Rapid Communications

The Rapid Communications section is intended for the accelerated publication of important new results. Manuscripts submitted to this section are given priority in handling in the editorial office and in production. A Rapid Communication may be no longer than $3\frac{1}{2}$ printed pages and must be accompanied by an abstract. Page proofs are sent to authors, but, because of the rapid publication schedule, publication is not delayed for receipt of corrections unless requested by the author.

Critical ultrasonic attenuation in superfluid helium: Mixing of order-parameter and fluctuation contributions

Richard A. Ferrell* and Jayanta K. Bhattacharjee†
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, and Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 7 October 1980)

The mixing of the Landau-Khalatnikov (order-parameter) and fluctuation contributions to the critical ultrasonic attenuation is calculated in the ϵ expansion. The resulting scaling function agrees with the experimental data over nine octaves.

When a system undergoes a second-order phase transition and enters a state of broken symmetry, this change is characterized by a nonvanishing average value of the order parameter $\langle\phi\rangle \neq 0$. A pressure wave propagating through the system produces a local modulation of $\langle\phi\rangle$. If the relaxation rate γ is comparable to the angular frequency ω of the wave, the modulation of $\langle\phi\rangle$ will lag behind the input pressure signal. The Landau-Khalatnikov[1] (LK) theory of ultrasonic attenuation is based on this hysteresis. The LK attenuation exhibits a characteristic relaxational maximum when $\gamma = \omega$. It vanishes at the critical point, where $\langle\phi\rangle = 0$. The actual critical ultrasonic attenuation, however, by no means vanishes at the critical point. This discrepancy can be attributed to the effect of fluctuations. Being a mean-field theory, the LK approach neglects fluctuations. Williams and Rudnick,[2] in interpreting their measurements of the ultrasonic attenuation in liquid ^{4}He II, achieved success by simply superposing the LK component on top of the fluctuation component. They assumed the latter to be the same function of the reduced temperature below the λ point as it was above, where there is no LK contribution. This approach, however, has been found by Tozaki and Ikushima[3] to fail at higher frequencies. It has clearly become necessary to have a theory of critical ultrasonic attenuation which does not *ad hoc* assume a linear superposition of the order-parameter (LK) and fluctuation contributions. It is essential instead to take the fluctuations into account from the outset. The purpose of this short note is to present such a theory. Our principal result is the scaling function shown in Fig. 2, which fits the experimental data over a frequency range that spans three orders of magnitude. From a more abstract point of view, the general significance of our work is that there is an inescapable mixing of the two components which necessarily invalidates any approach that assumes a linear superposition. This "hybridization" also occurs in the intensity of light scattering[4,5] and can be expected to occur in a wide range of phenomena not restricted to superfluid helium.

Recall that the origin of the ultrasonic attenuation is a lagging temperature signal[6] resulting from a complex frequency-dependent specific heat $C_P(\omega, \gamma)$. Thus the attenuation in one wavelength is

$$\alpha\lambda = (\text{const})\,\mathrm{Im}\frac{-1}{C_P(\omega,\gamma)} \quad . \tag{1}$$

For $(\omega, \gamma) > 1$ MHz liquid helium is in the van Hove precritical regime[7,8] where the kinetic coefficient for order-parameter relaxation equals its noncritical background value B_ψ. Therefore the temperature dependence of the relaxation rate is given by

$$\gamma = B_\psi \kappa^2 = B_\psi \kappa_0^2 |t|^{4/3} \quad , \tag{2}$$

where t is the reduced temperature, κ^{-1} is the correlation length, and κ_0 is a nonuniversal constant of the fluid that has to be determined empirically and independently above and below the λ point.

The specific heat that we need for Eq. (1) is related to the correlation function G_Q of the quadratic field $Q = Q_1 + Q_2$, where the first- and second-order components are $Q_1 = 2\langle\phi\rangle\phi_L$ and $Q_2 = \phi^2 = \phi_L^2 + \phi_T^2$.

$\phi_{L,T}$ are the longitudinal and transverse fluctuations of the order parameter, respectively. The correlation function of Q between times 1 and 2 is consequently composed of three terms,

$$G_Q(12) = \langle Q(1)Q(2)\rangle = G_{11} + 2G_{12} + G_{22} \ , \quad (3)$$

where

$$G_{11} = \langle Q_1(1)Q_1(1)\rangle = 4\langle\phi\rangle^2\langle\phi_L(1)\phi_L(2)\rangle \equiv 4\langle\phi\rangle^2 G(12) \ . \quad (4a)$$

The mixing of the two contributions to Q is clearly exhibited by the cross term

$$2G_{12} = \langle Q_1(1)Q_2(2)\rangle + \langle Q_2(1)Q_1(2)\rangle = 2\langle\phi\rangle[\langle\phi_L(1)\phi^2(2)\rangle + \langle\phi^2(1)\phi_L(2)\rangle] \ . \quad (4b)$$

The purely fluctuational component is

$$G_{22} = \langle Q_2(1)Q_2(2)\rangle_c = \langle\phi^2(1)\phi^2(2)\rangle_c \ . \quad (4c)$$

(The subscript on the above equilibrium average indicates the connected part.) We now work with the Fourier transform and convert all correlation functions to response functions. In the broken symmetry state there is a three-point vertex which produces the mixing and interaction effects illustrated by the graphs of Fig. 1. We treat these effects in the ϵ expansion. Graph (a) involves only the noninteracting response function γg_0 corresponding to correlation function

$$g_0 = \frac{1}{\gamma - i\omega} \ . \quad (5)$$

The zero-order part of Eq. (4a) becomes therefore the response function

$$g_{11}^0 = 4\langle\phi\rangle^2 g_0 = C_0\gamma g_0 = C_0\frac{\gamma}{\gamma - i\omega} \ , \quad (6)$$

of the LK form. The proportionality of $\langle\phi\rangle^2$ to γ, which is valid to lowest order in the ϵ expansion, is expressed by the constant C_0. As expected, Eq. (6) vanishes at the λ point, where $\gamma = 0$. In the thermodyanic limit $\omega \to 0$ it adds the constant C_0 to the specific heat for all $\gamma > 0$ (i.e., for all temperatures below the λ point). This is the finite-frequency form

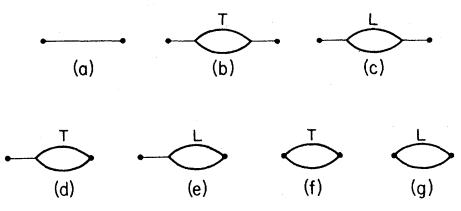

FIG. 1. Low-order contributions to the specific heat below the λ point.

of the familiar mean-field step-function contribution of the order parameter to the specific heat.

The interaction corrections to Eq. (4a) corresponding to graphs (b) and (c) become

$$\Delta g_{11} = 4\langle\phi\rangle^2\Delta g \propto (\gamma g_0)^2 L_{T,L} \ , \quad (7)$$

where $L_{T,L}$ represent the "bubbles" of graphs (f) and (g). Each "leg" of a graph corresponds to a factor of γg_0. Thus Eq. (4b) becomes

$$g_{12} \propto \gamma g_0 L_{T,L} \ . \quad (8)$$

Paying due attention to numerical factors, we find

$$g_Q^T(\omega,\gamma) = (1-\gamma g_0)^2\Delta L_T + \mathrm{Re}L_T(-i\gamma,\gamma) \quad (9a)$$

and

$$g_Q^L(\omega,\gamma) = (1-3\gamma g_0)^2\Delta L_L + L_L(-i\gamma,\gamma) \quad (9b)$$

for the total contribution to the specific heat from all of the low-order graphs of Fig. 1 involving transverse fluctuations and longitudinal fluctuations, respectively. We have made a subtraction at the pole $\omega = -i\gamma$ as this is, by definition, the renormalized value.

A particularly satisfactory feature of Eq. (9a) is that, although ΔL_T does not have a thermodynamic limit, Eq. (9a) does. This is because of the vanishing of the prefactor of ΔL_T as $\omega \to 0$. For this reason we do not have to sum a series of bubbles, as is often done in this kind of problem. It would be natural to represent superfluid helium, which has one longitudinal and one transverse mode, by an equal mixture of Eqs. (9a) and (9b). We, however, follow a different approach which adheres more closely to the ϵ expansion. We take as the hallmark of the superfluid not that it has just one transverse component of the order parameter, but rather that its critical specific-heat exponent vanishes (or very nearly so). It is well known that to describe this situation to first order in the ϵ expansion we need three transverse components. We therefore argue that the correct weighting of Eqs. (9a) and (9b) is $\frac{3}{4}$ and $\frac{1}{4}$, respectively. At this point a computational convenience enters. We have found that the results are not significantly changed (error of a few percent) by using 100% of Eq. (9a). Therefore, from now on, for the sake of simplicity, we drop Eq. (9b) and use only Eq. (9a). A further simplification, justified by a detailed study to be published elsewhere, is to neglect the oscillatory nature of the intermediate lines in graph (f). Thus, ignoring second sound effects and treating the lines as relaxation only, we obtain $L_T(\omega,\gamma) = L_\lambda(\omega)$, independent of γ, where $L_\lambda(\omega)$ is the λ-point specific heat. The subtraction is consequently

$$\Delta L_T = L_\lambda(\omega) - \mathrm{Re}L_\lambda(-i\gamma) = \ln\frac{\gamma}{-i\omega} \ . \quad (10)$$

Denoting $\mathrm{Re}L_\lambda(\omega)$ by $L_{1\lambda}(\omega)$ and writing $g_Q^T(\omega,\gamma)$ as $L(\omega,\gamma)$ we find from substituting Eq. (10) into

Eq. (9a)

$$L(\omega,\gamma)=\frac{C_0\gamma}{\gamma-i\omega}+\frac{\omega^2}{(\omega+i\gamma)^2}\ln\frac{\gamma}{-i\omega}+\ln\frac{\omega}{\gamma}$$
$$+L_{1\lambda}(\omega)=\frac{C_0}{1-i\Omega}-\frac{\Omega^2}{(\Omega+i)^2}\ln(-i\Omega)$$
$$+\ln\Omega+L_{1\lambda}(\omega)\ , \qquad (11)$$

where we have introduced the scaled frequency[9] $\Omega=\omega/\gamma$. With $L=L_1+iL_2$, the imaginary part gives the scaling function

$$F(\Omega)=\frac{2}{\pi}L_2(\Omega)$$
$$=\frac{2}{\pi}\frac{C_0\Omega}{1+\Omega^2}+\Omega^2\frac{\Omega^2-1}{(\Omega^2+1)^2}$$
$$+\frac{4}{\pi}\frac{\Omega^3}{(\Omega^2+1)^2}\ln\Omega\ , \qquad (12a)$$

which is shown by the solid curve in Fig. 2(b). The dashed curve shows the first term of Eq. (12a) alone, the LK contribution. The real part of L does not scale, being a function of two variables

$$L_1(\omega,\gamma)=\frac{C_0}{1+\Omega^2}+\frac{\pi}{2}\frac{\Omega^3}{(\Omega^2+1)^2}$$
$$-\Omega^2\frac{\Omega^2-1}{(\Omega^2+1)^2}\ln\Omega+\ln\Omega+L_{1\lambda}(\omega)\ . \qquad (12b)$$

The thermodynamic limit of Eq. (12b) is $L_1(0,\gamma)=C_0+L_{1\lambda}(\gamma)$. Although the ϵ expansion gives a value for C_0 of roughly the right size, it is obviously better to determine C_0 empirically from the thermodynamic data. Having determined in this way $C_0=5.4$ as well as $\kappa_0=1.5\times10^8$ cm^{-1}, we have been able to account very well for the ultrasonic data on the basis of Eq. (1), which becomes $\alpha\lambda\propto-\mathrm{Im}L^{-1}=L_2/|L|^2$. Thus, neglecting the very slight temperature dependence of the sound velocity, we find the ratio of the attenuation below the λ point to that at the λ point to be

$$\frac{\alpha}{\alpha_\lambda}=r^{-1}F(\Omega)\ , \qquad (13)$$

where $r=|L|^2/|L|^2_\lambda$. Because the factor r^{-1} does not scale, the product depends on the frequency of the measurement. At higher frequencies, $|L|^2$ is a relatively strong function of Ω and tends to cancel out the maximum of $F(\Omega)$. For this reason the 1-GHz data of Commins and Rudnick[10] as well as the 1.7-GHz data of Lambert, Legros, and Salin[11] hardly exhibit any maximum in α/α_λ, as seen in Fig. 2(a) which exhibits the raw data. This contrast with the pronounced maximum in the 3-MHz data of Williams and Rudnick.[2] The effect of frequency on the shape of α/α_λ is also clearly evident in the data of Tozaki and Ikushima,[3] and has been remarked upon by them.

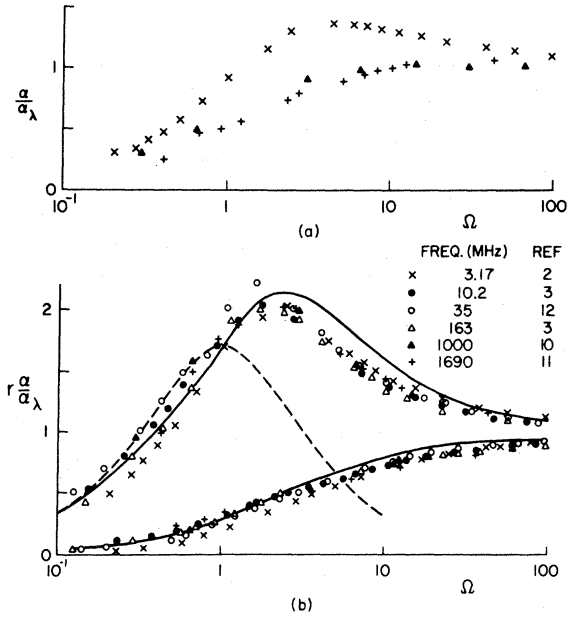

FIG. 2. (a) Unprocessed experimental data showing dependence of shape on frequency and absence of scaling. (b) Scaling functions $F(\Omega)$ from Eq. (12a) and processed experimental attenuation data $r\alpha/\alpha_\lambda$, where $r=|L|^2/|L_\lambda|^2$. Inclusion of r restores scaling. The dashed curve shows the Landau-Khalatnikov contribution. The lower curve shows the scaling above the lambda point (Ref. 6).

In order to reveal the underlying regularity in the apparently disparate data described above, it is essential to subject them to a "processing." This is simply the removal of the offending nonscaling factor r^{-1} from the right-hand side of Eq. (13) by transferring it to the left. The "processed" values $r\alpha/\alpha_\lambda$ for the various runs referred to above, as well as for the 35-MHz run of Roe, Meyer, and Ikushima[12] for their 0.7% mixture, are shown in Fig. 2(b). It will be noted that not only do the data thereby coalesce into a single scaling function, but also that this function is in satisfactory agreement with the predicted scaling function of Eq. (12a).

In closing we are happy to acknowledge support from the National Science Foundation under Grant No. PHYS 77-27084 to the Institute for Theoretical Physics of the University of Califorinia, Santa Barbara, and under Grants No. DMR 78-01172, No. 79-00908, and No. 79-10819 to the University of Maryland. We are also grateful to Dr. D. Salin and to Dr. H. Meyer for providing us with their experimental data.

*Permanent address: University of Maryland, College Park, Md. 20742.
†Present address: University of Maryland, College Park, Md. 20742.

[1]L. D. Landau and I. M. Khalatnikov, Sov. Phys. Dokl. 96, 469 (1951) [English translation, *Collected Papers of L. D. Landau,* edited by D. ter Haar (Gordon and Breach, New York, 1965), p. 626].
[2]R. D. Williams and I. Rudnick, Phys. Rev. Lett. 25, 276 (1970). See also R. D. Williams, Ph.D. thesis (University of California, Los Angeles, 1970) (unpublished).
[3]K. Tozaki and A. Ikushima, J. Low Temp. Phys. 32, 379 (1978).
[4]R. A. Ferrell and J. K. Bhattacharjee, in Proceedings of Colloque Pierre Curie, Paris, September 1980, edited by N. Boccara (unpublished).
[5]J. F. Nicoll, University of Maryland, Institute for Physical Science and Technology Publication No. BN935 (1980) and Phys. Lett. 80A, 317 (1980).
[6]R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. Lett. 44, 403 (1980).
[7]R. A. Ferrell and J. K. Bhattacharjee, in *Proceedings of the International Conference on Critical Dynamics, Geneva, 1978,* edited by C. P. Enz (Springer Verlag, Berlin, 1979).
[8]R. A. Ferrell and J. K. Bhattacharjee, in *Proceedings of the Second US-USSR Symposium on Light Scattering, New York, 1978,* edited by H. Z. Cummins (Plenum, New York, 1979).
[9]This definition of Ω is twice that of Ref. 6.
[10]D. E. Commins and I. Rudnick, in *Proceedings of the Thirteenth International Conference on Low Temperature Physics, Boulder, Colorado, 1973,* edited by K. D. Timmerhaus, W. J. O'Sullivan, and E. F. Hammel (Plenum, New York, 1974), Vol. I. See also D. E. Commins, Ph.D. thesis (University of California, Los Angeles, 1972) (unpublished).
[11]D. Salin (private communication). See also B. Lambert, P. Legros, and D. Salin, J. Phys. (Paris) Lett. 41, 507 (1980).
[12]D. B. Roe, H. Meyer, and A. Ikushima, J. Low Temp. Phys. 32, 67 (1978).