Effects of a random symmetry-breaking field on topological order in two dimensions

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The effect of a random symmetry-breaking field ($V=\int d\vec{x}[h_{1p}\cos p\theta(\vec{x})+h_{2p}\sin p\theta(\vec{x})]$, h_{1p} and h_{2p} random) on topological order in two-dimensional systems is studied. Such a field would simulate the interaction of the idealized two-dimensional system with an underlying substrate if the substrate were to contain patches (islands) in which short-range order is retained rather than being a single crystalline surface. These conditions might well pertain when charge-density waves are studied in chemisorption and physisorption experiments. The transition is studied using a sine-Gordon field theory modified to include the effects of disorder. It is shown that the spin-wave behavior of the planar model is stable against such perturbations provided $4<2\pi K_{\rm eff}<\frac{1}{2}p^2$. That is, planar-model critical behavior should be observable for threefold-symmetric ($p=3$) perturbations.

I. INTRODUCTION

In recent years the classical two-dimensional Heisenberg model has attracted a great deal of attention.[1-5] In the continuum limit it is represented by a reduced Hamiltonian or "action" of the form

$$A[\theta]=\frac{H}{k_BT}=\frac{1}{2}K\int d^2x[\nabla\theta(\vec{x})]^2\ , \qquad (1.1)$$
$$0<\theta<2\pi\ .$$

The properties of this system have been analyzed in great detail. It was first pointed out by Kosterlitz and Thouless[1] that the excitations of the planar model include, in addition to spin waves, vortices which interact among themselves via a logarithmic two-dimensional Coulomb interaction. These topological excitations determine the nature of the phase transition in the system. Below the critical temperature bound vortex-antivortex pairs populate an ordered phase coexisting with the spin-wave excitations. Above T_c, these pairs dissociate. Kosterlitz[2] subsequently analyzed the transition quantitatively. He constructed a simple renormalization-group transformation which contains a line of fixed points, corresponding physically to a line of critical points terminating at a finite T_c.

The importance of the planar model goes far beyond the original magnetic system. Indeed the first verification of the unique nature of the Kosterlitz-Thouless transition was the experimental measurement of the universal jump in superfluid density[6] at the transition in ^{4}He films on a Mylar substrate.[7] In ^{4}He films $\theta(\vec{x})$ is the phase angle of the superfluid order parameter. Other examples may be found in the charge-density waves observed in chemisorption[8-10] and physisorption systems[11-13]; again the order parameter is characterized by a phase angle $\theta(\vec{x})$. More complicated systems include the two-dimensional melting of floating solids as discussed recently by Halperin and Nelson[14,15] and Young.[16] To describe this problem, however, a two-component phase angle $\theta_\mu(\vec{x})$ is needed. These diverse physical systems can be represented by the Hamiltonian of Eq. (1.1) only in the idealized situation of a perfectly inert smooth substrate whose only role is to confine the system to strictly two-dimensional behavior. In actual practice the substrate represents an additional potential of symmetry lower than $O(2)$. The substrate will also contain impurities, imperfections, and other types of disorder.

The effect of disorder on the Kosterlitz-Thouless-type transition is of particular interest. It has been found[17] that a small amount of bond disorder does not affect the transition. On the other hand, a number of authors[18-20] have concluded that a random field (site disorder), such as that generated by a quenched random distribution of impurities, would couple linearly to the phase angle and destroy conventional long-range order in space dimension $d<4$. However, it is not clear whether this argument applies to the special case of $d=2$ where the phase transition is controlled by topological excitations. One can also visualize types of disorder which preserve much of the short-range order of the system. For example, consider a charge-density wave in which the displacement vector of the particles has long-wavelength modulations of the form $e^{i\theta(\vec{x})}$ around the commensurate value $e^{i\vec{q}\cdot\vec{x}}$ with $\vec{q}=\vec{G}/p$,[21,22] $\vec{G}$ being a reciprocal-lattice vector of the substrate and p is a positive integer. For example, for rare gases absorbed on graphite,[22–24] $p=3$. The effect of the substrate potential is then represented by a term of the form

$$V=\int\left[h_{1p}(\vec{x})\cos p\theta(\vec{x})+h_{2p}(\vec{x})\sin p\theta(\vec{x})\right]d\vec{x}\ . \tag{1.2}$$

This term is just the type of symmetry-breaking field discussed recently by José *et al.*[4] If the substrate contains patches (islands) of different sizes rather than being a single crystalline surface, then the effective interaction with the substrate can be described by a potential of the form Eq. (1.2), but with random coefficients $h_{1p}(x)$ and $h_{2p}(x)$. In the simplest model one can describe this randomness by the distribution

$$P[\vec{h}]=\exp\left(-\frac{1}{2a^2g_0^2}\int d\vec{x}\,\vec{h}_p^2(\vec{x})\right)\ . \tag{1.3}$$

Here a is the lattice spacing. We might note that the potential generated by a completely random array of impurities can be represented by Eqs. (1.2) and (1.3) with $p=1$.

For ordered symmetry-breaking fields José *et al.*[4] have shown that $p=4$ is the important dividing line. They show that the spin-wave behavior of the planar model is stable against symmetry-breaking perturbations provided

$$4<2\pi K_{\mathrm{eff}}<\tfrac{1}{4}p^2\ . \tag{1.4}$$

The effective coupling which is model dependent is such that $K_{\mathrm{eff}}\to K$ as $T\to 0$; at the transition $2\pi K_{\mathrm{eff}}=4$. There is therefore no stability region for uniaxial or threefold-symmetric perturbations $p=2$ and 3. A real planar system with sixfold-symmetric perturbation, $p=6$, however would be expected to show typical planar-model behavior for

$$4<2\pi K_{\mathrm{eff}}<9\ . \tag{1.5}$$

Below the lower critical point the system would presumably select one of the six favored directions. With our previous discussion in mind it is also important to know what is the corresponding behavior for a random symmetry-breaking field. This question will be addressed in the remainder of this article. We find that the result corresponding to Eq. (1.4) is

$$4<2\pi K_{\mathrm{eff}}<\tfrac{1}{2}p^2\ . \tag{1.6}$$

That is, planar-model critical behavior should be observable for threefold-symmetric ($p=3$) perturbations.

II. PURE SINE-GORDON MODEL

Rather than work with the effective Hamiltonian Eqs. (1.1) and (1.2) we prefer to use the equivalent action[5,25–27]

$$A[\phi]=A_0[\phi]+A_{\mathrm{imp}}[\phi]\ , \tag{2.1}$$

where

$$A_0[\phi]=\tfrac{1}{2}(\vec{\nabla}\phi)^2+(\alpha_0/\beta_0^2a^2)\cos\beta_0\phi \tag{2.2}$$

is the well-known sine-Gordon action for which a systematic field-theoretic renormalization-group treatment has been developed recently by Amit *et al.*[5] Here[28]

$$\beta_0^2=(2\pi)^2K \tag{2.3}$$

and the vortex fugacity z is given by

$$z=\alpha_0/2\beta_0^2\ . \tag{2.4}$$

The original phase angle $\theta(\vec{x})$ is proportional to the dual of the scalar field $\phi(\vec{x})$

$$\partial_\mu\tilde{\phi}(\vec{x})=\epsilon_{\mu\nu}\partial_\nu\phi(\vec{x})\ , \tag{2.5}$$

$$\frac{2\pi}{\beta_0}\tilde{\phi}(\vec{x})=i\theta(\vec{x})=\int_{-\infty}^{x_1}\frac{\partial\phi}{\partial z_2}\bigg|_{x_2}dz_1\ ; \qquad \vec{x}=(x_1,x_2)\ . \tag{2.6}$$

Here $\epsilon_{\mu\nu}$ is the completely antisymmetric tensor in two dimensions. The random symmetry breaking field is given in terms of $\tilde{\phi}(\vec{x})$ as

$$A_{\mathrm{imp}}=\frac{h_{1p}}{a^2}\cosh\left(\frac{2\pi p\tilde{\phi}}{\beta_0}\right)-i\frac{h_{2p}}{a^2}\sinh\left(\frac{2\pi p\tilde{\phi}}{\beta_0}\right)\ . \tag{2.7}$$

Before we discuss the effects of disorder it is useful to summarize the main points of the renormalization scheme for the sine-Gordon theory Eq. (2.2). First we note that the critical line of the spin-wave vortex gas model starts at $(\alpha_0,\beta_0^2)=(0,8\pi)$. Therefore, in order to develop a field-theoretic renormalization-group treatment of the Kosterlitz-Thouless phase transition it is necessary to understand how to renormalize (remove the singularities as $a\to 0$) the sine-Gordon theory near $\beta_0^2=8\pi$. Coleman[25] had shown that normal ordering was sufficient to render the sine-Gordon theory finite for $\beta_0^2<8\pi$. However, as β_0^2 approaches 8π the scale dimension of the operator $\cos(\beta_0\phi)$ approaches 2 ($\cos\beta_0\phi$ becomes a marginal operator) and the sum of individually finite terms in the perturbation series at a given order in α_0 develop a logarithmic divergence as the lattice spacing goes to zero. In an important paper Amit *et al.*[5] have shown how these divergences may be removed by wave-function renormalization, to third order in powers of renormalized coupling constants α and $\delta=\beta^2/8\pi-1$.

The result of this renormalization procedure, which we will sketch below was to rederive and extend Kosterlitz's recursion relations.

The essence of the calculation is quite simple: an expression for the bare, one-particle irreducible, two-point function $\Gamma^{(2)}$ is derived by graphical methods. The divergences of $\Gamma^{(2)}$ are located, expanded in α_0 and $\delta_0=\beta_0^2/8\pi-1$ and then removed by appropriate renormalization. The diagrammatic convention is given in Fig. 1 while in Fig. 2 we give the diagrams contributing to order α_0^2. In Fig. 1 the dashed line represents the sum of an odd number of propagators, $[\sinh I(\vec{x})-I(\vec{x})]$, while the wavy line represents the sum of an even number, $[\cosh I(\vec{x})-1]$:

$$I(\vec{x})=\beta_0^2 G(\vec{x}) \ . \tag{2.8}$$

The propagator which is defined by[5]

$$G(\vec{x},a)=\int \frac{d^2p}{(2\pi)^2}\frac{e^{i\vec{p}\cdot\vec{y}}}{p^2+m_0^2}\bigg|_{y^2=x^2+a^2} = \frac{1}{2\pi}K_0[m(x^2+a^2)^{1/2}] \tag{2.9}$$

has Fourier transform

$$G(\vec{q})=1/(q^2+m_0^2) \tag{2.10}$$

and asymptotic form

$$G(\vec{x},a)=-\frac{1}{4\pi}\ln cm_0^2(x^2+a^2), \quad |\vec{x}|m_0 << 1 \ , \tag{2.11}$$

where $c=\frac{1}{4}e^{2\gamma}$, γ is Euler's constant and again use has been made of a as an ultraviolet regulator. The infrared behavior has been regulated by adding a mass term $\frac{1}{2}m_0^2\phi^2$ to A_0. This does not affect the critical behavior and the mass term may be removed at the end of the calculation.

The general form of $\Gamma^{(2)}$ is

$$\Gamma^{(2)}(\vec{q})=q^2+m^2-\Sigma(q,m) \ , \tag{2.12}$$

where Σ is the sum of all one-particle irreducible contributions to the self-energy. The leading-order $O(\alpha_0)$ contributions to Σ given by the sum of all tadpole graphs are represented by the circle in Fig. 1.

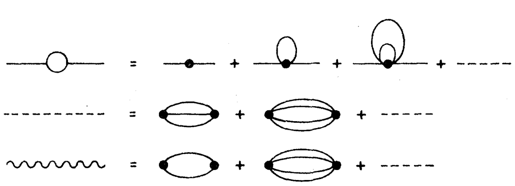


FIG. 1. Diagrammatic convention for the renormalized α_0 vertex, and the two diagonal propagators $[\cosh I(x)-1]$ and $[\sinh I(x)-1]$ that enter the theory.

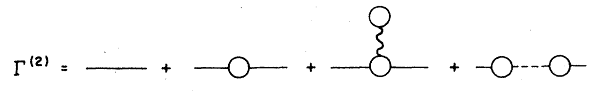


FIG. 2. Diagrams contributing to the one-particle irreducible two-point function $\Gamma^{(2)}$ of the pure sine-Gordon theory to order α^2.

Explicitly

$$\Sigma_{\alpha_0}=\frac{\alpha_0}{a^2}\exp[-\tfrac{1}{2}I(x=0)] = \frac{\alpha_0}{a^2}\exp\frac{\beta_0^2}{8\pi}\ln(cm_0^2a^2) \ . \tag{2.13}$$

It is convenient to define

$$J_\alpha=cm_0^2(cm_0^2a^2)^{\beta_0^2/8\pi-1} \tag{2.14}$$

so that

$$\Sigma_{\alpha_0}=\alpha_0 J_\alpha \tag{2.15}$$

exhibiting clearly $\beta_0^2/8\pi=1$ as the critical "dimensionality" at this order. Summation of the $O(\alpha_0^2)$ graphs gives

$$\Gamma^{(2)}(\vec{q})=q^2+m_0^2-\alpha_0J_\alpha + \frac{1}{\beta_0^2}(\alpha_0J_\alpha)^2 \times\int d\vec{x}\{[\cosh I(\vec{x})-1] - e^{i\vec{q}\cdot\vec{x}}[\sinh I(\vec{x})-I(\vec{x})]\} \ . \tag{2.16}$$

To locate the divergences in $\Gamma^{(2)}$ we examine the value of $\Gamma^{(2)}$ and its derivatives with respect to q at the point $q^2=0$. Using Eq. (2.11) it is easily seen that $\Gamma^{(2)}(q^2=0)$ and $\partial\Gamma^{(2)}/\partial q^2|_{q^2=0}$ are divergent at $\beta_0^2=8\pi$, but all higher derivatives are ultraviolet convergent, at least to this order. To second order in (α_0,δ_0) the divergences are

$$\Gamma^{(2)}(q^2=0)=m_0^2-\alpha_0cm_0^2[1+\delta_0\ln(cm_0^2a^2)+\cdots] \tag{2.17}$$

and

$$\frac{\partial\Gamma^{(2)}}{\partial q^2}\bigg|_{q^2=0}=1-\tfrac{1}{64}\alpha_0^2\ln(cm_0^2a^2)+\cdots \ . \tag{2.18}$$

As noted by Amit *et al.*[5] these infinities may be removed by two independent renormalization constants. Defining renormalized parameters, α, β, and m and a renormalized field by

$$\alpha_0=Z_\alpha\alpha, \quad \beta_0^2=Z_\phi^{-1}\beta^2 \ , \quad m_0^2=Z_\phi^{-1}m^2, \quad \phi^2=Z_\phi\phi_R^2 \ . \tag{2.19}$$

Z_α and Z_ϕ are chosen in such a way that the renormalized vertex function

$$\Gamma_R^{(2)}(q,\alpha,\delta,m^2,\kappa)=Z_\phi\Gamma^{(2)}(q,\alpha_0,\delta_0,m_0^2,a) \quad (2.20)$$

is finite order by order in the double expansion in powers of α and δ; κ is a mass scale necessary to define the renormalized theory. To this order the renormalization constants are found to be

$$Z_\phi=1+\tfrac{1}{64}\alpha^2\ln(\kappa^2a^2) \quad (2.21)$$

and

$$Z_\alpha=1-\delta\ln(\kappa^2a^2)\ . \quad (2.22)$$

As the bare parameters cannot depend on the mass scale, Kosterlitz's recursion relations are found by differentiating Eqs. (2.21) and (2.22) with respect to κ

$$\kappa\frac{\partial\alpha}{\partial\kappa}=2\alpha\delta,\quad \kappa\frac{\partial\delta}{\partial\kappa}=\tfrac{1}{32}\alpha^2\ . \quad (2.23)$$

III. SINE-GORDON MODEL WITH RANDOM FIELDS

With this introduction we may now turn to a discussion of the effect of site disorder on topological order. Technically, we investigate whether the sine-Gordon theory with random symmetry-breaking field can still be renormalized at $\beta_0^2=8\pi$.

We study the self-energy of the impurity averaged sine-Gordon function

$$\langle\langle\phi(x)\phi(y)\rangle_h\rangle_{av}=\left\langle\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}\ln Z[h,J]\right\rangle_{av}\Big|_{J=0}\ . \quad (3.1)$$

$Z[h,J]$ is the generating functional of the sine-Gordon theory

$$Z[h,J]=\int D[\phi]\times\exp\left\{-\int\left[\tfrac{1}{2}(\partial\phi)^2+\frac{\alpha_0}{\beta_0^2a^2}\cos\beta_0\phi+J\phi+\frac{h_{1p}}{a^2}\cosh\left(\frac{2\pi p\tilde\phi}{\beta_0}\right)-i\frac{h_{2p}}{a^2}\sinh\left(\frac{2\pi p\tilde\phi}{\beta_0}\right)\right]\right\}\ . \quad (3.2)$$

The $\langle\cdots\rangle_{av}$ denotes impurity averaging and $\langle\cdots\rangle_h$ is the thermal average for a specified random field. It is convenient to rewrite $\ln Z$ in Eq. (3.1) using the replica trick[28]

$$\ln Z=\lim_{n\to 0}\frac{Z^n-1}{n} \quad (3.3)$$

hence

$$\langle\langle\phi(x)\phi(y)\rangle_h\rangle_{av}=\lim_{n\to 0}\frac{1}{n}\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}\langle Z^n[h,J]\rangle_{av}|_{J=0}\ . \quad (3.4)$$

Carrying out the average over the random fields

$$\langle Z^n[h,J]\rangle_{av}=\int D[\vec\phi]\exp\left\{-\int\left[\sum_{i=1}^{n}\left(\tfrac{1}{2}(\partial\phi_i)^2+\frac{\alpha_0}{\beta_0^2a^2}\cos\beta_0\phi_i\right)+J\sum_i\phi_i-\frac{g_0^2}{2a^2}\sum_{i\neq j=1}^{n}\cosh\left(\frac{2\pi p\tilde\phi_i}{\beta_0}\right)\cosh\left(\frac{2\pi p\tilde\phi_j}{\beta_0}\right)+\frac{g_0^2}{2a^2}\sum_{i\neq j=1}^{n}\sinh\left(\frac{2\pi p\tilde\phi_i}{\beta_0}\right)\sinh\left(\frac{2\pi p\tilde\phi_j}{\beta_0}\right)\right]\right\} \quad (3.5)$$

and

$$\langle\langle\phi(x)\phi(y)\rangle\rangle_{av}=\lim_{n\to 0}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle\phi_i(x)\phi_j(y)\rangle_{\langle Z^n\rangle_{av}}\ , \quad (3.6)$$

where $\langle\cdots\rangle_{\langle Z^n\rangle_{av}}$ means thermal average for the replica field theory. In deriving Eq. (3.6) we have used the fact that $\lim_{n\to 0}\langle Z^n\rangle_{av}=1$. To derive an expression for $\langle\langle\phi(x)\phi(y)\rangle_h\rangle_{av}$ we start from the Dyson equation for the replica field theory

$$\vec G_{ij}(\vec q)=\vec G_{ij}^0(\vec q)+\vec G_{ik}^0(\vec q)\Gamma_{kl}^{(2)}\vec G_{lj}(\vec q)\ , \quad (3.7)$$

where $\Gamma_{kl}^{(2)}$ is the sum of all one-particle irreducible (1PI) graphs; the propagators are matrices in $(\phi,\tilde\phi)$ space

$$\vec G_{ij}=\begin{vmatrix}\langle\phi_i\phi_j\rangle & \langle\phi_i\tilde\phi_j\rangle\\ \langle\tilde\phi_i\phi_j\rangle & \langle\tilde\phi_i\tilde\phi_j\rangle\end{vmatrix}\ , \quad (3.8)$$

where average over $\langle Z^n \rangle_{av}$ is understood,

$$\vec{G}^{0}_{ij} = \delta_{ij} \begin{pmatrix} G & \tilde{G} \\ \tilde{G} & G \end{pmatrix} = \delta_{ij} \vec{G}^{0} \ . \tag{3.9}$$

The function G was defined in Eqs. (2.9)–(2.11) and

$$\tilde{G}(\vec{x},a) = \int_{-\infty}^{x_1} dz_1 \left. \frac{\partial G}{\partial z_2} \right|_{z_2 = x_2}$$
$$= -\frac{1}{2\pi} \frac{x_2}{(x_2^2 + a^2)^{1/2}} \tan^{-1} \frac{x_1}{(x_2^2 + a^2)^{1/2}} \ ; \quad |x| m_0 \ll 1 \ . \tag{3.10}$$

The matrix two-point function

$$\vec{\Gamma}^{(2)}_{ij} = \begin{pmatrix} \Gamma^{(2)}_{ij} & \tilde{\Gamma}^{(2)}_{ij} \\ \tilde{\Gamma}^{(2)}_{ij} & \tilde{\Gamma}'^{(2)}_{ij} \end{pmatrix} , \tag{3.11}$$

where $\Gamma^{(2)}_{ij}$ has two amputated ϕ legs, $\tilde{\Gamma}^{(2)}_{ij}$ one ϕ and one $\tilde{\phi}$, and $\tilde{\Gamma}'^{(2)}_{ij}$ two amputated $\tilde{\phi}$ legs. To all orders in perturbation theory the general matrix structure of the two-point function is

$$\vec{\Gamma}^{(2)}_{ij} = \begin{pmatrix} \Gamma^{(2)}_{ij} & (n\delta_{ij} - 1)\tilde{\Gamma}^{(2)} \\ (n\delta_{ij} - 1)\tilde{\Gamma}^{(2)} & (n\delta_{ij} - 1)\tilde{\Gamma}'^{(2)} \end{pmatrix} . \tag{3.12}$$

The details of the derivation of this result are given in the Appendix.

It then follows immediately that

$$\sum_{ij=1}^{n} \vec{G}_{ij} = n\vec{G}^{0} + \sum_{l} \vec{G}^{0} \sum_{i} \vec{\Gamma}^{(2)}_{il} \sum_{j} \vec{G}_{ij} \tag{3.13}$$

which as a result of the structure of $\vec{\Gamma}^{(2)}_{ij}$ reduces to

$$\sum_{ij=1}^{n} \vec{G}_{ij} = n \begin{pmatrix} G & \tilde{G} \\ \tilde{G} & G \end{pmatrix} + \begin{pmatrix} G & \tilde{G} \\ \tilde{G} & G \end{pmatrix} \begin{pmatrix} \Gamma^{(2)} & 0 \\ 0 & 0 \end{pmatrix} \sum_{ij} \vec{G}_{ij} \ , \tag{3.14}$$

where $\Gamma^{(2)} = \sum_{i=1}^{n} \Gamma^{(2)}_{ij}$, independent of j by replica symmetry. In particular if we select the (1-1) element

$$\langle\langle \phi\phi \rangle\rangle_{av} = G + G \left(\lim_{n \to 0} \Gamma^{(2)} \right) \langle\langle \phi\phi \rangle\rangle_{av} \ . \tag{3.15}$$

The self-energy of $\langle\langle \phi\phi \rangle\rangle_{av}$ is therefore $\lim_{n \to 0} \Gamma^{(2)}$.

In the remainder of this section we will determine $\Gamma^{(2)}$ to second order in α_0 and lowest order in the impurity concentration g_0^2. The leading-order contribution in α is of course the sum of the tadpole graphs in Fig. 1 which "normal order" the α vertex. The result is

$$\frac{\alpha_0}{a^2} \cos\beta_0\phi \to \alpha_0 J_\alpha {:}\cos\beta_0\phi{:} \ , \tag{3.16}$$

where J_α is given in Eq. (2.14). In a similar way at first order in g_0^2 we sum the tadpole graphs to normal order the impurity vertex. We find

$$\frac{g_0^2}{2a^2} \cosh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_i \right) \cosh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_j \right)$$
$$= \tfrac{1}{2} g_0^2 J_g {:}\cosh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_i \right){::}\cosh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_j \right){:} \ , \tag{3.17}$$

where

$$J_g = a^{-2} \exp\left[-\left(\frac{2\pi p}{\beta_0} \right)^2 G(0,a) \right]$$
$$= c m_0^2 (c m_0^2 a^2)^{(\pi p^2/\beta_0^2) - 1} \tag{3.18}$$

and

$$\frac{g_0^2}{2a^2} \sinh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_i \right) \sinh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_j \right)$$
$$= \tfrac{1}{2} g_0^2 J_g {:}\sinh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_i \right){::}\sinh\left(\frac{2\pi p}{\beta_0} \tilde{\phi}_j \right){:} \ . \tag{3.19}$$

The graphical notation for the cosh and sinh vertices is given in Fig. 3. The terms of order g_0^2 only contribute to $\tilde{\Gamma}'^{(2)}_{ij}$ and therefore do not affect $\langle\langle \phi\phi \rangle\rangle$.

The first nontrivial contribution to $\Gamma^{(2)}_{ij}$ appears at order $\alpha_0 g_0^2$. There is only one graph at this order which is shown in Fig. 4. In this diagram the slashed wavy line connecting an impurity vertex with an α vertex represents the sum of an even number of off-diagonal propagators Eq. (3.10). Diagrams of this type correlate the impurities with the vortices on the dual lattice. To this order, when $n \to 0$,

$$\Gamma^{(2)}(\vec{q}) = q^2 + m_0^2 - \alpha_0 J_\alpha + \alpha_0 g_0^2 J_\alpha J_g \int d^2x \, [\cos p\Theta(x) - 1] \ , \tag{3.20}$$

where

$$\Theta(x) = 2\pi \tilde{G}(x,a) \ . \tag{3.21}$$

We have also obtained this result by direct calculation

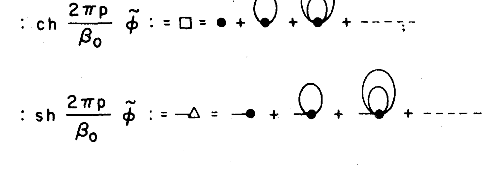


FIG. 3. Diagrammatic convention for the renormalized random field vertices.

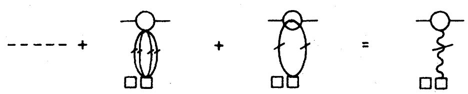


FIG. 4. Contributions to $\Gamma^{(2)}$ at order αg^2. The slashed lines indicate off-diagonal propagators.

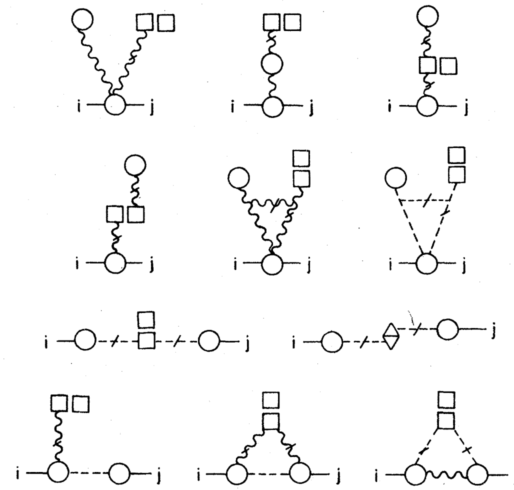


FIG. 5. Contributions to $\Gamma^{(2)}$ at order $\alpha_0^2 g_0^2$.

without making use of the replica trick (see comments in the Appendix).

As we noted in Sec. II, in the pure sine-Gordon theory additional divergences arise at order α_0^2 when $\beta_0^2 \geq 8\pi$. These terms which were given in Eq. (2.16) were made finite by wave funtion renormalization. To investigate whether the wave-function renormalization must be modified in the present situation we must consider the $\alpha_0^2 g_0^2$ contribution to $\Gamma^{(2)}$. The graphs contributing to $\Gamma^{(2)}_{\alpha_0^2 g_0^2}$ are given in Fig. 5. Our result for $\Gamma^{(2)}$ to order α_0^2 and lowest order in g_0^2 is

$$\begin{aligned}
\Gamma^{(2)}(q) = {} & q^2 + m_0^2 - \alpha_0 J_\alpha + \alpha_0 g_0^2 J_\alpha J_g \int d\vec{x}\,[\cos p\Theta(\vec{x}) - 1] \\
& - \frac{\alpha_0^2}{\beta_0^2} J_\alpha^2 \int d\vec{x}\,\{ e^{i\vec{q}\cdot\vec{x}}[\sinh I(x) - I(x)] - [\cosh I(x) - 1] \} \\
& - \frac{\alpha_0^2 g_0^2}{\beta_0^2} J_\alpha^2 J_g \int d\vec{x} \int d\vec{y}\,(e^{i\vec{q}\cdot\vec{x}} \{ [\cosh I(x) - 1] \sin p\Theta(\vec{y}) \sin p\Theta(\vec{x}-\vec{y}) \\
& \qquad - \sinh I(x) [\cos p\Theta(\vec{y}) - 1][\cos p\Theta(\vec{x}-\vec{y}) - 1] \\
& \qquad - 2[\sinh I(x) - I(x)][\cos\Theta(\vec{y}) - 1] \} \\
& \qquad + 2[\cos p\Theta(\vec{x}) - 1][\cos p\Theta(\vec{y}) - 1] + 2[\cosh I(x) - 1][\cos p\Theta(\vec{y}) - 1] \\
& \qquad + [\cosh I(x) - 1][\cos p\Theta(\vec{y}) - 1][\cos p\Theta(\vec{x}-\vec{y}) - 1] \\
& \qquad - \sinh I(x) \sin p\Theta(\vec{y}) \sin p\Theta(\vec{x}-\vec{y})) \ . \qquad (3.22)
\end{aligned}$$

The renormalization of $\Gamma^{(2)}$ and the conclusions that can be drawn are given in Sec. IV.

IV. STABILITY OF TOPOLOGICAL ORDER

We have calculated $\Gamma^{(2)}(\vec{q})$ to second order in α_0 and lowest order in g_0^2. The critical behavior is determined by those contributions which diverge as $a \to 0$. As we discussed in Sec. II Amit *et al.*[5] have shown that all the divergences of the pure sine-Gordon field can be absorbed by two renormalization constants Z_ϕ and Z_α, Eqs. (2.21) and (2.22), provided the theory is also expanded as a power series in

$$\delta_0 = \beta_0^2/8\pi - 1 \ . \qquad (4.1)$$

The divergences in the pure theory were analyzed in Sec. II.

In the presence of a random symmetry-breaking field additional divergences may appear. Therefore it may be necessary to introduce another renormalization constant Z_g, so that

$$g_0^2 = Z_g g^2 \ . \tag{4.2}$$

We notice that due to the nature of the configurational average (Appendix) any additional terms involve at least one power of the vortex fugacity α_0. Consequently, we are searching for critical behavior in terms which couple the vortices to the impurities.

The lowest-order contribution is momentum independent

$$\Gamma^{(2)}_{\alpha_0 g_0^2} = \alpha_0 g_0^2 J_\alpha J_g \int d\overline{x}\,[\cos p\Theta(\overline{x}) - 1] \tag{4.3}$$

so any extra divergences can be absorbed into Z_g. The integral is finite. In the region $|\overline{x}| m_0 < 1$ the angular propagator is defined by Eqs. (2.5), (2.9), and (3.10)

$$\partial_\mu \Theta = \epsilon_{\mu\nu} \partial_\nu K_0 \ , \tag{4.4}$$

which can be integrated to give in polar coordinates (r, θ)

$$\Theta = \frac{m_0 r^2}{(r^2 + a^2)^{1/2}} K_1(m_0 (r^2 + a^2)^{1/2}) \ . \tag{4.5}$$

Outside this region $\Theta \propto e^{-m_0 r}$ and the integrand vanishes exponentially with increasing r. Hence, the integral is proportional to the area of a circle with radius in $1/m_0$ and we write

$$\int d\overline{x}\,[\cos p\Theta(\overline{x}) - 1] = A / c m_0^2 \ , \tag{4.6}$$

where A is a finite constant. So

$$\Gamma^{(2)}_{\alpha_0 g_0^2} = A\,\alpha_0 g_0^2 c m_0^2 (c m_0^2 a^2)^{(\beta_0^2/8\pi) + (p^2\pi/\beta_0^2) - 2} \ . \tag{4.7}$$

Now, to order α_0, wave-function renormalization is not necessary ($Z_\phi = 1$) and

$$Z_\alpha = (\kappa^2 a^2)^{1 - (\beta_0^2/8\pi)} \ . \tag{4.8}$$

Thus, doing the α renormalization in Eq. (4.7) gives

$$\Gamma^{(2)}_{\alpha_0 g_0^2} = A\,\alpha g_0^2 c m^2 (c m^2 a^2)^{(p^2\pi/\beta_0^2) - 1} (c m^2/\kappa^2)^{(\beta_0^2/8\pi) - 1} \tag{4.9}$$

and we see that there is an additional divergence associated with the impurities when $\beta_0^2 > p^2\pi$. In order to make Eq. (4.9) finite we must define

$$Z_g = (\kappa^2 a^2)^{1 - (p^2\pi/\beta_0^2)} \tag{4.10}$$

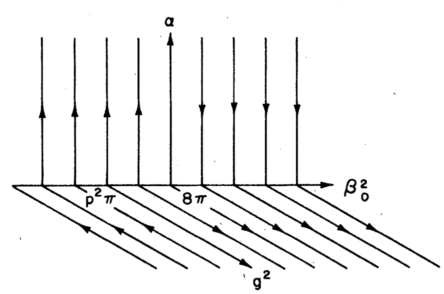
which implies that the renormalized coupling g^2 satisfies the flow equation

$$\frac{\partial g^2}{\partial \ln\kappa} = -g^2 \frac{\partial \ln Z_g}{\partial \ln\kappa} = 2\left(\frac{p^2\pi}{\beta_0^2} - 1\right) g^2 \ . \tag{4.11}$$

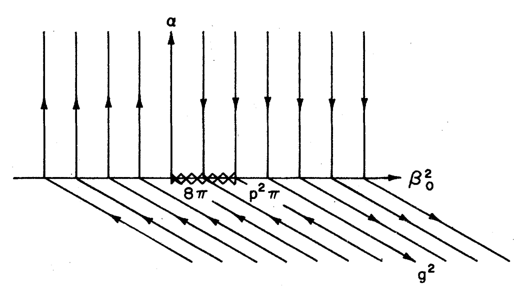
We see that g^2 becomes relevant for $\beta_0^2 > p^2\pi$.

This result should be interpreted as specifying the value of p for which a Kosterlitz-Thouless phase transition survives at $\beta_0^2 = 8\pi$. If $p^2 < 8$ which means $p = 1$, or 2 (uniaxial disorder), then as the temperature is lowered, the impurities become relevant (at $\beta_0^2 = p^2\pi$) before the vortices have a chance to condense (at $\beta_0^2 = 8\pi$). The transition which occurs at $p^2\pi$ is of unknown variety and separates a high-temperature phase dominated by vortices from a low-temperature phase of vortices and impurities. This situation is shown in Fig. 6(a) where the flow in α at lowest order comes from Eq. (4.8). We anticipate that there are additional divergences in the higher order terms which require that this theory be renormalized around $\beta_0^2 = p^2\pi$ rather than 8π. We can only say that the resulting critical behavior is not Kosterlitz-Thouless.

Consequently, for the remainder of this paper we will assume that $p \geqslant 3$ (concentrating on the most physically interesting value $p = 3$). Then we see from Eq. (4.11) that, at lowest order, the impurities are irrelevant until $\beta_0^2 > 9\pi$, a lower temperature than the conventional XY transition. The flows of the coupling constants are shown in Fig. 6(b). If we



(a) p ≤ 2



(b) p ≥ 3

FIG. 6. (a) Lowest-order renormalization-group flows for $p^2 < 8$. (b) Lowest-order renormalization-group flows for $8\pi < \beta_0^2 < p^2\pi$ showing a Kosterlitz-Thouless fixed line.

can believe these low-order results, they predict a Kosterlitz-Thouless fixed line for

$$8\pi < \beta_0^2 < p^2\pi \quad (4.12)$$

which can be written in terms of $K_{\rm eff}=\beta_0^2/4\pi^2$ to give

$$4 < 2\pi K_{\rm eff} < \tfrac{1}{2}p^2 \ . \quad (4.13)$$

Comparing with the result of Ref. 4, Eq. (1.6), we see that the effect of disorder is to make the spin-wave behavior of the planar model stable against an underlying threefold symmetry in some finite temperature range. Notice that relative to an ordered p-fold field the smallest value of p for which an XY fixed line is stable is halved (from $p=6$). We cannot say much about the low-temperature phase $\beta_0^2 > p^2\pi$. Here the impurities are relevant, and flow is to zero-vortex fugacity and some large value of the disorder. It is tempting to interpret this as the spins becoming locked in to the underlying field in a phase which bears some resemblance to a spin-glass.

However, we have obtained these conclusions from the lowest-order flow equations which are known to be unreliable. Hence, before we can be confident about the above interpretation we must investigate the divergence structure at higher orders.

It is important to check that the impurities do not modify the wave-function renormalization carried out by Amit *et al.* The lowest order in which there might be momentum-dependent divergences proportional to g_0^2 is $\alpha_0^2 g_0^2$. The momentum-independent part of this term is

$$\Gamma^{(2)}_{\alpha_0^2 g_0^2}(0) = -\frac{\alpha_0^2 g_0^2}{\beta_0^2} J_\alpha^2 J_g \int d\vec{x}\, d\vec{y}\, \{ (e^{-I(x)}-1)[\cos p\Theta(\vec{y})\cos p\Theta(\vec{x}-\vec{y}) + \sin p\Theta(\vec{y})\sin p\Theta(\vec{x}-\vec{y}) - 1] + 2I(x)[\cos p\Theta(\vec{x})-1] + 2[\cos p\Theta(\vec{x})-1][\cos p\Theta(\vec{y})-1] \} \quad (4.14)$$

and the integral is finite. The divergence in J_α^2 is canceled by the renormalization of α in Eq. (4.8), so this term is proportional to $g_0^2 J_g$, just like Eq. (4.9) and requires no additional renormalization. The momentum-dependent part which contains a potential new divergence is

$$\Gamma^{(2)}_{\alpha_0^2 g_0^2}(q) = -\frac{\alpha_0^2 g_0^2}{2\beta_0^2} J_\alpha^2 J_g \int d\vec{x}\, d\vec{y}\, (e^{i\vec{q}\cdot\vec{x}}-1) e^{I(x)} [1-\cos p\Theta(\vec{y})\cos p\Theta(\vec{x}-\vec{y}) - \sin p\Theta(\vec{y})\sin p\Theta(\vec{x}-\vec{y})] \ . \quad (4.15)$$

The only region where the integral might diverge is $|\vec{x}| < \Delta$ where $\Delta m_0 << 1$. There

$$I(x) = -\frac{\beta_0^2}{4\pi}\ln c m_0^2 (x^2+a^2), \quad |\vec{x}| m_0 << 1 \ . \quad (4.16)$$

Thus, consider the integral

$$J = \int_{|\vec{x}|<\Delta} d\vec{x}\, d\vec{y}\, \frac{e^{i\vec{q}\cdot\vec{x}}-1}{[c m_0^2 (x^2+a^2)]^{\beta_0^2/4\pi}} [1-\cos p\Theta(\vec{y})\cos p\Theta(\vec{x}-\vec{y}) - \sin p\Theta(\vec{y})\sin p\Theta(\vec{x}-\vec{y})] \ . \quad (4.17)$$

Integrating by parts on $|\vec{x}|$ and using the fact that

$$\frac{\partial\Theta(x)}{\partial|\vec{x}|} = 0, \quad |\vec{x}| m_0 < 1 \ , \quad (4.18)$$

which follows directly from the Cauchy-Riemann equations defining Θ, it is easy to see that the small $|\vec{x}|$ part of J is multiplied by

$$1-\cos p\Theta(\vec{y})\cos p\Theta(-\vec{y}) - \sin p\Theta(\vec{y})\sin p\Theta(-\vec{y}) \ , \quad (4.19)$$

which vanishes since $\Theta(\vec{y})=\Theta(-\vec{y})$. Hence, we find that the dependence on the cutoff of $\Gamma^{(2)}_{\alpha_0^2 g_0^2}$ is exactly the same as that of $\Gamma^{(2)}_{\alpha_0 g_0^2}$. This is a crucial observation because it means that the wave-function renormalization carried out by Amit *et al.* is sufficient also for weak disorder. We might also note that the term order $\alpha_0 g_0^4$ has exactly the same dependence on the cutoff as $\alpha_0 g_0^2$.

Hence, in our expansion to second order in α_0 and lowest order in g_0^2, all contributions from the impurities are finite provided $p \geq 3$. No renormalization of g_0^2 is required and the flow equation for the renormalized couplings are just those of Kosterlitz as derived by Amit *et al.*, Eq. (2.23). Consequently for small positive δ flow is to the Kosterlitz-Thouless fixed point $\alpha^* = \delta^* = 0$ and the expansion in powers of α and δ can b expected to converge.

However, from the lowest-order results for the impurity terms we might expect this expansion to break down at some point. As we have just noted to the order considered the Kosterlitz equations are independent of g^2. Since this flow is towards $\alpha^* = 0$ for $\delta > 0$, any terms in Eq. (4.11) which go like $g^2 \times$ power of α^2 can be set to zero. Hence, Eq. (4.11) is valid to lowest order in g^2 and all orders in $\delta > 0$ and can be used as a guide to the region of validity of the δ expansion. We see that, when $p = 3$, g^2 flows to zero (impurities are irrelevant) provided $\delta < \frac{1}{8}$. At $\delta = \frac{1}{8}$ the impurity vertices become marginal, and for larger values of δ are relevant. Hence, we predict that the Kosterlitz-Thouless fixed line terminates at $\delta = \frac{1}{8}$ confirming our lowest-order result. It is comforting that this occurs for such a small value of δ when $p = 3$. We are inclined to trust the δ expansion in this range.

This result enables us to predict the exponent η for $p = 3$ at the upper and lower transition temperatures, $T_1(g^2)$ and $T_2(g^2)$, corresponding to $\delta = 0$ and $\frac{1}{8}$, respectively. From Ref. 4,

$$\eta = \frac{1}{2\pi K_{\text{eff}}} \quad . \tag{4.20}$$

Hence, we find that $\eta = \frac{1}{4}$ as T goes to $T_1(g^2)$ from below and $\eta = \frac{2}{9}$ as T goes to $T_2(g^2)$ from above. The g^2-T phase diagram for $p = 3$ is shown in Fig. 7.

ACKNOWLEDGMENTS

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APPENDIX

Here we prove that the matrix two-point function $\Gamma^{(2)}$ has the structure of Eq. (3.12), i.e., we show that $\tilde{\Gamma}_{ij}^{(2)}$ and $\tilde{\Gamma}_{ij}^{\prime(2)}$ vanish when the $n \to 0$ limit is taken. The essence of this result is the two following observations:

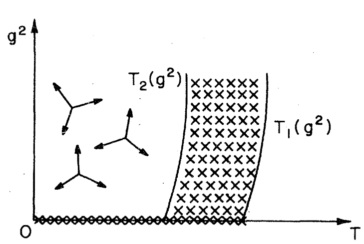

FIG. 7. g^2-T phase diagram for $p = 3$.

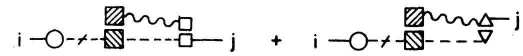

FIG. 8. Typical diagrams contributing to $\tilde{\Gamma}^2$, the diagrams shown here combine to give a factor of $(1 - n\delta_{ij})$.

(i) The coupling between the spins and the random impurity field, Eq. (2.7) is rotationally invariant and this symmetry is preserved by the impurity distribution Eq. (1.3).

(ii) Note that there is an alternative to the $n \to 0$ trick for calculating Green's functions of the quenched system: Calculate the connected Green's functions for a fixed but arbitrary impurity configuration and then average the result over the impurities.[18]

The replica method we have used provides a considerable simplification because it builds in the symmetries of point (i) which result in a massive cancelation of diagrams if Green's functions are calculated prior to averaging. However, when $n \neq 0$ the replica method includes some diagrams which would be disconnected before averaging. The $n \to 0$ limit removes them [see Eq. (3.3)]. We show that the only diagrams allowed by (i) to contribute to $\tilde{\Gamma}_{ij}^{(2)}$ and $\Gamma_{ij}^{(2)}$ would be disconnected if we worked with a fixed-impurity distribution. These diagrams combine to give factors of $1 - n\delta_{ij}$ which vanish when $n \to 0$ according to the prescription of Sec. III. We carry out the proof, which amounts to enumerating all possible diagrams, only for the off-diagonal two-point function $\tilde{\Gamma}_{ij}^{(2)}$. The proof for $\tilde{\Gamma}_{ij}^{\prime(2)}$ is similar.

Rotational invariance is manifest in the replica formulation of Eq. (3.5), where the interaction with the

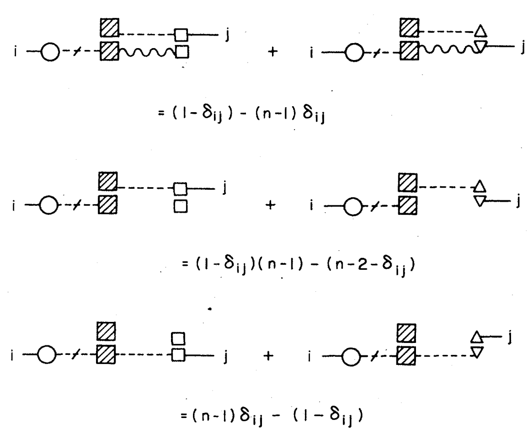

FIG. 9. Remaining contributions to $\tilde{\Gamma}^2$.

random impurity field is

$$\frac{g_0^2}{2a^2}\sum_{\substack{i,j\\ i\neq j}}\left\{\cosh\frac{2\pi p}{\beta_0}\tilde{\phi}_i\cosh\frac{2\pi p}{\beta_0}\tilde{\phi}_j - \sinh\frac{2\pi p}{\beta_0}\tilde{\phi}_i\sinh\frac{2\pi p}{\beta_0}\tilde{\phi}_j\right\} , \qquad (A1)$$

by the requirement that $i \neq j$. This implies that diagrams in which both "boxes" of an impurity vertex are connected to the same vertex are not allowed, because the propagators are diagonal in replica space and would require $i = j$. Consequently, the only allowed diagrams would be disconnected before averaging.

In Fig. 8 we show a pair of diagrams which combine to give a factor $1-n\delta_{ij}$. Here the shaded boxes are taken to be identical in the two graphs and are connected together only by the impurity averaging. Otherwise the notation is conventional. The first graph in which both external legs are connected to the same "black box" is proportional to $(n-1)\delta_{ij}$, since there is a free sum over the replica label of the upper box except that it must not be i. In the second graph we must have $i \neq j$ and it has the opposite sign, so is proportional to $-(1-\delta_{ij})$. Adding we obtain the factor $1-n\delta_{ij}$. The remaining diagrams and the way in which they combine to give $1-n\delta_{ij}$ are shown in Fig. 9.

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