Johnson noise in ideal type-II superconducting films

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The Johnson-noise power spectrum is calculated for an ideal type-II superconducting film containing an array of thermally agitated vortices. The interactions between the vortices are taken into account via a wave-vector-dependent interaction matrix. We develop a theoretical method for the calculation of the voltage produced by a moving vortex in a measuring circuit with leads of finite radii. For a special measuring circuit, only transverse modes contribute to the measured voltage. Below a characteristic frequency determined by the interaction between the vortices, the viscous drag, and the geometry of the measuring circuit, the power spectrum is suppressed. The ac impedance, whose real part is proportional to the Johnson-noise power spectrum, is also calculated.

I. INTRODUCTION

It is by now well known that the motion of singly quantized vortices generates a voltage in a type-II superconductor. That is, such motion produces a voltage signal in a sensitive voltmeter whose terminals are connected by a pair of leads to two contacts on the specimen. The term flux-flow voltage usually is used to refer to the voltage induced by motion of vortices from one side of a current-carrying specimen to the other. The time-averaged part of the flux-flow voltage was first investigated thoroughly by Kim, Hempstead, and Strnad.[1] The flux-flow noise voltage—the fluctuating component of the flux-flow voltage—was first examined by van Ooijen and van Gurp.[2]

In interpreting their flux-flow noise experiments, the latter authors found that the main features of the power spectrum could be understood by assuming that the voltage is produced by a sequence of random overlapping pulses produced by the motion of flux entities (either as singly quantized vortices or as bundles of these) moving across the specimen at constant speed. This interpretation was given some theoretical support by the work of Clem,[3] where general expressions for the autocorrelation function and power spectrum were obtained. Although this theory allowed for the possibility of time-dependent vortex velocities, most of the subsequent model calculations of Ref. 3 assumed constant vortex velocities or constant flux-line dislocation dipole velocities, such that the flux-flow noise voltage was regarded as being generated by the rigid motion of vortex density fluctuations moving with constant velocity. Such assumptions led to the predictions that the shape of the power spectrum versus frequency should depend strongly upon the measuring circuit configuration and that strong time-of-flight effects should be observed in cross-correlation experiments between separated contact pairs. Recent experiments[4] have shown that the power spectrum does depend upon the measuring circuit configuration, but not as strongly as predicted by Ref. 3, and that time-of-flight effects are not observed in cross-correlation experiments in which flux pinning centers break up correlations among vortices during their travel from one contact pair to the other.[5] Further experiments have shown that local velocity fluctuations are important[5] and that local flux-flow noise production depends very strongly upon details of the flux pinning mechanism.[6]

In this paper we lay some groundwork for a more complete theory of flux-flow noise including velocity correlations in a natural way. Our approach is intended to be consistent with approaches to the calculation of pinning forces in which the strength of the interactions between vortices in an array plays a crucial role. In such approaches, for a random array of pinning centers, the bulk pinning force decreases as the strength of interaction increases and averages to zero in the limit of a perfectly rigid vortex array. It is natural to extend these ideas to the question of flux-flow noise generation and to point out that, as an array of vortices is driven across an array of pinning centers, it is likely that local deformations of the vortex array build up and decay in the vicinity of the pinning centers. On occasion, many vortices may be involved, depending upon the spatial extent of the deformation, and, because the vortices are interacting with each other, the velocities of adjacent vortices are correlated.

We consider in this paper voltage noise generated when a regular array of vortices, a flux-line lattice (FLL), undergoes a buildup and decay of local FLL deformations. However, in contrast to the case of flux flow, where the FLL deformations can be generated as a consequence of the relative motion of the vortex array and the array of pinning centers, the FLL deformations are here generated by the random, thermally generated Langevin forces on the vortices. These forces may be interpreted as arising from the interaction between the vortices and the phonons of the crystalline lattice (CL). Our point of view here is that, in a classical description at zero temperature, all vortices would be at rest at their equilibrium positions in a perfect FLL; there being no vortex motion, no voltage would be produced. On the other hand, at nonzero temperature, thermal fluctuations excite modes of the FLL; the corresponding vortex motion generates a time-varying voltage, whose autocorrelation function and power spectrum are calculated in this paper.

Burgess[7] calculated the Johnson noise due to thermally excited motion of noninteracting vortices elastically tied to pinning centers with an elastic force constant per unit length K. Each vortex was assumed to have a mass M per unit length and to be subject to a viscous drag force per unit length of magnitude ηv, where v is the vortex speed. According to his calculations, the power spectrum as a function of frequency ω was predicted (a) to be proportional to ω^2 for $\omega << \omega_0 = K/\eta$, (b) to attain a constant, plateau value, characterized by the flux-flow resistance, for $\omega_0 << \omega << \tau^{-1} = \eta/M$, and (c) to be proportional to ω^{-2} for $\omega >> \tau^{-1}$. In the absence of pinning (taking $\omega_0 = K/\eta = 0$), one might expect the plateau region to persist to zero frequency. However, our results show that including the vortex-vortex interactions in a two-dimensional (not accounting for bending), FLL in a thin, type-II superconducting film yields a plateau only in the range $\tau_R^{-1} << \omega << \tau^{-1}$, where $\tau_R^{-1} = K_t/\eta R^2$. Here, $K_t = \phi_0 c_{66}/B$ describes the elastic response of the FLL to shear, and R is the contact radius. For much lower frequencies ($\omega << \tau_R^{-1}$), the power spectrum is predicted to be suppressed and to be proportional to ω.

This paper is organized as follows. In Sec. II, we calculate an expression for the voltage measured by a given measuring circuit across a superconductor when a vortex is moving in the superconductor. In Sec. III, we present a formalism by which noise voltages can be calculated, taking into account the interactions between the vortices. The theory is then applied to Johnson noise. In Sec. IV, the ac impedance of the vortex lattice is calculated and is compared to the Johnson-noise power spectrum. A discussion of the results is given in Sec. V.

II. MEASURED VOLTAGE IN SUPERCONDUCTORS

A. Resolution function

The measured voltage generated by a moving vortex depends upon both the geometric configuration of the measuring circuit and the velocity of the vortex. In this section we derive a gauge-invariant expression for this voltage, allowing for finite radius of the leads. Our derivation is a generalization of that given in Ref. 8, which applies only to measuring circuits with leads of negligibly small radius.

Consider a type-II superconductor with two normal leads attached to it, as sketched in Fig. 1. The other ends of the leads are connected to terminals A and B of a sensitive voltmeter. The measured voltage generated by the motion of a vortex, which threads the superconductor, is[3]

$$V = V_A - V_B$$
$$= -\int_{A\,[C_M]}^{a} d\vec{l} \cdot \vec{\nabla}\phi - \int_{a\,[C_S]}^{b} d\vec{l} \cdot \vec{\nabla}\psi' - \int_{b\,[C_m]}^{B} d\vec{l} \cdot \vec{\nabla}\phi \ . \quad (1)$$

Here ψ' is the electrochemical potential per unit charge in the superconductor, ϕ is the same quantity in the normal leads, C_S is a path inside the superconductor that connects the two points a and b, one under each contact, and C_M is a path that connects A and B through the leads and the voltmeter. Assuming no contact voltage between the superconducting specimen and the leads, V is independent of C_S and C_M. In the normal leads we can write

$$\vec{e} = -\vec{\nabla}\phi - \frac{1}{c}\partial\vec{a} \ , \quad (2)$$

where $\vec{e}$ is the electric field, c is the speed of light in vacuum, $\vec{a}$ is the vector potential, and ∂ represents a time derivative. In the superconductor the electrochemical potential per unit charge can be written as[3]

$$\psi' = \frac{\phi_0}{2\pi c}\partial\gamma \ , \quad (3)$$

where ϕ_0 is the flux quantum and γ is the phase of the order parameter. Combining Eqs. (1) through

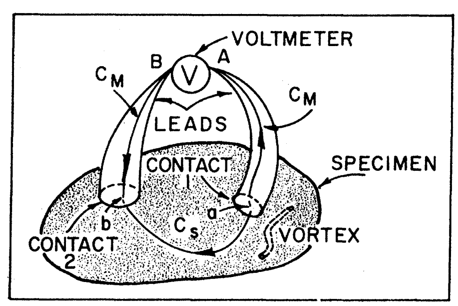

FIG. 1. Measuring circuit with thick leads, for which the measured voltage V is described by Eq. (1).

(3) yields

$$V=-\frac{\phi_0}{2\pi c}\int_{a\,[C_S]}^{b} d\vec{l}\cdot\vec{\nabla}\partial\gamma-\frac{1}{c}\int_{a\,[C_M]}^{A} d\vec{l}\cdot\partial\vec{a}-\frac{1}{c}\int_{B\,[C_M]}^{b} d\vec{l}\cdot\partial\vec{a}-\int_{a\,[C_M]}^{A} d\vec{l}\cdot\vec{e}-\int_{B\,[C_M]}^{b} d\vec{l}\cdot\vec{e}\ . \quad (4)$$

For leads with negligible resistance, the last two terms in Eq. (4) are very small, and the equation is the same as Eq. (2.1) of Ref. 3 and Eq. (1) of Ref. 4 with ψ' replaced by $(\phi_0/2\pi c)\partial\gamma$.

Next we look at the quantity $I_M V$, where I_M is a virtual current flowing through the leads and the superconductor from A to B. Assuming that there is no accumulation of charge anywhere, we can visualize a continuous "current tube," through which a fixed amount of current, dI_M, flows. Choosing the paths C_S and C_M to lie along a current tube, we have

$$dI_M V=\frac{\phi_0}{2\pi c}dS_1\hat{n}_M\cdot\vec{j}_M\frac{d\gamma_a}{dt}-\frac{\phi_0}{2\pi c}dS_2\hat{n}_M\cdot\vec{j}_M\frac{d\gamma_b}{dt}+dS_{M1}\hat{n}_M\cdot\vec{j}_M\int_A^a d\vec{l}\cdot\left(\frac{\partial\vec{a}}{c}+\vec{e}\right)+dS_{M2}\hat{n}_M\cdot\vec{j}_M\int_b^B d\vec{l}\cdot\left(\frac{\partial\vec{a}}{c}+\vec{e}\right)\ , \quad (5)$$

where dS_1 and dS_2 are surface elements in contacts 1 and 2, respectively, dS_{M1} is the cross section of the current tube between A and a, dS_{M2} is the cross section of the current tube between b and B. In all terms on the right-hand side (RHS) of Eq. (5),

$$dS\hat{n}_M\cdot\vec{j}_M\equiv dI_M\ , \quad (6)$$

where dS is any surface element specified above. Integrating Eq. (5) over all current tubes yields

$$I_M V=-\frac{\phi_0}{2\pi c}\int_{\substack{\text{both}\\ \text{contacts}}} dS\,\hat{n}\cdot\vec{j}_M\partial\gamma+\frac{1}{c}\int_{\text{leads}} d^3r\,\vec{j}_M\cdot\partial\vec{a}+\int_{\text{leads}} d^3r\,\vec{j}_M\cdot\vec{e}\ , \quad (7)$$

where $\hat{n}$ is an outward normal to the superconductor. Integrating by parts yields

$$\frac{\phi_0}{2\pi c}\frac{d}{dt}\int_{\text{inside}} d^3r\,\vec{j}_M\cdot\vec{\nabla}\gamma=\frac{\phi_0}{2\pi c}\frac{d}{dt}\int_{\substack{\text{both}\\ \text{contacts}}} dS\,\hat{n}\cdot\vec{j}_M\gamma-\frac{\phi_0}{c}\frac{d}{dt}\int_{\text{cut}} dS\,\hat{n}_{\text{cut}}\cdot\vec{j}_M\ . \quad (8)$$

The integral on the left-hand side (LHS) is over the inside of the superconductor. The integral in the second term on the RHS is over a cut surface bounded by the closed curve formed by the vortex axis and an arbitrary curve C on the surface of the superconductor connecting the top (t) and bottom (b) of the vortex, as shown in Fig. 2.

We now consider the interaction energy

$$\Delta U=\int_{\substack{\text{all}\\ \text{space}}} d^3r\,\frac{1}{4\pi}\vec{b}_M\cdot\vec{b}_1+\int_{\text{inside}} d^3r\,\frac{4\pi\lambda^2}{c^2f^2}\vec{j}_M\cdot\vec{j}_1\ , \quad (9)$$

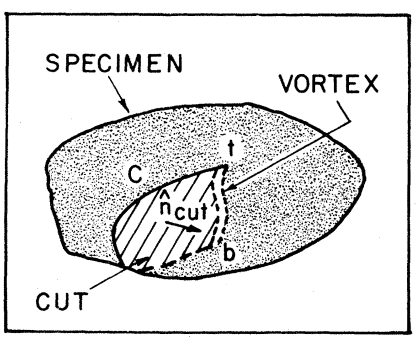

FIG. 2. View of specimen showing the cut surface appearing in Eq. (8); t and b denote the top and bottom of the vortex, respectively. The voltmeter and leads are not shown.

where f is the magnitude of the reduced order parameter and the subscript 1 denotes the contribution from the vortex. Writing

$$\vec{b}_M\cdot\vec{b}_1/4\pi=\vec{\nabla}\cdot\vec{a}_M\times\vec{b}_1/4\pi+\vec{j}_1\cdot\vec{a}_M/c\ ,$$

we can show that $\Delta U=0$. Next we evaluate ΔU by writing

$$\vec{b}_M\cdot\vec{b}_1/4\pi=\vec{\nabla}\cdot\vec{a}_1\times\vec{b}_M/4\pi+\vec{j}_M\cdot\vec{a}_1/c\ .$$

This leads to the equation

$$\frac{\phi_0}{2\pi c}\int_{\text{inside}} d^3r\,\vec{j}_M\cdot\vec{\nabla}\gamma=\frac{1}{c}\int_{\text{leads}} d^3r\,\vec{j}_M\cdot\vec{a}_1\ , \quad (10)$$

where we have used the fact that $\vec{j}_1$ is parallel to the specimen surface.

Now we make the important assumption that $\vec{j}_M$ is independent of time both under the contacts and in the leads. This means that the skin depth associated with the characteristic frequency of the flux movements has to be large by comparison with the size of the leads, and the "backflow current" around the vortex core is being ignored.

With this assumption and the aid of Eqs. (8) and (10), we can rewrite Eq. (7) as

$$V=-\frac{\phi_0}{I_M c}\frac{d}{dt}\int_{\text{cut}} dS\,\hat{n}_{\text{cut}}\cdot\vec{j}_M+\frac{1}{I_M}\int_{\text{leads}} d^3r\,\vec{j}_M\cdot\vec{e}\ . \quad (11)$$

If we have low-resistance leads, we can drop the second term on the RHS, and Ampere's law and Stokes's theorem then enable us to write

$$V=\frac{\hbar}{2e}\frac{d\theta}{dt} , \quad (12)$$

where, ignoring the contribution from along the vortex axis,

$$\theta=\frac{c}{2I_M}\int_{b\,C}^{t} d\vec{l}\cdot\vec{b}_M . \quad (13)$$

The integral in Eq. (13) is along the boundary C of the cut from the bottom (b) to the top (t) of the vortex, as sketched in Fig. 2. This is the same as Eq. (4) in Ref. 8 for the case of vanishing lead radius. Note that in the case of vanishing lead radius, $\vec{j}_M$ in the leads and under the contacts is time independent for all practical purposes.

We can also write

$$V=\frac{\phi_0}{4\pi I_M}[\vec{b}_M(\vec{\rho}_t)\cdot\partial\vec{\rho}_t-\vec{b}_M(\vec{\rho}_b)\cdot\partial\vec{\rho}_b] , \quad (14)$$

where $\vec{\rho}_t$ and $\vec{\rho}_b$ are the coordinates of the top and bottom of the vortex, respectively.

For the case of a flat slab or thin film of uniform thickness, we can usually assume the top and bottom of a vortex move at the same velocity. The voltage is then simply

$$V=\vec{g}(\vec{\rho})\cdot\vec{v} , \quad (15)$$

where $\vec{\rho}$ is the position of the vortex, now a two-dimensional vector, $\vec{v}\equiv\partial\vec{\rho}$, and

$$\vec{g}(\vec{\rho})\equiv\frac{\phi_0}{4\pi I_M}[\vec{b}_{Mt}(\vec{\rho})-\vec{b}_{Mb}(\vec{\rho})] . \quad (16)$$

Here $\vec{b}_{Mt}$ and $\vec{b}_{Mb}$ are the values of $\vec{b}_M$ at the top and bottom of the slab (film), respectively.

B. Measured voltage and noise

For a specimen containing a number of vortices, the total measured voltage is a superposition of contributions from individual vortices, such that

$$V=\sum_i V_i . \quad (17)$$

Here,

$$V_i=\frac{\hbar}{2e}\frac{d\theta_i}{dt} , \quad (18)$$

where θ_i is defined as in Eq. (13), but according to the position of vortex i.

From now on, we shall only consider the case of a flat slab or thin film, so that we can extend Eq. (18) to

$$V=\sum_i\vec{g}(\vec{\rho}_i)\cdot\vec{v}_i . \quad (19)$$

According to Ref. 9, we can define a vortex-current density,

$$\vec{J}(\vec{\rho},t)=\sum_i\vec{v}_i(t)\delta_2(\vec{\rho}-\vec{\rho}_i(t)) . \quad (20)$$

Conservation of the number of vortices leads to the continuity equation

$$\frac{\partial n}{\partial t}+\vec{\nabla}\cdot\vec{J}=0 , \quad (21)$$

where n is the vortex density

$$n(\vec{\rho},t)=\sum_i\delta_2(\vec{\rho}-\vec{\rho}_i(t)) . \quad (22)$$

$\vec{J}(\vec{\rho},t)$ contains all the information about vortex dynamics. The voltage then can be written as

$$V(t)=\int d^2\rho\,\vec{g}(\vec{\rho})\cdot\vec{J}(\vec{\rho},t) , \quad (23)$$

where the integral is over the whole specimen. The time-averaged voltage is

$$\langle V(t)\rangle_t=\int d^2\rho\,\vec{g}(\vec{\rho})\cdot\langle\vec{J}(\vec{\rho},t)\rangle_t , \quad (24)$$

and the noisy part of the voltage is

$$\delta V(t)\equiv V(t)-\langle V(t)\rangle_t = \int d^2\rho\,\vec{g}(\vec{\rho})\cdot\delta\vec{J}(\vec{\rho},t) , \quad (25)$$

where

$$\delta\vec{J}(\vec{\rho},t)=\vec{J}(\vec{\rho},t)-\langle\vec{J}(\vec{\rho},t)\rangle_t . \quad (26)$$

The usual measured quantities are the autocorrelation function and the power spectrum. The autocorrelation function is given by

$$\Psi_V(s)\equiv\langle\delta V(t)\delta V(t+s)\rangle_t = \int d^2\rho\int d^2\rho'\sum_{\alpha,\beta}g_\alpha(\vec{\rho})g_\beta(\vec{\rho}\,') \times K_{\alpha\beta}(\vec{\rho},\vec{\rho}\,',s) , \quad (27)$$

where

$$K_{\alpha\beta}(\vec{\rho},\vec{\rho}\,',s)\equiv\langle\delta J_\alpha(\vec{\rho},t)\delta J_\beta(\vec{\rho}\,',t+s)\rangle_t \quad (28)$$

is the vortex-current correlation function. Here, α and β refer to the Cartesian coordinates. The power spectrum, $W_V(\omega)$, is twice the Fourier transform of the autocorrelation function.

We thus see that the voltage and its autocorrelation function consist of two parts, one depending only on the measuring circuit geometry, and the other on the vortex dynamics. This is true if we assume the measuring circuit does not influence the motion of the vortices. The above formalism can be easily applied to the case of cross-correlation measurements, where two pairs of leads, placed at different parts of the specimen, are used to measure the noise.

We now turn our attention to the vortex dynamics, keeping in mind that we want to take the interactions between vortices into account. Since the interaction force on a vortex in general depends on the position of all other vortices, a treatment in reciprocal space is much simpler mathematically than in real space. This is done in the next section and the results are applied to a special kind of noise, Johnson noise.

III. THEORY INCLUDING THE INTERACTION BETWEEN VORTICES: APPLICATION TO JOHNSON NOISE

A. Expansion of vortex positions in normal modes

Consider a slab or thin film, with no bending of vortices allowed. The positions of vortices are described by two-dimensional vectors. If the vortices form a perfect lattice, each vortex can be uniquely identified by its equilibrium position within the lattice. We attach a reference frame to the flux-line lattice (FLL frame). The equilibrium position of a vortex is measured in this frame. There is also a laboratory frame (lab frame), which coincides with the FLL frame if the lattice is not moving as a whole. The position of the vortex with equilibrium position $\vec{1}$ is, in the FLL frame,

$$\vec{\rho}_0(\vec{1},t)=\vec{1}+\vec{s}(\vec{1},t) \quad , \tag{29}$$

where $\vec{s}(\vec{1},t)$ is the deviation from the equilibrium position. The interaction energy between the vortices can be written in quadratic form in the harmonic approximation

$$V(\vec{s}(\vec{1},t))=\frac{1}{2}\sum_{\vec{1},\vec{1}'}\tilde{G}(\vec{1},\vec{1}')\vec{s}(\vec{1},t)\vec{s}(\vec{1}',t) \quad , \tag{30}$$

where $\tilde{G}(\vec{1},\vec{1}')$ is the elastic matrix. The interaction force on vortex $\vec{1}$ is then

$$\vec{f}_{el}(\vec{1},t)=-\frac{\partial}{\partial\vec{s}(\vec{1},t)}V(\vec{s}(\vec{1},t))$$
$$=-\sum_{\vec{1}'}\tilde{G}(\vec{1},\vec{1}')\vec{s}(\vec{1}',t) \quad . \tag{31}$$

Since $\tilde{G}(\vec{1},\vec{1}')$ depends only on $\vec{1}-\vec{1}'$ and is real and symmetric,

$$\tilde{G}(\vec{1}-\vec{1}')=\tilde{G}(\vec{1}'-\vec{1})=\tilde{G}(\vec{h}) \quad , \tag{32}$$

we can define a dynamical matrix, as in a crystal lattice,

$$\tilde{D}(\vec{q})=\sum_{\vec{h}}\tilde{G}(\vec{h})e^{i\vec{q}\cdot\vec{h}} \quad . \tag{33}$$

$\tilde{D}$ is also real and symmetric. The basis vectors $\hat{\epsilon}(\vec{q}\lambda)$ that diagonalize $\tilde{D}$ are called polarization vectors

$$\tilde{D}(\vec{q})\hat{\epsilon}(\vec{q}\lambda)=D_{q\lambda}\hat{\epsilon}(\vec{q}\lambda) \quad . \tag{34}$$

There are two polarizations, which at long wavelengths can be identified as transverse $(\lambda=t)$ or longitudinal $(\lambda=l)$. As usual, the polarization vectors are orthogonal:

$$\hat{\epsilon}(\vec{q}\lambda)\cdot\hat{\epsilon}(\vec{q}\lambda')=\delta_{\lambda\lambda'} \tag{35}$$

$$\sum_{\lambda}\epsilon_i(\vec{q}\lambda)\epsilon_j(\vec{q}\lambda)=\delta_{ij} \tag{36}$$

From

$$\sum_{\vec{q}}e^{i\vec{q}\cdot\vec{h}}=N\delta_{\vec{h},0} \quad ,$$

where N is the number of vortices and the sum is over the first Brillouin zone (BZ), we obtain

$$\tilde{G}(\vec{h})=\frac{1}{N}\sum_{\vec{q},\,\text{1st BZ}}\tilde{D}(\vec{q})e^{-i\vec{q}\cdot\vec{h}} \quad . \tag{37}$$

The orthogonality of the polarization vectors yields further

$$G_{ij}(\vec{h})=\frac{1}{N}\sum_{\substack{\vec{q},\lambda\\ \text{1st BZ}}}D_{q\lambda}\epsilon_i(\vec{q}\lambda)\epsilon_j(\vec{q}\lambda)e^{\pm i\vec{q}\cdot\vec{h}} \quad . \tag{38}$$

In the following, all summations on $\vec{q}$ are over the first BZ unless otherwise stated.

Now we expand $\vec{s}(\vec{1},t)$ in the basis $\hat{\epsilon}(\vec{q}\lambda)$:

$$\vec{s}(\vec{1},t)=\sum_{\vec{q},\lambda}e^{i\vec{q}\cdot\vec{1}}\hat{\epsilon}(\vec{q}\lambda)Q_{q\lambda}(t) \quad . \tag{39}$$

The coefficients of expansion, $Q_{q\lambda}(t)$, are the normal-mode amplitudes. The inverse relation is

$$Q_{q\lambda}(t)=\frac{1}{N}\sum_{\vec{1}}e^{-i\vec{q}\cdot\vec{1}}\hat{\epsilon}(\vec{q}\lambda)\cdot\vec{s}(\vec{1},t) \quad . \tag{40}$$

All the above relations are derived for the FLL frame.

According to Eq. (22), the noise measured can be expressed in terms of the vortex-current correlation function. If the dimensions of the measuring circuit are large by comparison with the intervortex spacing, we can treat the vortex lattice as a continuum. This is equivalent to saying that we only consider modes with q much smaller than those at the zone boundary. To first order in small quantities, the change in the vortex-current density is, assuming the displacements of the vortices from their equilibrium positions to be small,

$$\delta\vec{J}(\vec{\rho},t)=n_0\delta\vec{v}(\vec{\rho}-\vec{v}_0t,t)+\vec{v}_0\delta n(\vec{\rho}-\vec{v}_0t,t) \quad , \tag{41}$$

where n_0 is the equilibrium vortex density, $\delta\vec{J}$ and $\vec{\rho}$ are vectors in the lab frame, and $\delta\vec{v}$ and δn are referred to the FLL frame, which is moving with velocity $\vec{v}_0$. We can identify $\delta\vec{v}$ as just $\partial\vec{s}$, and δn as $-n_0\vec{\nabla}\cdot\vec{s}$. Therefore, in terms of normal modes in the FLL frame,

$$\delta\vec{J}(\vec{\rho},t)=n_0\sum_{\vec{q},\lambda}[\hat{\epsilon}(\vec{q}\lambda)\partial Q_{q\lambda}(t)-\vec{v}_0 i\vec{q}\cdot\hat{\epsilon}(\vec{q}\lambda)Q_{q\lambda}(t)]\times\exp[i\vec{q}\cdot(\vec{\rho}-\vec{v}_0t)] \quad (42)$$

The measured noise voltage is, from Eq. (28),

$$\delta V(t)=\sum_{\vec{q},\lambda}\delta V_{q\lambda}(t) , \quad (43)$$

with

$$\delta V_{q\lambda}(t)=[F_{q\lambda}\partial Q_{q\lambda}(t)+G_{q\lambda}Q_{q\lambda}(t)]\times\exp(-i\vec{q}\cdot\vec{v}_0t) , \quad (44)$$

where

$$F_{q\lambda}\equiv n_0\int d^2\rho\,\vec{g}(\vec{\rho})\cdot\hat{\epsilon}(\vec{q}\lambda)e^{i\vec{q}\cdot\vec{\rho}} ,$$
$$G_{q\lambda}\equiv -n_0\int d^2\rho\,\vec{g}(\vec{\rho})\cdot\vec{v}_0e^{i\vec{q}\cdot\vec{\rho}}[i\vec{q}\cdot\vec{\epsilon}(\vec{q}\lambda)] . \quad (45)$$

The Doppler factor $\exp(-i\vec{q}\cdot\vec{v}_0t)$ in Eq. (44) follows from the fact that the normal modes are defined with respect to the moving FLL frame, whereas the measuring circuit is fixed in the lab frame.

There are two terms in the expression of the noise voltage. The first term is proportional to $\delta\vec{v}$, or $\partial Q_{q\lambda}$, while the second term is proportional to δn, or $Q_{q\lambda}$. This is another way of saying that flux-flow noise can be produced in two ways: by local velocity fluctuations or by density fluctuations being carried along with the FLL.[5,10,11]

We now specify the measuring circuit geometry and calculate the resolution function. We consider an infinite film of type-II superconducting material with thickness d_f. A perpendicular magnetic field generates a flux density B in the film. The measuring circuit consists of two identical low resistance leads of radius R attached to the film. The leads rise perpendicularly to the film and connect to a voltmeter far away from the film. The distance between the two contacts, $\rho_{ab}\equiv|\vec{\rho}_a-\vec{\rho}_b|$ is much larger than R (Fig. 3). From Eq. (16) we obtain the resolution function,

$$\vec{g}(\vec{\rho})=-\frac{\phi_0}{4\pi}[\vec{b}'_{Ma}(\vec{\rho})+\vec{b}'_{Mb}(\vec{\rho})] , \quad (46)$$

where $\vec{\rho}$ is the position of the top of the vortex and

$$\vec{b}'_{Ma}(\vec{\rho})=\frac{2}{c|\vec{\rho}-\vec{\rho}_a|^2}\hat{z}\times(\vec{\rho}-\vec{\rho}_a) \quad (47)$$

for $|\vec{\rho}-\vec{\rho}_a|>R$ and $\hat{z}\equiv\vec{B}/B$. $\vec{b}'_{Mb}$ is also given by Eq. (47) with subscript a replaced by b and the addition of a minus sign on the RHS. We assume that the resistivity of the normal leads is much smaller than the flux-flow resistivity of the superconductor,

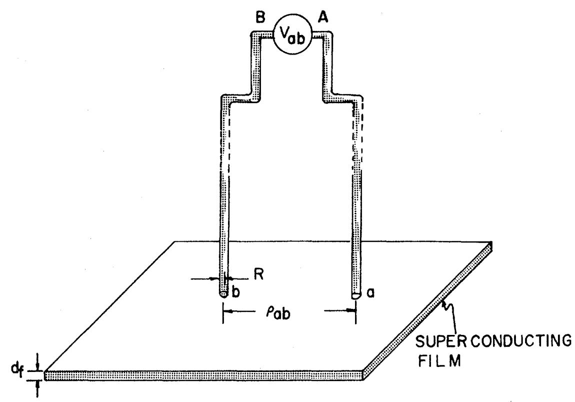

FIG. 3. Measuring circuit assumed for calculation of Johnson noise.

so that

$$\vec{b}'_{Ma}(\vec{\rho})\approx 0 \quad (48)$$

for $|\vec{\rho}-\vec{\rho}_a|<R$. The same applies to $\vec{b}'_{Mb}$. Equation (45) then yields

$$F_{q\lambda}=\frac{B}{c}[\hat{z}\cdot\hat{\epsilon}(\vec{q}\lambda)\times\hat{q}]\frac{\exp(i\vec{q}\cdot\vec{\rho}_a)-\exp(i\vec{q}\cdot\vec{\rho}_b)}{iq}\times J_0(qR)\delta_{\lambda,t} , \quad (49)$$

$$G_{q\lambda}=\frac{B}{c}(\hat{z}\cdot\vec{q}\times\vec{v}_0)[\exp(i\vec{q}\cdot\vec{\rho}_a)-\exp(i\vec{q}\cdot\vec{\rho}_b)]\times J_0(qR)\delta_{\lambda,l} , \quad (50)$$

where J_0 is the zeroth-order Bessel function. For this measuring circuit, where the two leads run perpendicularly from the film up to a large distance away, the only contribution to the noise voltage is from the transverse modes if $\vec{v}_0$ is zero. For other measuring circuit configurations, however, this will not in general be the case.

B. Vortex dynamics

The phenomenological force-balance equation for a vortex is

$$M\partial^2\vec{\rho}(\vec{l},t)=-\eta\partial\vec{\rho}(\vec{l},t)-\sum_{\vec{l}'}\tilde{G}(\vec{l},\vec{l}')\vec{s}(\vec{l}',t)+\vec{f}_{\text{ext}}(\vec{l},t) . \quad (51)$$

Here M is the mass per unit length of a vortex, as defined by Bardeen and Stephen,[12] η is the viscous drag coefficient per unit length, and $\vec{\rho}(\vec{l},t)$ is the position in the lab frame of the vortex whose equilibrium position in the FLL frame is $\vec{l}$. $\vec{f}_{\text{ext}}(\vec{l},t)$ is the external force per unit length on the vortex, excluding the viscous drag forces and the interaction force. It includes, for instance, the elementary pinning forces and the Lorentz force from the transport current. Assuming that the whole FLL is moving with an average velocity $\vec{v}_0$, which is time independent, we obtain with the help of Eq. (51),

$$M\partial^2\vec{s}(\vec{l},t)+\eta\vec{v}_0+\eta\partial\vec{s}(\vec{l},t)+\sum_{\vec{l}'}\tilde{G}(\vec{l},\vec{l}')\vec{s}(\vec{l}',t)=\vec{f}_{\rm ext}(\vec{l},t) \quad . \quad (52)$$

Taking the time average of Eq. (52) yields

$$\eta\vec{v}_0=\vec{f}_d+\langle\vec{f}(\vec{l},t)\rangle \quad , \quad (53)$$

and

$$M\partial^2\vec{s}(\vec{l},t)+\eta\partial\vec{s}(\vec{l},t)+\sum_{\vec{l}'}\tilde{G}(\vec{l},\vec{l}')\vec{s}(\vec{l}',t)=\delta\vec{f}(\vec{l},t) \quad . \quad (54)$$

Here, $\langle \cdots \rangle$ denotes an average over time, $\vec{f}=\vec{f}_{\rm ext}-\vec{f}_d$, where $\vec{f}_d$ is the constant Lorentz force due to the constant transport current, and $\delta\vec{f}=\vec{f}-\langle\vec{f}\rangle$. We have assumed that $\langle\partial\vec{s}(\vec{l},t)\rangle$ and $\langle\partial^2\vec{s}(\vec{l},t)\rangle=0$ for all $\vec{l}$, and that $\vec{f}(\vec{l},t)$ is distributed such that $\langle\vec{s}(\vec{l},t)\rangle$ is independent of $\vec{l}$. $\sum_{\vec{l}'}\tilde{G}(\vec{l},\vec{l}')=0$ has also been used.

Equation (53) is just the familiar dynamic force balance equation. Applied to flux pinning where $\vec{f}$ is the elementary pinning force, this equation states that the dynamic pinning force is just the time average of the elementary pinning force on a vortex. Equation (53) also shows that the deviation of the positions of the vortices from equilibrium is determined by the deviation of the applied force from its time-averaged value. Therefore, for flux-flow noise induced by pinning, the fluctuating part of the pinning force is the governing factor. There are also other sources of the applied force. A thermal gradient will generate a force on the vortices. If the thermal gradient fluctuates with time, voltage noise can be produced. Even with no thermal gradient, at nonzero temperature each vortex is in thermal equilibrium with its surroundings and undergoes thermally induced random motion about its equilibrium position. If the time-averaged velocity of the vortex, $\vec{v}_0$, is not zero, then the dissipation can be separated into two parts. The part that is proportional to v_0^2 is associated with the viscous drag coefficient η, with a corresponding viscous drag force, $-\eta\vec{v}_0$. The other part of the dissipation is associated with a random force, which is called the Langevin force, as in the case of the Brownian movements of colloidal particles. The associated noise is called Johnson noise. It is present at nonzero temperatures, whether $\vec{v}_0$ is zero or not.

Expanding Eq. (54) in normal modes, we obtain

$$M\partial^2Q_{q\lambda}(t)+\eta\partial Q_{q\lambda}(t)+D_{q\lambda}Q_{q\lambda}(t)=\delta f_{q\lambda}(t) \quad , \quad (55)$$

where

$$\delta f_{q\lambda}(t)=\frac{1}{N}\sum_{\vec{l}}e^{-i\vec{q}\cdot\vec{l}}\hat{\epsilon}(\vec{q}\lambda)\cdot\delta\vec{f}(\vec{l},t) \quad . \quad (56)$$

The solution of Eq. (55) is

$$Q_{q\lambda}(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}g_\lambda(\vec{q},\omega)A_\lambda(\vec{q},\omega)e^{-i\omega t} \quad , \quad (57)$$

where

$$A_\lambda(\vec{q},\omega)=\int_{-\infty}^{\infty}dt\,A_{q\lambda}(t)e^{i\omega t} \quad , \quad (58)$$

$$A_{q\lambda}(t)\equiv\frac{1}{M}\delta f_{q\lambda}(t) \quad , \quad (59)$$

$$g_\lambda(\vec{q},\omega)=-(\omega^2+i\omega/\tau-\omega_{q\lambda}^2)^{-1} \quad , \quad (60)$$

$$\omega_{q\lambda}^2\equiv D_{q\lambda}/M \quad , \quad (61)$$

$$\tau\equiv M/\eta \quad . \quad (62)$$

Hence, if $\delta\vec{f}$ is known, we can calculate $Q_{q\lambda}$ and $\partial Q_{q\lambda}$, and the associated noise voltage.

C. Johnson noise

As an example, we shall apply the above procedure to Johnson noise in an ideal film with no transport current at constant temperature. In this case, the FLL is stationary and $\vec{v}_0=0$. The force we are concerned with is the Langevin force and has the property that it is uncorrelated in direction, space, and time, and has zero time average:

$$\langle f_i(\vec{l},t)f_{i'}(\vec{l}',t')\rangle=\beta\delta_{ii'}\delta_{l,l'}\delta(t-t') \quad , \quad (63)$$

$$\langle\vec{f}(t)\rangle=0 \quad . \quad (64)$$

Since $\vec{v}_0=0$, if we consider the measuring circuit shown in Fig. 3, the only contribution to the noise voltage is from the transverse modes. The autocorrelation function is then

$$\Psi_V(s)=\sum_{\vec{q},\vec{q}'}F_{qt}F_{q't}^{*}\langle\partial Q_{qt}(t)\partial Q_{q't}^{*}(t+s)\rangle \quad . \quad (65)$$

By using Eq. (57), this can be shown to be

$$\Psi_V(s)=\frac{\beta}{M^2N}\sum_{\vec{q}}|F_{qt}|^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2|g_t(\vec{q},\omega)|^2e^{-i\omega s} \quad . \quad (66)$$

Equation (63) has been used to eliminate the cross terms with $\vec{q}\neq\vec{q}'$. The power spectrum then follows directly

$$W_V(\omega)=\frac{2\beta}{M^2N}\sum_{\vec{q}}|F_{qt}|^2\omega^2|g_t(\vec{q},\omega)|^2e^{-i\omega s} \quad . \quad (67)$$

The constant β can be shown from equipartition to be given by

$$\beta = \frac{2k_B T \eta}{d_f} , \tag{68}$$

where k_B is the Boltzmann constant. Using Eq. (49) to calculate $|F_{qt}|^2$, the complete expression for the power spectrum is found to be

$$W_V(\omega) = \frac{4k_B T B^2}{N d_f \eta \tau^2 c^2} \sum_{\vec{q}} \frac{\sin \frac{1}{2}\vec{q}\cdot\vec{\rho}_{ab}}{(\frac{1}{2}q)^2} J_0^2(qR) \times \frac{\omega^2}{(\omega^2-\omega_{qt}^2)^2+(\omega/\tau)^2} . \tag{69}$$

Since the power spectrum depends on D_{qt} only through ω_{qt}, and D_{qt} is proportional to q^2 for both a bulk and a thin film superconductor,[13,14] we can write

$$\omega_{qt} = s_t q , \tag{70}$$

where

$$s_t = (K_t/M)^{1/2} , \tag{71}$$

and

$$K_t = \phi_0 c_{66}/B = D_{qt}/q^2 . \tag{72}$$

The summation in Eq. (69) is over the first BZ. We can approximate the summation by an integral over a circle of equal area as the first BZ in reciprocal space. The radius of the circle, and thus the upper limit of integration, is of the order of the inverse of the intervortex spacing d. The factor $J_0^2(qR)$, however, provides a much lower cutoff in q, of the order of R^{-1}, if $R >> d$. The upper limit of integration can therefore be extended to infinity.

With the above substitutions, and the angular integrations done, we obtain

$$W_V(\omega) = \frac{4k_B T B \phi_0}{\pi d_f \eta \tau^2 c^2} \int_0^\infty dq \frac{1-J_0(q\rho_{ab})}{q} J_0^2(qR) \times \frac{\omega^2}{(\omega^2-s_t^2 q^2)^2+(\omega/\tau)^2} . \tag{73}$$

The power spectrum has the following limiting expressions:

$$W_V(\omega) = \begin{cases} C[\ker(\sqrt{\omega\tau_{ab}}) + \ln(\frac{1}{2}\gamma\sqrt{\omega\tau_{ab}})] , & \omega << \tau_R^{-1} , \quad (74a) \\ C \ln(\rho_{ab}/R)[1+(\omega\tau)^2]^{-1} , & \omega >> \tau_R^{-1} , \quad (74b) \end{cases}$$

where

$$C = 4k_B T B \phi_0/\pi d_f \eta c^2 , \tag{75}$$

$$\tau_{ab}^{-1} = \tau(s_t/\rho_{ab})^2 = K_t/\eta\rho_{ab}^2 , \tag{76}$$

$$\tau_R^{-1} = \tau(s_t/R)^2 = K_t/\eta R^2 . \tag{77}$$

ker is a Kelvin function and $\gamma = 1.78107 \cdots$. τ_{ab} and τ_R are the decay times for transverse modes of wave vector ρ_{ab}^{-1} and R^{-1}, respectively. The expressions for the power spectrum simplify further in the following limits:

$$W_V(\omega) = \begin{cases} C\frac{\pi}{16}\omega\tau_{ab}, & \omega << \tau_{ab}^{-1} , \quad (78a) \\ C \ln(\frac{1}{2}\gamma\sqrt{\omega\tau_{ab}}), & \tau_{ab}^{-1} << \tau_R^{-1} , \quad (78b) \\ C \ln(\rho_{ab}/R), & \tau_R^{-1} << \omega << \tau^{-1} , \quad (78c) \\ C \ln(\rho_{ab}/R)(\omega\tau)^{-2}, & \tau^{-1} << \omega . \quad (78d) \end{cases}$$

For $\omega >> \tau_R^{-1}$, $W_V(\omega)$ has the frequency dependence of the normal-state Johnson-noise power spectrum, except that τ is not in general equal to the normal collision time τ_n. We can write, in this frequency regime,

$$W_V(\omega) = 4k_B T R_f [1+(\omega\tau)^2]^{-1} , \tag{79}$$

where

$$R_f = \frac{\rho_f}{\pi d_f} \ln\left(\frac{\rho_{ab}}{R}\right) , \tag{80}$$

and $\rho_f = B\phi_0/\eta c^2$ is the flux-flow resistivity. Equation (79) is analogous to the Nyquist formula for the Johnson-noise power spectrum of a passive resistor at absolute temperature T. As the flux density approaches H_{c2}, ρ_f approaches ρ_n, and R_f approaches R_n, the normal-state resistance. At H_{c2}, the power

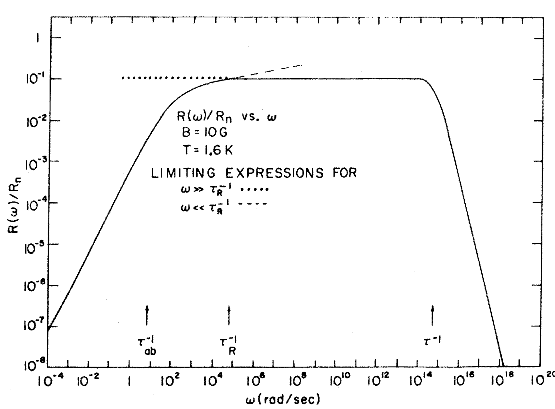

FIG. 4. Calculated Johnson-noise power spectrum, expressed in terms of the equivalent frequency-dependent normalized resistance. Parameters are chosen corresponding to the granular aluminum film of thickness 110 Å in Ref. 14.

spectrum is identical to the Nyquist formula except that according to the Bardeen-Stephen theory[12] $\tau = 2\tau_n$ at H_{c2}.

The power spectrum was calculated numerically from the 110-Å-thick oxygen-doped aluminum film used in Ref. 14. The result is shown in Fig. 4. The quantity plotted is $R(\omega)/R_n$, where $R(\omega)$ is the real part of the ac impedance, which is proportional to the Johnson-noise power spectrum. Also shown are the limiting expressions.

IV. IMPEDANCE OF THE FLUX-LINE LATTICE IN AN IDEAL TYPE-II SUPERCONDUCTING FILM IN THE MIXED STATE

We consider the same geometry as in the preceding section. An ac current $Ie^{-i\omega t}$ is fed into the film via the leads. It can be shown that the driving force on a vortex at $\vec{l}$ is given by

$$\vec{f}(\vec{l},t) = \frac{I\phi_0}{4\pi d_f}[\vec{b}'_{Ma}(\vec{l}) + \vec{b}'_{Mb}(\vec{l})]e^{-i\omega t} , \quad (81)$$

with $\vec{b}'_{Ma}$ given by Eq. (47). Substituting this into Eq. (57) to obtain $\partial Q_{q\lambda}$ and using Eq. (44) to calculate the voltage, we obtain the ac impedance as

$$Z(\omega) = \frac{1}{MNd_f}\sum_{\vec{q}}|F_{qt}|^2[-i\omega g_t(\vec{q},\omega)] . \quad (82)$$

The real and imaginary parts, $Z = R + iX$, are given explicitly by

$$R(\omega) = \frac{1}{d_f\eta\tau^2 N}\sum_{\vec{q}}|F_{qt}|^2\frac{\omega^2}{(\omega^2-\omega_{qt}^2)^2+(\omega/\tau)^2} , \quad (83)$$

$$X(\omega) = \frac{1}{d_f\eta\tau N}\sum_{\vec{q}}|F_{qt}|^2\frac{\omega(\omega^2-\omega_{qt}^2)}{(\omega^2-\omega_{qt}^2)^2+(\omega/\tau)^2} . \quad (84)$$

$R(\omega)$, which corresponds to the resistance, is equal to $W_V(\omega)/4k_BT$ for a fixed temperature. This fact can also be derived independently from the fluctuation-dissipation theorem.[15] $X(\omega)$ has the following limiting expression at low frequencies,

$$X(\omega) = -\frac{B\phi_0}{d_f\eta c^2}\left[\frac{\pi}{4} + \mathrm{kei}(\sqrt{\omega\tau_{ab}})\right] , \quad (85)$$

$$\omega << \tau_R^{-1} ,$$

where kei is a Kelvin function. A plot of $X(\omega)$ is shown in Fig. 5.

V. DISCUSSION

If we compare the frequency dependence of the Johnson-noise power spectrum with that of the normal-state expression (Nyquist formula), we see that there is a suppression of the power spectrum at low frequencies ($\omega < \tau_R^{-1}$) in the superconducting state. This is a result of the shear interaction between the vortices. As the flux density B approaches H_{c2}, the shear modulus and the parameter K_t decrease to zero, and there is less and less suppression, as shown in Fig. 6. Finally, when $B = H_{c2}$, $K_t = 0$, and there is no suppression. For $B < H_{c2}$, if ω is less than the inverse of the decay time of a normal mode, that mode will not contribute to the power spectrum or the real part of the impedance; hence, a suppression occurs at low frequencies.

It is important to note that the resolution function $\vec{g}(\vec{\rho})$ (and hence $|F_{qt}|^2$) plays a large part in determining the range of important wave vectors. For the present geometry, the important wave vectors are those which satisfy $\rho_{ab}^{-1} \lesssim q \lesssim R^{-1}$. For $\omega > \tau_R^{-1}$, the

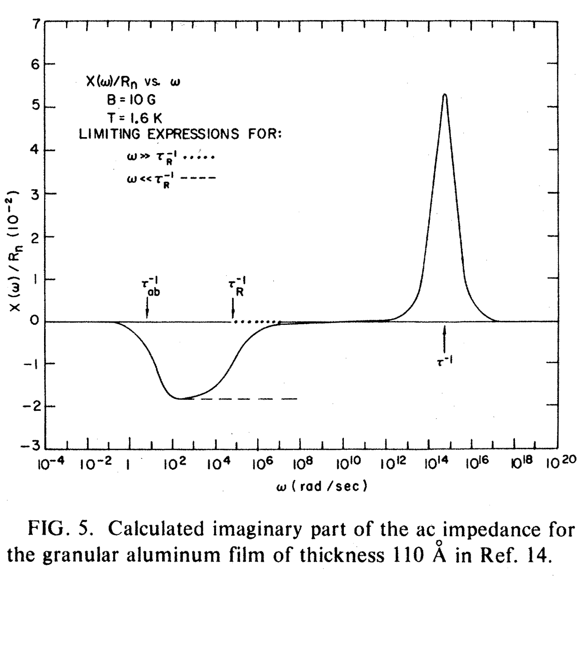

FIG. 5. Calculated imaginary part of the ac impedance for the granular aluminum film of thickness 110 Å in Ref. 14.

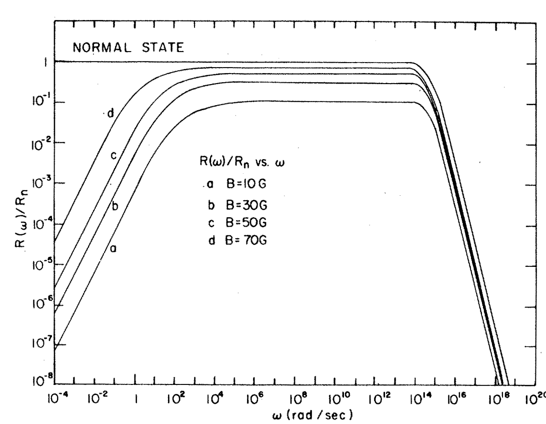

FIG. 6. Calculated Johnson-noise power spectrum, expressed in terms of the equivalent frequency-dependent normalized resistance, for various values of the magnetic induction. Parameters are chosen corresponding to the granular aluminum film of thickness 110 Å in Ref. 14.

effect of the measuring circuit dominates and the power spectrum "saturates" to a constant value. For $\omega \lesssim \tau_R^{-1}$, the effect mentioned in the previous paragraph dominates. At even higher frequencies $\omega > \tau^{-1}$, inertial effects come in, and the power spectrum starts to decrease with frequency.

Comparing our expression for the impedance with those obtained earlier,[16–19] the major difference is that we have included the effects of the interactions among vortices. Instead of assuming a restoring force of the form

$$\vec{F}(\vec{l}) = -k\vec{s}(\vec{l}) \ , \tag{86}$$

where k is a scalar force constant, as was done by previous authors, we have used the interaction tensor, Eq. (31). The force constant is now wave-vector dependent. As a consequence, the frequency dependence of $Z(\omega)$ is modified at low frequencies: instead of $R(\omega) \propto \omega^2$ we now have $R(\omega) \propto \omega$ for $\omega < \tau_{ab}^{-1}$.

ACKNOWLEDGMENTS

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[1]Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. 139, A1163 (1965).
[2]D. J. van Ooijen and G. J. van Gurp, Phys. Lett. 17, 230 (1965).
[3]J. R. Clem, Phys. Rev. B 1, 2140 (1970).
[4]P. Jarvis and J. G. Park, J. Phys. F 4, 1238 (1974).
[5]C. Heiden, D. Kohake, W. Krings, and L. Ratke, J. Low Temp. Phys. 27, 1 (1977).
[6]F. Habbal and W. C. H. Joiner, Phys. Lett. 60A, 439 (1977).
[7]R. E. Burgess, Physica (Utrecht) 55, 369 (1971).
[8]J. R. Clem, J. Phys. (Paris) 39, C6-619 (1978).
[9]J. R. Clem, in *Noise in Physical Systems,* edited by D. Wolf (Springer, Berlin, 1978), p. 214.
[10]H. Dirks, C. Heiden, and D. Kohake, in *Noise in Physical Systems,* edited by D. Wolf (Springer, Berlin, 1978), p. 229.
[11]K. Beckstette and C. Heiden, in *Noise in Physical Systems,* edited by D. Wolf (Springer, Berlin, 1978), p. 234.
[12]J. Bardeen and M. J. Stephen, Phys. Rev. 140, A1197 (1965).
[13]A. L. Fetter and P. C. Hohenberg, Phys. Rev. 159, 330 (1967).
[14]A. T. Fiory, Phys. Rev. B 8, 5039 (1973).
[15]R. Kubo, Rep. Prog. Phys. 29, 255 (1966).
[16]M. Rabinowitz, J. Appl. Phys. 42, 88 (1971).
[17]J. L. Gittleman and B. Rosenblum, Phys. Rev. Lett. 16, 734 (1966).
[18]G. E. Possin and K. W. Shepard, Phys. Rev. 171, 458 (1968).
[19]Yu. K. Krasnov, Fiz. Nizk. Temp. 3, 1022 (1977) [Sov. J. Low Temp. Phys. 3, 498 (1977)].