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Two-dimensional Ising model in random magnetic fields

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The statistical mechanics of finite $L \times L$ Ising square lattices in a field $\pm h$ of random sign is investigated numerically by a modified recursive transfer matrix method for $6 \leq L \leq 16$. Our results are consistent with the absence of a spontaneous magnetization for $h \neq 0$ even in the ground state. The singularities occurring at $T=0$ in the range $0 \leq h \leq 4J$, J being the exchange constant, are discussed in terms of a cluster expansion. For nonzero T, less than the critical temperature of the pure two-dimensional Ising model, and h sufficiently small, the system exhibits a nonzero spin-glass order parameter of the Mattis type although there is no magnetization. The ferromagnetic correlation function becomes long ranged for $h \rightarrow 0$ and is calculated from the domain-wall density.

I. INTRODUCTION

Recently there has been much interest in systems with quenched random disorder.[1] Particularly interesting is the case of random magnetic fields[2–12]: (i) For spin dimensionality $n > 2$ it has been shown that ferromagnetic order is unstable against arbitrarily weak random fields for systems with dimensionality d less than (or equal to) $d_c = 4$,[2,5,7] while in the Ising case ($n=1$) the lower critical dimensionality $d_c = 2$.[2] (ii) For $n \geq 2$ the critical exponents of d- dimensional systems are the same as those of the corresponding pure problem in $d-2$ dimensions,[4,6] and scaling laws are correspondingly modified,[3] while not much is known in the Ising case about the critical behavior at the physical dimensionalities $d = 2, 3$.[2,9] (iii) For strong enough values of the random field an interesting multicritical behavior is predicted.[8] (iv) An experimental realization of these systems can be provided by site-disordered antiferromagnets in a uniform magnetic field.[10,12]

The present paper is concerned with the Ising case in two dimensions, i.e., right at the predicted lower critical dimensionality. There one expects that no spontaneous magnetization will occur for nonzero random fields $\pm h$, although for $h=0$ a transition occurs to a ferromagnetic state at a critical temperature ($k_B T_c/J \simeq 2.27$ in the case of nearest-neighbor interactions.[13,14] So far not much is known about the detailed properties of this transition.

As more rigorous techniques are not available for this problem, we perform a numerical study applying a modified "recursive transfer matrix method" recently developed for the study of two-dimensional random systems.[15,16] This method has been described in detail in the context of the two-dimensional Edwards-Anderson spin-glass,[15] and hence will not be described here. In Sec. II we rather discuss the properties of our model at $T=0$, where for sufficiently large h a "cluster expansion" is used to discuss singularities of thermodynamic functions similar to those encountered already in the one-dimensional case.[17] Section III then contains our results at nonzero temperatures. In Sec. IV we give a calculation of the correlation function for small h which results from the estimated domain-wall density, and Sec. V contains our conclusions.

II. GROUND-STATE AND LOW-TEMPERATURE PROPERTIES OF THE RANDOM FIELD MODEL

We consider the Ising Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i h_i S_i, \quad S_i = \pm 1 \quad , \tag{1}$$

where J is the exchange constant, the summation $\langle i,j \rangle$ runs over all pairs of nearest neighbors, and h_i is a (frozen-in) random variable distributed according to the probability

$$P(h_i) = \tfrac{1}{2}[\delta(h_i - h) + \delta(h_i + h)] \quad . \tag{2}$$

Consider now a finite domain of n sites, and denote by $\frac{1}{2}n(1+y)$ the number of sites within the domain where the random field has a negative sign. Thus y denotes the (small) relative deviation in the number of minus signs from its mean value. It is easy to show that for large enough n the probability density distribution for having a particular y is Gaussian,

$$p(y) = (n/2\pi)^{1/2} \exp(-\tfrac{1}{2}ny^2) \quad . \tag{3}$$

If the domain has a quadratic shape and the spins surrounding it point up, the energy cost for overturning the domain, neglecting edge effects, is (at any temperature $T < T_c$)

$$\Delta E = 4f_s\sqrt{n} - 2nyhM \quad , \tag{4}$$

where f_s is the interface free energy and M the spontaneous magnetization per spin at $h=0$. From the exact solution[13,14] we have

$$f_s = 2J - k_B T \ln[(1+u)/(1-u)] \quad , \quad u = e^{-2J/k_B T} \quad , \tag{5}$$

and

$$M = (1+u^2)^{1/2}(1-6u^2+u^4)^{1/8}/(1-u^2)^{1/2} \quad . \tag{6}$$

The probability that $\Delta E < 0$ is thus given by

$$P(\Delta E < 0) = \int_{2f_s/Mh\sqrt{n}}^{1} dy\, p(y) = (2\pi)^{-1/2}\int_{2f_s/Mh}^{\sqrt{n}} \exp(-\tfrac{1}{2}z^2)dz \quad . \tag{7}$$

Equation (7) implies that $P(\Delta E > 0)$ is nonzero and becomes independent of n for large n. Thus

$$P(\Delta E < 0) = \tfrac{1}{2}\,\mathrm{erfc}(\sqrt{2}f_s/Mh) \quad , \tag{8}$$

which for $h \to 0$ reduces to ($x = \sqrt{2}f_s/Mh$)

$$P(\Delta E < 0) = \frac{1}{2\sqrt{\pi}x}\exp(-x^2) \quad . \tag{9}$$

Hence the density of domains is essentially given by, for small h, (neglecting preexponential factors for the moment)

$$\rho_d = P(\Delta E < 0) \propto \exp(-2f_s^2/M^2h^2) \quad . \tag{10}$$

Figure 1 illustrates the domain pattern which emerges on the basis of this argument. Large domains of up-spins contain smaller domains of down-spins, which in turn contain smaller domains of up-spins again, etc. The density of misoriented subdomains within each domain tends exponentially strong to zero as $h \to 0$, cf. Eq. (10). Figure 1 is schematic, of course, as the shapes of the various domains need not be regular. The implications of these results for the correlation function are investigated in Sec. IV. The expected qualitative behavior is shown in Fig. 2(a).

The correlation function decays from unity to the square of the magnetization of the pure system on a length which is basically the correlation length of the pure problem, ξ^{pure}. On a much larger length scale, governed by the correlation length $\xi^{\text{random}}(h)$ of the disordered system, the correlation function decays to zero.

It is important to note that this decay does not imply the absence of *any* order. Since the distribution

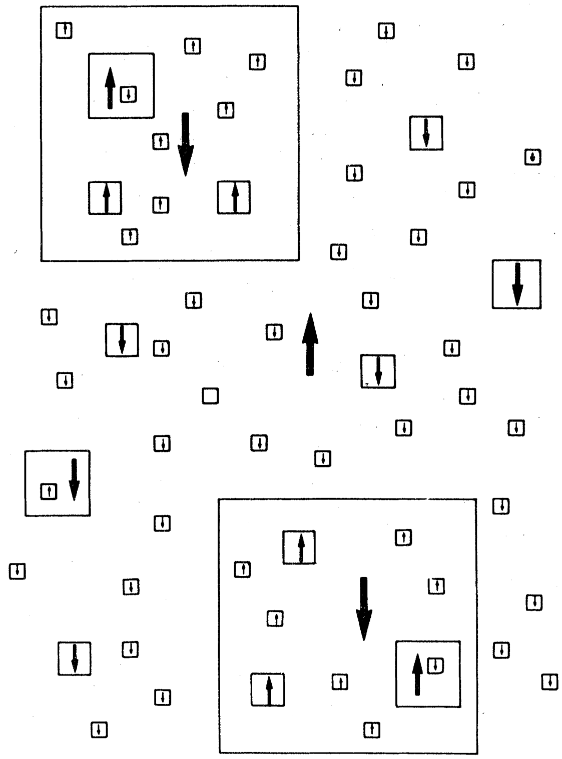

FIG. 1. Ground-state domain pattern of the two-dimensional Ising lattice in a small random magnetic field (schematic). Arrows indicate orientation of the domains.

of random fields is quenched, the regions where significant excess of one sign of the random field occurs are also fixed. Even if the domain pattern of Fig. 1 is degenerate in energy with other domain arrangements, we nevertheless expect to have a nonzero Edwards-Anderson order parameter[18] q,

$$q = [\langle S_i\rangle_T^2]_{\text{av}} \neq 0 \quad , \tag{11}$$

where $[\cdots]_{\text{av}}$ denotes an average over the random field configuration [taken with Eq. (2)]. Since[15] ($R \equiv \vec{r}_i - \vec{r}_j$)

$$\lim_{R\to\infty} [\langle S_iS_j\rangle_T^2]_{\text{av}} = q^2 \quad , \tag{12}$$

the correlation function $[\langle S_iS_j\rangle_T^2]_{\text{av}}$ does not decay towards zero for large distances [Fig. 2(b)]. We speculate that, in analogy to $[\langle S_iS_j\rangle_T]\rangle_{\text{av}}$, the decay is governed by two different lengths: at a length of essentially ξ^{pure} the correlation reaches a value of about M^4 (M is the spontaneous magnetization of the pure system), while the spin-glass order parameter q is reached on the length scale $\xi^{\text{random}}(h)$ (which diverges as $h \to 0$). For $h \to 0$ we have $q \to M^2$ [Fig. 2(c)], while at some critical field $h_c(T)$ the spin-glass order breaks down (note that M_r^2, the magnetization in the random field directions trivially has to be sub-

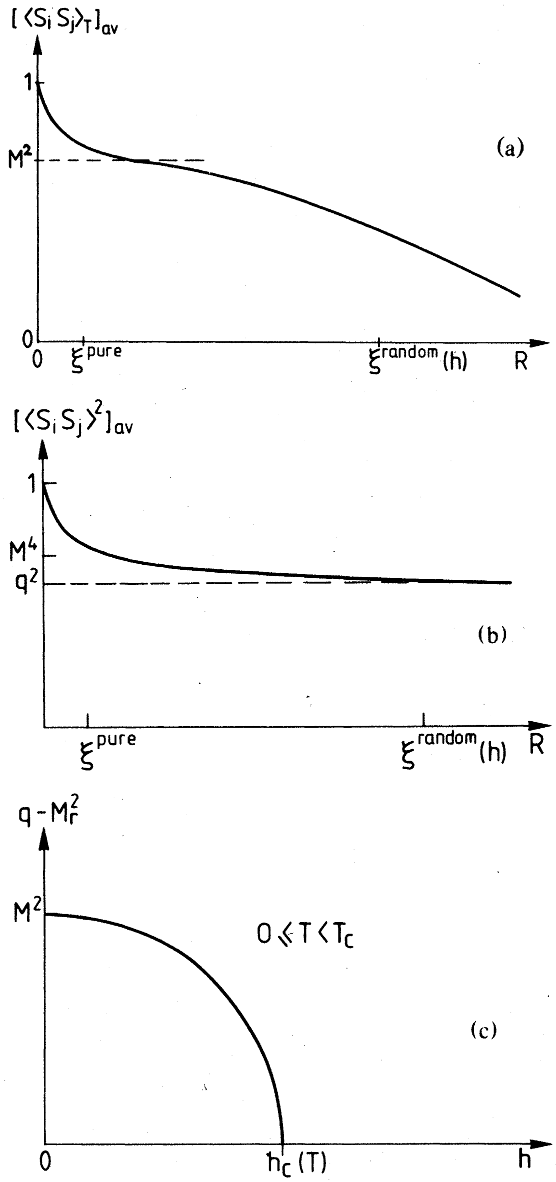

FIG. 2. (a) Expected decay of the averaged spin-correlation function at a temperature at which, for $h=0$, a spontaneous magnetization M occurs (schematic). (b) Expected decay of the averaged squared correlation (schematic). (c) Expected variation of the Edwards-Anderson order parameter (schematic).

tracted in the order-parameter definition).

Obviously, using $\epsilon_1 = \text{sgn}(h_i)$ and transforming Eq. (1) with $S_1' = \epsilon_i S_i$ one obtains a Mattis spin-glass[19] in a homogeneous field

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j - h \sum_i S_i \ , \quad J_{ij} = \epsilon_i \epsilon_j J \ . \quad (13)$$

We hence suggest that for $d=2$ the behavior of this model is quite nontrivial for $h \neq 0$: the order parameter[20] $\vec{\psi}$,

$$\psi^{(l)} = \frac{1}{N} \sum_i \phi_i^{(l)} \langle S_i \rangle_T \ , \quad (14)$$

$\phi_i^{(l)}$ being the spin orientation of S_i in the lth ground state of the system, is not related to the spontaneous magnetization of the model Eq. (1), as for higher dimensionalities. On the other hand, we expect that spin-glass order parameters for the model Eq. (13) do not vanish as they do for Edwards-Anderson spin-glasses.[15]

Finally, we note that our treatment [Eqs. (3)–(10)] does not imply that for a ferromagnet in an arbitrarily weak staggered (nonrandom) magnetic field h_s breaks up into domains also. In this case, the excess number of sites with fields of one sign within a domain can never be larger in comparison with its surface area, in contrast to the random field case. Hence Eq. (3) does *not* apply to staggered fields.

While these considerations of the behavior in weak random fields clearly are fairly qualitative, the behavior in strong random fields can be analyzed much more precisely. For $h > 4J$ all spins follow the random field, hence the internal energy E per spin just is $E=-h$, the magnetization per spin M_r measured along the local direction of the random field is unity, while the ferromagnetic susceptibility χ_F, defined via

$$k_B T \chi_F = \left[\sum_{ij} \langle S_i S_j \rangle_T - \langle S_i \rangle_T \langle S_j \rangle_T \right]_{\text{av}} \Big/ N \ , \quad (15)$$

yields for $T \to 0$

$$k_B T \chi_F = \sum_i (1 - \langle S_i \rangle^2) \big/ N = 0 \ . \quad (16)$$

At $h=4J$, however, spins for which following the random field would involve breaking four bonds become effectively decoupled [Fig. 3(a)]. Since the probability of the configuration of random fields shown in Fig. 3(a) is $(\frac{1}{2})^5$, and there are two such configurations [the second just has all the random fields of Fig. 3(a) reversed], we find that at $h=4J$ a fraction $\frac{1}{16}$ of the spins is loose, i.e., has $\langle S_i \rangle_0 = 0$. Hence we obtain for M_r, the susceptibility and entropy

$$M_r = \tfrac{15}{16}, \quad k_B T \chi_f = \tfrac{1}{16} \ ,$$
$$S/k_B = \tfrac{1}{16}(\ln 2) \quad (h=4J) \ . \quad (17)$$

For $3J < h < 4J$ the central spin of the configuration shown in Fig. 3(a) is no longer loose but rather

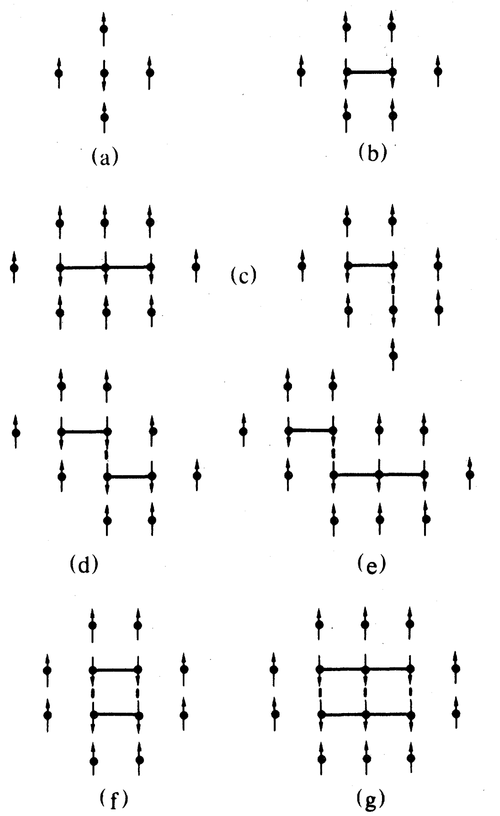

FIG. 3. Local random field configurations which lead to singularities at $T=0$ as described in the text. Arrows denote direction of random field at each site. Sites which form a loose cluster at the singular field are connected by straight lines.

aligned antiparallel to the random field. Hence

$$M_r=\frac{7}{8},\quad E=-h-\frac{1}{16}(4J-h)\ ,$$
$$k_BT\chi_f=0,\quad S/k_B=0\ . \tag{18}$$

Similarly, for $h=3J$ a cluster of two spins in the configuration shown in Fig. 3(b) becomes effectively decoupled from the rest. Since the probability of this random field configuration is $(\frac{1}{2})^8$, and there are four equivalent configurations, we have

$$M_r=\frac{27}{32},\quad k_BT\chi_f=\frac{1}{16}\ ,$$
$$S/k_B=\frac{1}{64}(\ln 2)\quad (h=3J)\ . \tag{19}$$

For $8J<h<3J$ these "dimers" are no longer loose but rather aligned antiparallel to the random field. Hence

$$M_r=\frac{13}{16},$$
$$E=-h-(4J-h)/16-\frac{1}{64}(6J-2h)\ , \tag{20}$$
$$k_BT\chi_f=S/k_B=0\ .$$

For $h=\frac{1}{3}8J$ we have loose "trimers" [Fig. 3(c)],

$$M_r=\frac{401}{512},\quad k_BT\chi_f=\frac{45}{512}\ ,$$
$$S/k_B=\frac{5}{512}(\ln 2)\quad (h=\tfrac{1}{3}8J)\ , \tag{21}$$

while for $\frac{1}{2}5J<h<\frac{1}{3}8J$

$$M_r=\frac{193}{256},$$
$$E=-h-\frac{1}{16}(4J-h)-\frac{1}{64}(6J-2h)-(8J-3h)\frac{5}{512}\ . \tag{22}$$

At $h=\frac{1}{2}5J$ we have loose "quadrumers" [Fig. 3(d)],

$$M_r=\frac{747}{1024},\quad k_BT\chi_f=\frac{25}{256}\ ,\quad S/k_B=\frac{25}{4096}(\ln 2)\ , \tag{23}$$

while for $\frac{1}{5}12J<h<\frac{1}{2}5J$ these quadrumers are antiparallel to their random field and thus

$$M_r=\frac{361}{512},$$
$$E=-h-\frac{1}{16}(4J-h)-\frac{1}{64}(6J-2h)$$
$$-(8J-3h)\frac{5}{512}-(10-4h)\frac{25}{4096}\ . \tag{24}$$

At $h=\frac{1}{5}12J$ clusters of five spins become loose [Fig. 3(c)],

$$M_r=\frac{22\,479}{32\,768},\quad k_BT\chi_f=\frac{3\,125}{32\,768}\ ,\quad S/k_B=\frac{125}{32\,768}(\ln 2) \tag{25}$$

while for $\frac{1}{3}7J<h<\frac{1}{5}12J$

$$M_r=\frac{10\,927}{15\,384}\ ,$$
$$E=-h-\frac{1}{16}(4J-h)-\frac{1}{64}(6J-2J)-(8J-3h)\frac{5}{512}$$
$$-(10J-4h)\frac{9}{1024}-(12J-5h)\frac{125}{32\,768}\ , \tag{26}$$

etc.

It is easy to realize that the critical values R_c of the ratio $R\equiv h/J$ at which singularities due to loose clusters of spins occur are given by

$$R_c=p_n^{(\gamma)}/n\ , \tag{27}$$

where $p_n^{(\gamma)}$ is the *bond perimeter* of a cluster of type γ which contains *n lattice sites.* The bond perimeter is the number of nearest-neighbor bonds connecting sites of a cluster with neighboring sites outside of the cluster.[21] The maximum perimeter for each n is obtained for *fully ramified clusters* (i.e., for which internal bonds do not form any closed loops). It is given by

$$p_n^{(\text{ramified})}=2n+2\rightarrow R_c^{(\text{ramified})}=2+2/n\ . \tag{28}$$

As a result, there is an infinite number of singularities in the regime $2<R\leqslant 4$. In between these

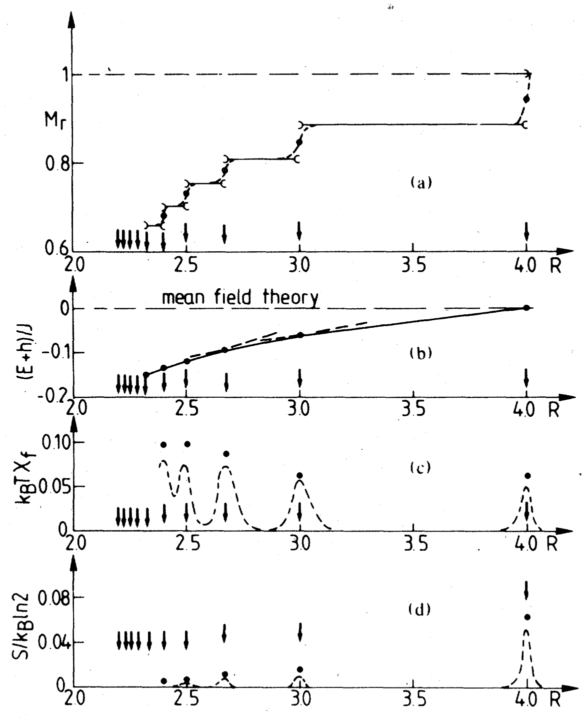

FIG. 4. Variation of (a) M_r, (b) internal energy, (c) susceptibility, and (d) entropy with $R \equiv h/J$. Arrows indicate where singularities occur at $T=0$. Dashed curves in (a), (c), and (d) indicate qualitatively the behavior at nonzero $T \ll T_c$.

singularities, all thermodynamic functions are constant (as M_r, $k_B T \chi_f$, S) or vary linearly with R (as E), cf. Fig. 4. At nonzero temperature these singularities are rounded off, but at low enough temperatures we still expect a very nonmonotonic behavior of $k_B T \chi$, S (as well as $\partial M_r/\partial h$, $\partial E/\partial h$), which reflects these singularities in the ground state (Fig. 4). This behavior is similar to the one-dimensional case[17]; there, however, singularities occur at $h/J = 2/n$, $n = 1, 2, 3, \ldots$, etc., the behavior as described here extends down to $h \to 0$. In our case the behavior of thermodynamic functions changes qualitatively at the mean-field critical field,[8] $R_c = 2$. For $0 < R \leqslant 2$ singularities occur at every rational number. This conclusion follows from an examination of random field clusters containing closed loops [e.g., Figs. 3(f) and 3(g)]. The most compact clusters are squares [Fig. 3(f)]. Their perimeter is (note that only these n are possible for which $\sqrt{n}$ is integer)

$$p_n^{(\text{square})} = 4\sqrt{n} \to R_c^{(\text{square})} = 4/\sqrt{n} \quad . \tag{29}$$

Just as the ramified clusters produce infinitely many singularities for $R \to 2^+$, the squares produce infinitely many singularities for $R \to 0^+$. Similarly, for rectangles [Fig. 3(g)] of linear dimensions m,k, we have $(m > k > 1)$

$$n = mk; \quad p_n = 2m + 2k \to R_c \frac{2k+2m}{mk} \quad , \tag{30}$$

while for general $n = mk + l$ the most compact configuration is given by a rectangle of linear dimensions m,k where one side has a kink a distance l apart from the edge with linear dimension $m+1$. The perimeter then is $(k > 1)$

$$p_n^{(\text{compact})} = 2m + 2k + 2 \to R_c = \frac{2k+2m+2}{mk+l} \quad . \tag{31}$$

Introducing further kinks at the surface of the cluster increases the perimeter in steps of 2 per kink. Hence, starting from the minimum perimeter given by Eq. (31) *all even integers* appear in the numerator in Eq. (31), up to the maximum value which is reached for fully ramified clusters, Eq. (28). Ramified clusters with just one closed loop have a perimeter

$$p_n = 2n \to R_c = 2 \quad , \tag{32}$$

which implies that there are no other singular values for $R > 2$ than the one given in Eq. (28). For $R < 2$, however, for every rational $R = \nu/\mu$, ν, μ integers, clusters can be constructed which are loose at this R: we choose $n = j\nu$, $p_n = j\mu$, where j is an even integer. The equations $j\nu = mk + l$, $j\mu = 2k + 2m + 2i$, where i is an integer counting the numbers of kinks of the cluster, obviously always have solutions for R in the given range. However, a pronounced effect on thermodynamic functions is expected only for those R where fairly many *small* clusters contribute, as the probability of finding a given cluster decreases rapidly with increasing cluster size. From the above it is clear that pronounced effects are expected at $R = 2$ and $\frac{5}{3}$ [where the cluster of Fig. 3(g) contributes], $R = \frac{4}{3}$ [Eq. (27), $n = 9$], $R = \frac{3}{2}$ [Eq. (28), $k = 2, m = 4$].

While the cluster expansion easily yields accurate results on ground-state properties for $R > 2$, for $R < 2$ the above considerations do not yield any explicit predictions. The modified recursive transfer matrix approach[15,16] can be applied here conveniently. Note that Fig. 5 shows that the ground-state energy strongly deviates from the mean-field prediction for $R \gtrsim 1$. A value $U/J = -2$ is reached as $h \to 0$ asymptotically. Figure 5 implies that the domain effects of Fig. 1, which destroy the ferromagnetic state for small h, contribute only marginally to the internal energy. However, these data do not give any indication as to the location of the critical field $h_c(0)$.

Figure 6 shows the behavior of $k_B T \chi_f$ and M_r. These results actually refer to a finite temperature ($k_B T/J = 0.2$, i.e., about 9% of the critical temperature of the pure system), but we expect the results at $T = 0$ to be fairly similar. The results for M_r show

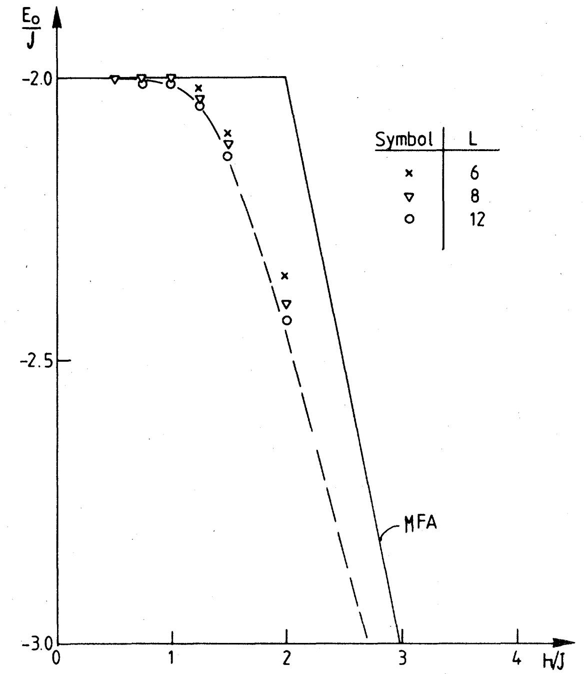

FIG. 5. Ground-state energy plotted vs random field for various $L \times L$ lattices. The dashed curve indicates the expected behavior of the infinite system and the full curve is the mean-field result (MFA).

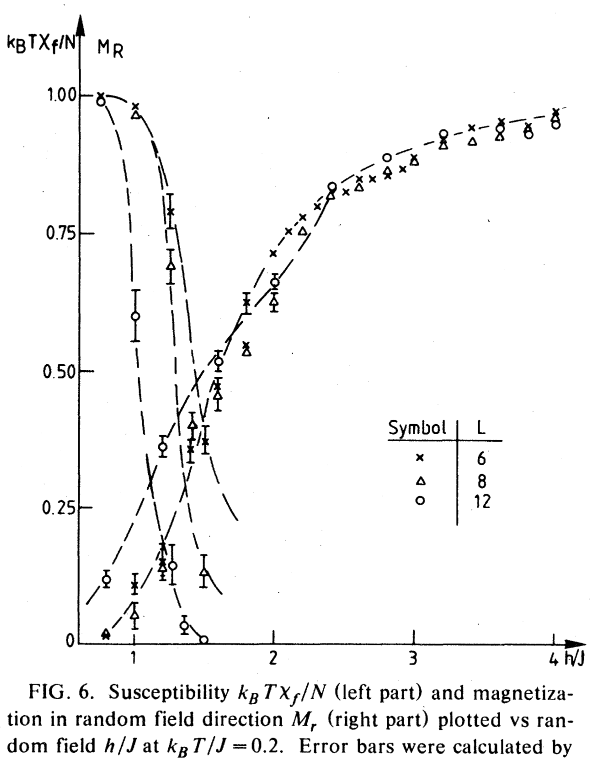

FIG. 6. Susceptibility $k_B T\chi_f/N$ (left part) and magnetization in random field direction M_r (right part) plotted vs random field h/J at $k_B T/J=0.2$. Error bars were calculated by averaging over about 70 realizations of an 8×8 lattice, about 40 realizations of a 6×6 lattice and about 30 of a 12×12 lattice.

very little size dependence, and we believe that they reflect quite faithfully the behavior of the infinite lattice. It is seen that the singularities for $h/J>2$ are already smoothed out at the considered temperature. Some irregularities in the M_r vs h curve for $1<h/J<2.5$ are probably a remnant of the $T=0$ singularities. For $h/J \lesssim 1$ M_r is very small. The statistical inaccuracy of the results (as well as the smallness of the lattices studied) prevents us from stating whether M_r has a singularity at the critical field $h_c(T)$ for $T>0$ as well, or not. While mean-field theory implies a transition to the ferromagnetic state at $h/J=2$ (Ref. 8), our results [Figs. 4(c) and 6] indicate that the susceptibility there is still fairly small, and instead increases strongly for $h/J\approx 1$. The saturation of $k_B T\chi/N$ seen in Fig. 6 for small h implies that there is a nearly perfect degree of ferromagnetic correlation (for the lattice sizes $L=6, 8$ considered). This is consistent with Eq. (10). However, Fig. 7 shows that the fields at which $k_B T\chi_f$ saturates decrease with increasing L, consistent with the absence of a spontaneous magnetization M ($\lim_{N\to\infty} k_B T\chi_f/N = M^2$) for $h>0$.

Note that $\langle S_i\rangle_T \equiv 0$ for $T>0$ in zero magnetic field, as finite systems have no phase transitions. Therefore for $h\to 0$ the "susceptibility" $k_B T\chi_f/N$ is not dominated by the susceptibility of the pure Ising system which is obtained there below T_c *in the thermodynamic limit,* but $k_B T\chi_F/N$ converges smoothly towards the square of the spontaneous magnetization M: we do not include any nonzero *homogeneous* magnetic field H here, and thus what we denote as "ferromagnetic susceptibility" here and in the following is in fact

$$k_B T\chi_F \equiv \sum_{ij} [\langle S_i S_j\rangle_T]_{\rm av}\Big/N \;,$$

and hence

$$\lim_{N\to\infty} k_B T\chi_f/N = \lim_{N\to\infty} \sum_{ij} [\langle S_i S_j\rangle_T]_{\rm av}\Big/N^2 = M^2 \;.$$

Figure 7(c) illustrates this behavior comparing data for zero random field and $L=6, 12, 16$ to exact results for M^2 (Ref. 14) for T less than the critical temperature T_c of the infinite system. Above T_c, the susceptibility χ_f in the thermodynamic limit, as taken from high-temperature series expansion estimates,[22] is used to compute the corresponding expressions of $k_B T\chi_f/N$. Clearly, our numerical calculations for rather small lattices cannot accurately describe the critical behavior of the infinite system, i.e., at T_c our susceptibility χ_f for any finite N is also finite. But one can see clearly from our calculations that a transition occurs from a disordered regime, where $\lim_{N\to\infty} k_B T\chi_f/N \to 0$, to an ordered regime where a nonzero order parameter exists in the thermodynamic limit. From the inflection point of the χ_f vs T curves one can estimate the transition temperature T_c of the

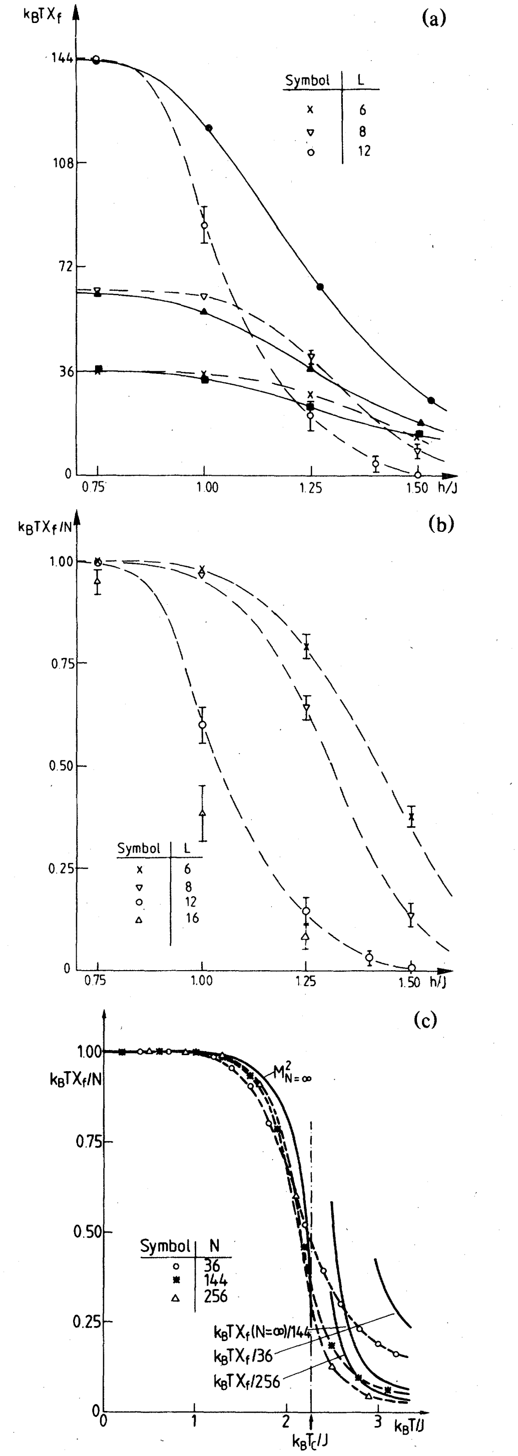

FIG. 7. (a) Susceptibility $k_BT\chi_f$ and (b) effective square of the spontaneous magnetization plotted vs random field at $k_BT/J = 0.2$ and various lattice sizes. Full curves are the calculations of Sec. IV. (c) "Susceptibility" $k_BT\chi_f/N$ for various N and zero random field plotted vs temperature. Full curves are drawn using the corresponding exact results (Refs. 14 and 24).

infinite system to within a few percent, since the variation with N of this inflection point is small. This is to be expected since $k_BT\chi_f/N \approx M^2$ as long as the correlation length ξ is distinctly smaller than the linear dimension L of the system and ξ is large only in the vicinity of T_c. Conversely, if we were to find that the inflection point shifts strongly to lower temperature as N is increased, such as occurs for a nonzero random field, we would conclude that we are still in the regime of the disordered phase but with a correlation length much larger than the linear dimension L.

III. SUSCEPTIBILITY AND SPECIFIC HEAT OF THE RANDOM FIELD ISING MODEL

The temperature variation of the susceptibility $k_BT\chi_f/N$ (i.e., normalized such that it converges towards M^2 for $N \to \infty$) is shown in Fig. 8 for several values of the random field. For $h/J = 0.5$ the behavior is hardly distinguishable from the case $h = 0$ (cf. Ref. 15), apart from a decrease of the "effective critical temperature" from $k_BT_c/J = 2.27$ (in the pure case) to $k_BT_c^{\text{eff}}/J \approx 2.0$ For $h/J = 0.75$ the behavior is similar, with $k_BT_c^{\text{eff}}/J \approx 1.8$. While the curves for $L = 6, 8$ are nearly identical for $T < T_c^{\text{eff}}$, the curve for $L = 12$ is already significantly depressed. This indicates that the data in fact do not converge to a nonzero $M^2(T)$ for $N \to \infty$, and the curves of Fig. 8(b) simply reflect the strong increase of the ferromagnetic correlation length rather than the existence of a spontaneous magnetization. At $h/J = 1.0$ the curve for $L = 12$ is already significantly depressed even for low temperatures [Fig. 8(c)], while the curves for $L = 6, 8$ remain nearly identical at low temperatures. At $h/J = 1.25$ the size dependence of $k_BT\chi_f/N$ is already very pronounced for all L considered. We believe that the ferromagnetic correlation length for $h/J = 1.0$ must saturate at about 12 lattice spacings, while for $h/J = 1.25$ it saturates at about 6 lattice spacings (or less).

Figure 9 shows the internal energy and Fig. 10 the specific heat of our model. For $h/J \gtrsim 1.0$ the size dependence of the specific heat is insignificant, and the data indicate the existence of a broad Schottky-like peak only. This is consistent with our above discussions of the susceptibility which implies that when the ferromagnetic correlation length does not become very large, there is not even a rounded phase transition but only short-range order. For $h/J = 0.5$, however, the peak of C at $k_BT_c^{\text{eff}}/J \approx 2.0$ grows with L and becomes sharper, and the same is true for $h/J = 0.75$ (Fig. 11). In both cases no saturation effect is seen. We interpret these findings as a transition to Mattis-like spin-glass order (Fig. 2). This situation would be compatible with Grinstein's modified scaling conjecture[3] $(d-2)\nu = 2 - \alpha$ only for $\nu \to \infty$, ν

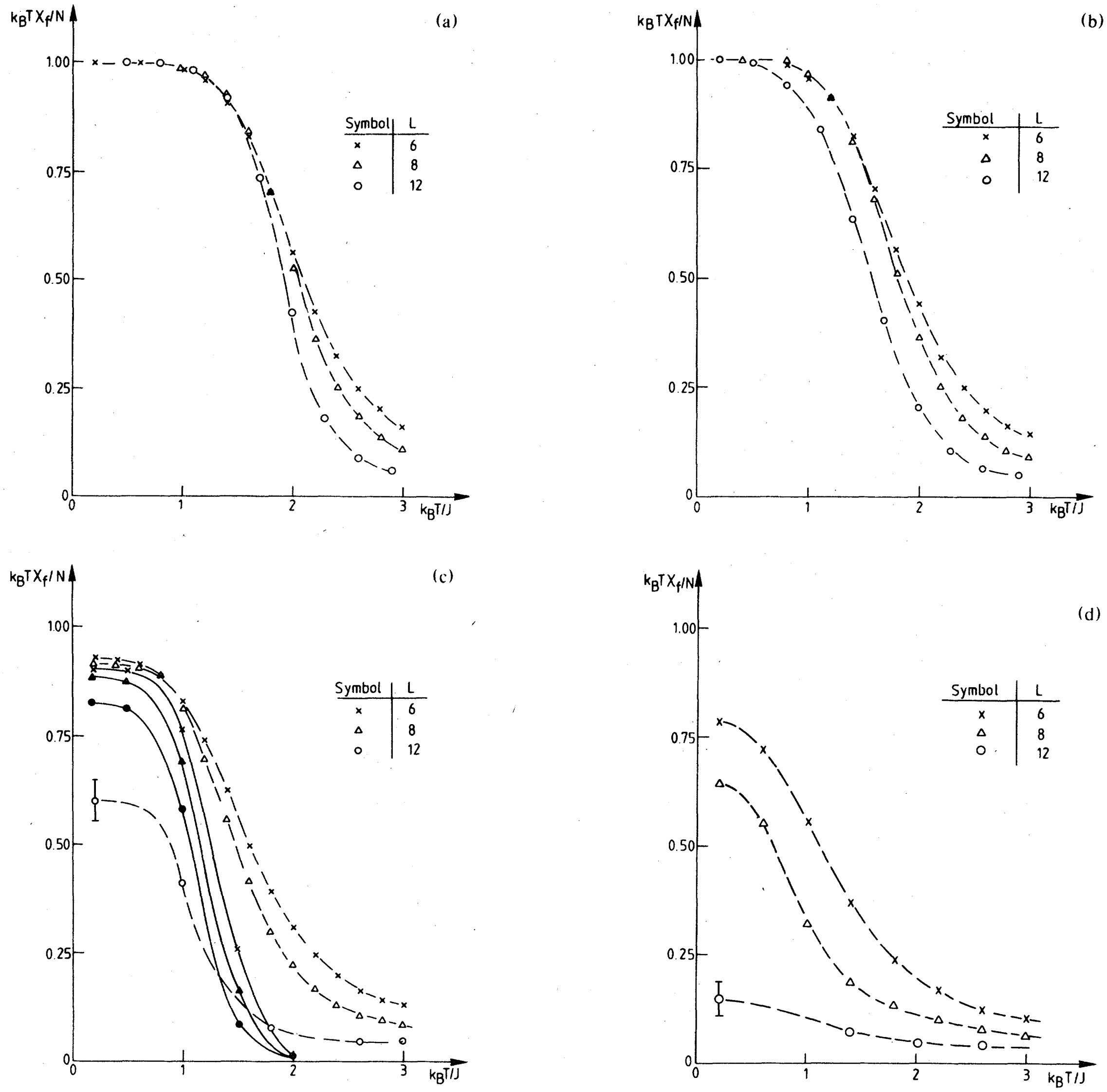

FIG. 8. Susceptibility k_BTX_f/N plotted vs temperature for (a) $h/J=0.5$, (b) 0.75, (c) 1.0, and (d) 1.25. Several lattice sizes are shown. The full curves on (c) are the calculations of Sec. IV.

being the correlation length exponent, α the specific-heat exponent, and d the dimensionality. No prediction for α arises in this case trivially, in contrast to the two-dimensional XY model where $\nu=\infty$ has been established,[23] and hence the standard scaling law $d\nu=2-\alpha$ implies $\alpha=-\infty$, i.e., the specific heat does not diverge, and only a broad Schottky-like peak remains.[24] It seems doubtful to us, however, whether Grinstein's scaling conjecture holds in our case.

Figure 12 compares our findings to those obtained from mean-field theory.[8] A true ferromagnetic state with a spontaneous magnetization occurs for $h=0$ only. For nonzero but sufficiently small h a "transition" to a quasiferromagnetic state is observed, where the magnetization still saturates completely within domains of fairly large size and we have nonzero spin-glass order parameters. As far as the ferromagnetic correlation function is concerned, the transition to this state may be either a rounded (gradual) transition or a phase transition to a state with a power law decay of the spin-spin correlation function. The tricritical point found in the mean-field treatment does not seem to have any significance for the interpretation of our numerical results.

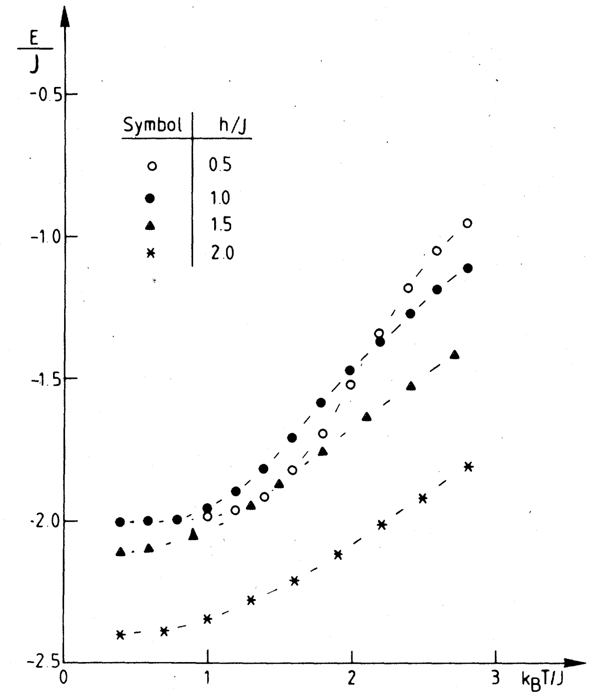

FIG. 9. Internal energy per spin plotted vs temperature for an 8 ×8 lattice at various values of the random field.

IV. SPIN-SPIN CORRELATION FUNCTION IN WEAK FIELDS

As discussed in Sec. II, the probability of finding, for the two-dimensional Ising model in the random field described by Eq. (2), a domain containing n reversed spins is given by $P=\frac{1}{2}\,\mathrm{erfc}(x)$ with $x=\sqrt{2}f_s/Mh$ for large n. For weak fields, where $x \gg 1$ and $P \ll 1$, the average size, σn, of a region containing one such reversed domain can be estimated by setting $(1-P)^{\sigma} \simeq e^{-1}$. Thus $\sigma = 1/P$. Noting the scale invariance of the system for distances much greater than $P^{-1/2}$, we conclude that in weak fields the interface (domain-wall) density μ perpendicular to any given direction is given by

$$\mu(h,T)=2P^{1/2}/(1-P^{1/2}) \simeq 2P^{1/2} \; . \tag{33}$$

We now define $P_j(l)$ as the probability of crossing j domain walls while moving along a path connecting two spins S_0, S_l, separated by l lattice units. Since the locations of the domain walls are essentially independent and they are uniformly distributed, $P_j(l)$ should be well represented by the Poisson distribution

$$P_j(l)=\frac{(\mu l)^j}{j!}e^{-\mu l} \; , \tag{34}$$

with μ given by Eq. (33). For a given distribution of domain walls at a temperature $T \ll T_c$ $(h=0)$, the

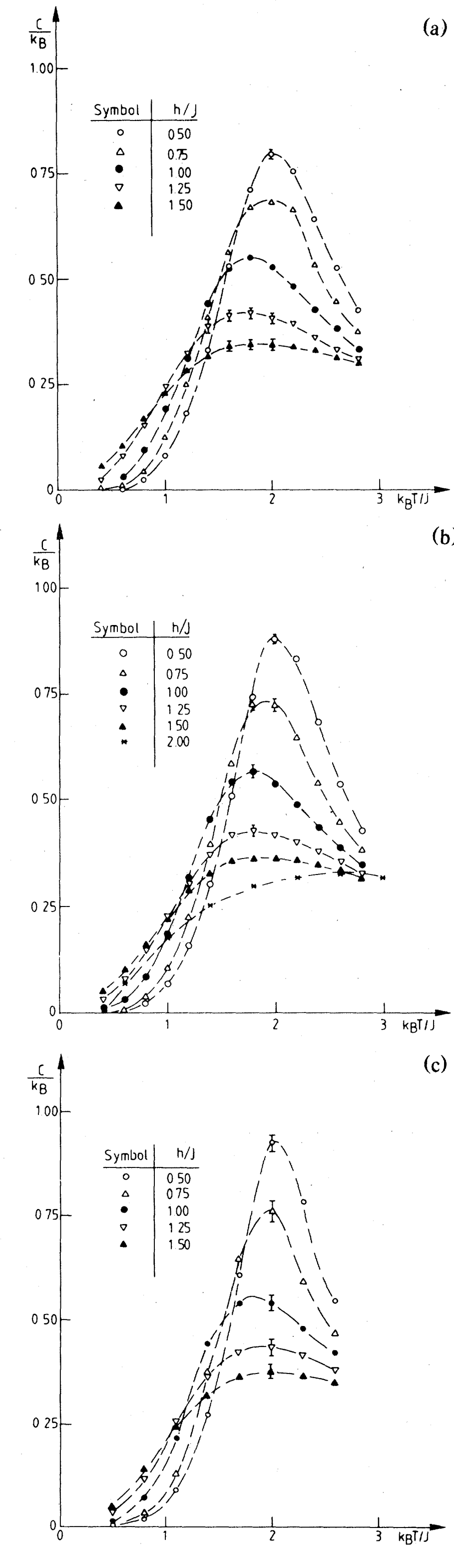

FIG. 10. Specific heat plotted vs temperature for several values of the random field; (a) $L=6$, averaged over 70 realizations, (b) $L=8$ averaged over 40 realizations, (c) $L=12$ averaged over 30 realizations.

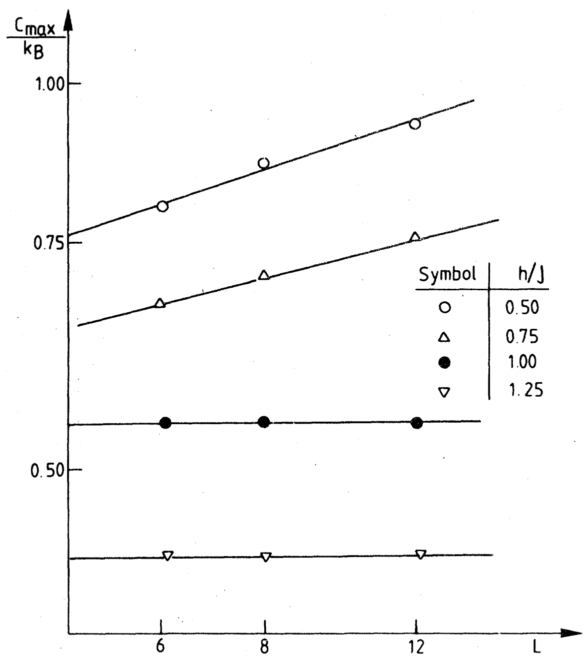

FIG. 11. Log-log plot of specific-heat maximum vs linear dimension of the system at various values of the random field.

spin-spin correlation function is simply

$$\langle S_0 S_l \rangle_T = (-1)^j M^2 \ , \qquad (35)$$

where M is given by Eq. (6). Using Eq. (34), we average $\langle S_0 S_l \rangle$ over all possible domain-wall configurations, obtaining

$$[\langle S_0 S_l \rangle_T]_{\text{av}} = M^2 \sum_{j=0}^{\infty} (-1)^j \frac{(\mu l)^j}{j!} e^{-\mu l} = M^2 e^{-2\mu l} \ . \qquad (36)$$

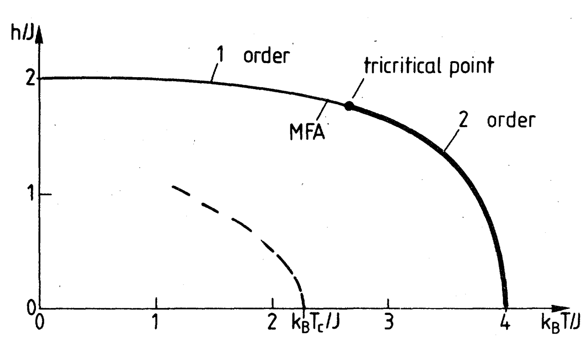

FIG. 12. Phase diagram of the random field model, as compared with the mean-field approximation (MFA) (Ref. 8). The broken line is our estimate for the transition paramagnet-Mattis spin-glass. Note that the line $h=0$ is special as for $h \to 0$, $T < T_c$ a transition to an ordered ferromagnetic state with nonzero spontaneous magnetization $M(T)$ occurs.

Using Eqs. (15) and (36) and taking $0 \le l \le R$, we obtain for the ferromagnetic susceptibility

$$k_B T \chi_f \simeq M^2 [1 - (2\mu R + 1) e^{-2\mu R}] / 2\mu^2 \ . \qquad (37)$$

This result should be valid in weak random fields and at low temperatures.

We have applied Eq. (37) to the same arrays studied numerically and discussed previously in Secs. II and III. The results, for χ_f vs h at $k_B T/\zeta = 0.2$ and χ_f vs T for $h/\zeta = 1$, are shown in Figs. 7(a) and 8(c), respectively. Considering first Fig. 7(a), we see that the analytic results compare reasonably well with those obtained numerically, particularly when one remembers that Eq. (37) can only be a first approximation for the small arrays and necessarily relatively strong random fields studied numerically. The calculated temperature dependence of χ_f shown in Fig. 8(c) is also in reasonable agreement with the numerical results for $T \lesssim 0.6 T_c$; at higher temperatures both the dependence of M on h and the fluctuation contribution to χ_f, which are not included in Eq. (37), will become important.

Returning to Eq. (35), we note that

$$\langle S_0 S_l \rangle_T^2 = (-1)^{2j} M^4 = M^4 \qquad (38)$$

for any configuration. Thus, as noted earlier, this Ising system exhibits spin-glass order, of the Mattis-type, at low temperatures in weak random fields.

This conclusion is corroborated by a direct numerical calculation of this correlation function with our modified transfer matrix approach, Fig. 13. It is seen

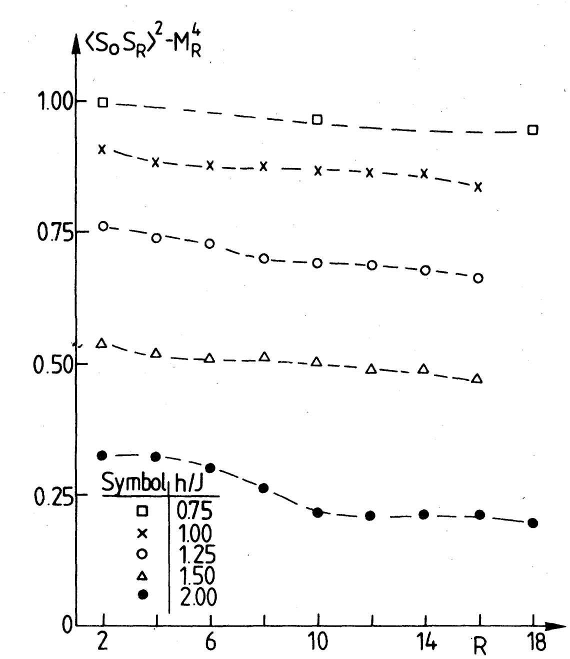

FIG. 13. Spin-glass correlation function $[\langle S_0 S_R \rangle^2]_{\text{av}} - M_r^4$ plotted vs distance R at $k_B T/J = 1.0$ for several values of the random field.

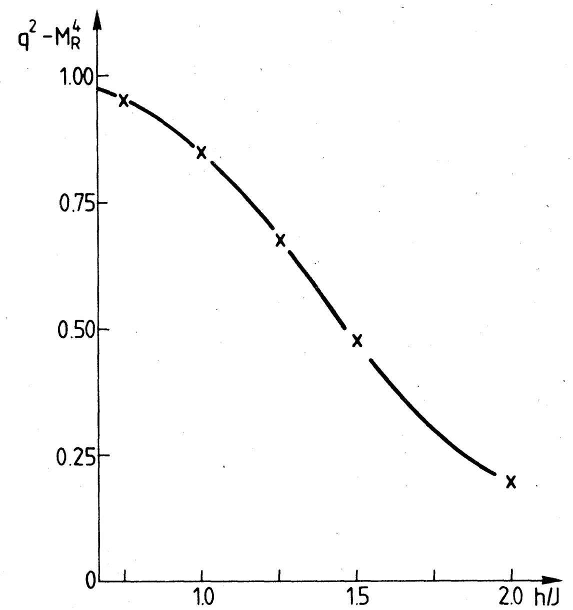

FIG. 14. Spin-glass order parameter $q - M_r^2$ resulting from Fig. 13, plotted vs h at $k_B T/J = 1.0$.

that for weak random fields the correlation function settles down at constant values if l exceeds a few lattice spacings. Thus the order parameter q of the Mattis spin-glass can be identified (cf. Fig. 2). Its variation with the field is shown in Fig. 14. The critical field at which the transition to the disordered state occurs agrees roughly with the location of the specific-heat maximum. This breakdown of the spin-glass order cannot be analyzed in terms of the above domain-wall considerations, however, which hold for weak random fields only.

V. CONCLUSIONS

In summary, we have shown that for $T = 0$ the system experiences an infinite number of (first-order) phase transitions as a function of random field in the range $2 < h/J \leq 4$, at the values $h/J = 2 + 2/n$, $n = 1, 2, \ldots$. In the range $0 < h/J \leq 2$ there exists even in any finite interval infinitely many (weak) singularities (whenever h/J is equal to a rational number in this range). A spontaneous magnetization does not exist even for small nonzero h in the ground state, but order of the Mattis spin-glass does exist. For $T > 0$ these singularities are wiped out at temperatures as low as about $\frac{1}{10}$ of the pure critical temperature. For T less than T_c of the pure system the transition from a paramagnetic state to Mattis order at $h_c(T)$ is located by observing a specific-heat singularity. In this ordered phase a strong ferromagnetic correlation develops as $h \rightarrow 0$. The (ferromagnetic) correlation function is calculated from the density of domain walls in the system and the resulting susceptibility is in fair agreement with the numerical calculations. Thus, unlike other two-dimensional spin-glasses, this system clearly exhibits spin-glass order at nonzero temperatures (note that there is no trivial relation to the pure Ising model, in contrast to the standard Mattis spin-glass). An investigation of the critical properties of the present spin-glass system would be of great interest.

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