_1 - \tfrac{3}{7}\psi_3 = 3\frac{\tfrac{1}{3}\tilde{F}_1 - \tfrac{1}{7}\tilde{F}_3}{1+\tfrac{2}{15}\tilde{F}_1+\tfrac{3}{35}\tilde{F}_3} \ . \tag{4.46}$$

We point out that this is true in the whole temperature regime and without any assumption on the Landau parameters. In particular, we have not assumed that the higher Landau parameters vanish, as it is usually done in the literature. Now, we can insert (4.46) into R_l and R_l [Eqs. (4.33), (4.34), (4.37), and (4.38)] and obtain finally:

$$R_l = \frac{(1+\tfrac{1}{3}\tilde{F}_1)(1+\tfrac{1}{7}\tilde{F}_3)}{1+\tfrac{2}{15}\tilde{F}_1+\tfrac{3}{35}\tilde{F}_3} \ , \tag{4.47}$$

$$R_t = \frac{1}{4}\frac{(1+\tfrac{1}{3}\tilde{F}_1)}{1+\tfrac{2}{15}\tilde{F}_1+\tfrac{3}{35}\tilde{F}_3}(4+\tfrac{1}{3}\tilde{F}_1+\tfrac{3}{7}\tilde{F}_3) \ . \tag{4.48}$$

These results are exact within the scope of the mean-field approximation. The crucial ratio of the various static susceptibilities of the hydrodynamic order parameters mentioned above [cf. Eq. (3.35)] is alterated to

$$\chi_{\delta\phi} : \chi_t : \chi_l = R_\phi^{-1} : 5R_t^{-1} : 10R_l^{-1} \ , \tag{4.49}$$

where $R_\phi = 1/(1+\tfrac{1}{3}\tilde{F}_1^s)$.

V. DISCUSSION

The susceptibilities studied above, have also been calculated by Brinkman and Smith[15] and by Cross.[17] Brinkman and Smith started from a Ginzburg-Landau functional for the triplet order parameter A_{ij} of ^{3}He and specialized it for the case of the B phase. Their result is restricted to temperatures near T_c. The inverse static susceptibility is obtained by differentiating the functional twice with respect to the variable $\delta\phi$ or $\delta\theta_i$. The BCS approximation expressions, calculated in Sec. III, are in complete agreement with theirs. They were the first who indicated that the ratio $\chi_l:\chi_t$ is temperature independent within the BCS approximation.

The work of Cross went far beyond that. He generalized a method developed by Werthamer[23] in order to calculate the current and free energy in s-wave superconductors to the case of superfluid ^{3}He. His starting point was the Gorkov equations, which he exploited to get an expansion of the Green's function in terms of the wave vector $\vec{k}$. Out of these he could determine the current and the free energy. Landau corrections have been taken into account, but he restricted himself to F_1^a and F_1^s. His results emerge as a special case from ours if we put $F_3 \equiv 0$.

We were concerned with the linear-response functions of the hydrodynamic order-parameter variables. Were it not for a terrible amount of algebraic work, we could, in principal, also evaluate the dynamic correlation functions by the same method. In that case, we would expect to recover all the hydrodynamic modes, predicted by the phenomenological theory, and beyond that, a lot of microscopic modes (i.e., modes with finite frequency in the limit $k\to 0$). As far as the static susceptibilities are concerned, we arrived, although on a quite different path, at results which are in complete agreement with those of Cross, if F_3^a is neglected. Our results for $F_3^a \neq 0$ are new.

We note some general features of the new results: (a) Within the mean-field approximation, the static susceptibilities of the hydrodynamic order-parameter variables in spin space involve only two Landau parameters F_1^a and F_3^a.

$$\chi_l(\vec{k}) = 10\left[\frac{m^*}{n\hbar^2}\right]\frac{1}{\rho_s^0 k^2}R_l^{-1} \ , \tag{5.1}$$

$$\chi_t(\vec{k}) = 5\left[\frac{m^*}{n\hbar^2}\right]\frac{1}{\rho_s^0 k^2}R_t^{-1} \ , \tag{5.2}$$

where n is the particle density of ^{3}He and

$$R_l = \frac{(1+\tfrac{1}{3}F_1^a)(1+\tfrac{1}{7}F_3^a)}{1+\tfrac{1}{3}F_1^a\rho_n^0+\tfrac{2}{15}F_1^a\rho_s^0+\tfrac{1}{7}F_3^a\rho_n^0+\tfrac{3}{35}F_3^a\rho_s^0+\tfrac{1}{21}F_1^aF_3^a(\rho_n^0)^2} \ , \tag{5.3}$$

$$R_t = \frac{1}{4}\frac{(1+\tfrac{1}{3}F_1^a)(1+\tfrac{1}{7}F_3^a\rho_n^0)}{1+\tfrac{1}{3}F_1^a\rho_n^0+\tfrac{2}{15}F_1^a\rho_s^0+\tfrac{1}{7}F_3^a\rho_n^0+\tfrac{3}{35}F_3^a\rho_s^0+\tfrac{1}{21}F_1^aF_3^a(\rho_n^0)^2}\left[4+\frac{1}{3}\frac{F_1^a\rho_s^0}{1+\tfrac{1}{3}F_1^a\rho_n^0}+\frac{3}{7}\frac{F_3^a\rho_s^0}{1+\tfrac{1}{7}F_3^a\rho_n^0}\right] \ . \tag{5.4}$$

(b) Below T_c, the total influence of temperature and the strong-coupling effects can be gathered defining new temperature-dependent Landau parameters

$$\tilde{F}_l = \frac{F_l\rho_s^0}{1+[1/(2l+1)]F_l\rho_n^0} \ . \tag{5.5}$$

Thus, we can preceive each change of temperature or pressure as the change of an effective Landau parameter $\tilde{F}_l$; This feature of the renormalized Landau scattering vertex Γ_{rs} will emerge in all expressions, which involve fluctuations in the normal channels.

(c) We are now prepared to write down the gradient part of the superfluid free energy, exact within the scope to mean-field theory

$$F_s = \frac{2}{8m}\rho_s(\nabla\phi)^2 + \frac{\rho_s^0}{10m^*} \times \{R_t[(\nabla_i\theta_\alpha)(\nabla_i\theta_\alpha) - (\nabla_i\theta_\alpha)(\nabla_\alpha\theta_i)] + \tfrac{1}{4}R_l[3(\nabla_i\theta_\alpha)(\nabla_\alpha\theta_i) - (\nabla_i\theta_i)^2]\} \;, \qquad (5.6)$$

where we have made use of the expression given by Cross.[17]

From the experimental point of view, the static susceptibilities of $\delta\vec{\theta}$ influence two measurable quantities. They enter the spin-wave velocities $c_\parallel^2$ and $c_\perp^2$ and the structure of magnetic solitons. As for the spin-space velocities, the linear hydrodynamics obtains[12]

$$c_\perp^2 = \gamma^2(\chi_M\chi_t)^{-1}, \quad c_\parallel^2 = \gamma^2(\chi_M\chi_l)^{-1} \;, \qquad (5.7)$$

where γ is the gyromagnetic ratio, χ_M the static susceptibility of the magnetization, and χ_l, χ_t

$$\chi_t(\vec{k}) = \chi_t/k^2, \quad \chi_l(\vec{k}) = \chi_l/k^2 \;, \qquad (5.8)$$

respectively. Thus, the ratio of $c_\perp^2$ and $c_\parallel^2$ depends only on F_1^q and F_3^q and the temperature

$$\frac{c_\perp^2}{c_\parallel^2} = \frac{2R_t}{R_l} \;. \qquad (5.9)$$

Equation (5.9) predicts a fixed relation of F_1^q and F_3^q. It may be used as a test of the validity of the mean-field approximation which was applied to derive it.

The influence of F_1^q and F_3^q may also be seen in the bound spin-wave states associated with the $\vec{n}$ solitons. Maki and Lin Liu[24] calculated the satellite lines which occur in the NMR spectrum, if the order parameter has undergone a spacial texture. Their starting point was the gradient free energy for the spin-space order parameter. The influence of the Landau coupling has been taken into account by a parameter $\lambda(T)$ which takes the form

$$\lambda = \frac{1+\frac{1}{2}\phi}{1+\frac{1}{4}\phi}, \quad \phi = \tfrac{1}{3}\tilde{F}_1 \qquad (5.10)$$

in a model where only F_1^q is retained. In the general case, it reads

$$\lambda = \frac{2R_t - R_l}{R_l} \;. \qquad (5.11)$$

$c_\perp^2$ is another Landau-coupling dependent quantity entering their theory. From measurements of λ and $c_\perp^2$ one can determine F_1^q and F_3^q.

Up to now, only F_1^q is estimated roughly making use of the exact sum rule for forward scattering and of the *sp* approximation. The value of F_1^q (~ -0.8) indicates that the spin-wave velocity should be reduced by some 10%, but we have to bear in mind that this estimation only is valid if all F_l^a for $l > 2$ vanish. Looking for the general behavior of $c_\perp^2 \sim R_t$ and $c_\parallel^2 \sim R_l$ we predict that $c_\parallel^2$ is more influenced by F_3^q than $c_\perp^2$. If we assume, for the moment, that $F_1^q = 0$ and $F_3^q = -3.5$ then $c_\parallel^2$ would be lowered by 29%, whereas $c_\perp^2$ by 11%. The qualitative structure persists if F_1^q is finite. F_1^q, however, does not distinguish in a considerable amount $c_\perp^2$ and $c_\parallel^2$. Its total influence, however, is much stronger than that of F_3^q. If we suppose $F_1^q = -1.5$ and $F_3^q = 0$ then both velocities $c_\perp^2$ and $c_\parallel^2$ are diminished by 50%.

Although it seems possible to extract from the NMR data the antisymmetric Landau parameters F_1^q and F_3^q we must bring in a jarring note. Up to this point, we have taken into account only the mean-field correction. Other contributions, which flow from the next order of the free-energy functional Φ, have been neglected. Serene and Rainer,[21,22] however, have pointed out that there may occur substantial errors if contributions due to the next order of the expansion of the functional in terms of T_c/T_F are neglected. Serene and Rainer calculated the corrections in the temperature regime near T_c. They proved to be always less than 10%, which, however, is the same amount, we expect for F_1^q and F_3^q. These contributions can be taken into account in our calculation if we include all diagrams of Φ up to the order $(T_c/T_F)^3$. By means of a thorough analysis comparable to that of Serene and Rainer[21,22] one has to discriminate the quasiparticle and incoherent contributions to the interaction vertex Γ. This program may be carried out in principle, but it implies a substantial amount of algebraic work.

Although this reduces our confidence in quantitative results, we think it, nevertheless, interesting to check the predictions we have made in regard of the temperature and the pressure dependence as well as the ratio of $c_\parallel^2/c_\perp^2$. The discrepancy between "exact" mean-field expressions and the experimental results were a further indication on the strong coupling behavior of ^{3}He.

ACKNOWLEDGMENT

I would like to thank Professor R. Graham for critical and useful discussions.

[1]A. J. Leggett, J. Phys. C 6, 3187 (1973).
[2]A. J. Leggett, Ann. Phys. (N.Y.) 85, 11 (1979).
[3]K. Maki, J. Low Temp. Phys. 16, 465 (1974).
[4]K. Maki and H. Ebisawa, J. Low Temp. Phys. 21, 475 (1975).
[5]K. Nagai, Prog. Theor. Phys. 54, 1 (1975).
[6]K. Maki, J. Low Temp. Phys. 24, 755 (1975).
[7]J. Hara and K. Nagai, J. Low Temp. 34, 351 (1979).
[8]L. Tewordt and N. Schophol, J. Low Temp. Phys. 37, 421 (1979).
[9]P. Wölfle, Phys. Rev. B 14, 89 (1976).
[10]R. Graham, Phys. Rev. Lett. 33, 1431 (1974).
[11]R. Graham and H. Pleiner, Phys. Rev. Lett. 34, 792 (1975).
[12]R. Graham and H. Pleiner, J. Phys. C 9, 279 (1976).
[13]H. Brand, M. Dörfle, and R. Graham, Ann. Phys. (N.Y.) 119, 434 (1979).
[14]M. Dörfle, H. Brand, and R. Graham, J. Phys. C 13, 3337 (1980).
[15]W. F. Brinkman and H. Smith, Phys. Rev. A 10, 2325 (1974).
[16]V. Ambegaokar, P. G. de Gennes, and D. Rainer, Phys. Rev. A 9, 2676 (1974).
[17]M. C. Cross, J. Low Temp. Phys. 21, 525 (1975).
[18]A. J. Leggett, Phys. Rev. 140, A1869 (1965).
[19]G. Baym and L. P. Kadanoff, Phys. Rev. 124, 287 (1961).
[20]G. Baym, Phys. Rev. 127, 1391 (1962).
[21]D. Rainer and J. W. Serene, Phys. Rev. B 13, 4745 (1976).
[22]J. W. Serene and D. Rainer, Phys. Rev. B 17, 2901 (1978).
[23]N. R. Werthamer, Phys. Rev. 132, 663 (1963).
[24]K. Maki and Y. R. Lin Liu, Phys. Rev. B 17, 3558 (1978).