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Real-space dynamic renormalization group. III. Calculation of correlation functions

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The real-space dynamic renormalization group method developed in previous papers is applied to the kinetic Ising model defined on a square lattice. In particular we extend the formalism to the calculation of space- and time-dependent equilibrium averaged correlation functions. We find that conventional methods for implementing the real-space renormalization group via cumulant expansions in terms of the intercell coupling lead to correlation functions which decay algebraically in space at large distances in the disordered phase in qualitative disagreement with the known exponential decay. We indicate how one can develop new perturbation theory expansion methods which lead to the proper exponential decay at large distances and also lead to good quantitative results for other observable quantities like the magnetization, susceptibility, and single spin time autocorrelation function. As the result of a first-order calculation we obtain excellent results for the static critical exponents and a value of $z = 1.79$ for the dynamic critical exponent. The critical exponents obtained from the correlation functions calculated using this method satisfy the proper static and dynamic scaling relations.

I. INTRODUCTION

In this paper we discuss the extension of the real-space dynamic renormalization group (RSDRG) method introduced by us earlier[1-4] to the central problem of the calculation of time and space dependent correlation functions. In particular, we will study the statics and dynamics of a two-dimensional kinetic Ising (KI) model defined on a square lattice. These calculations, which were summarized elsewhere,[5] represent, to the best of our knowledge, the first results, using the real-space renormalization-group (RSRG) method, of dynamical properties other than critical indices.

It has been understood[6] from the very beginning of its development that the RSRG method could be used to calculate thermodynamic quantities over the whole thermodynamic plane—not just near the critical point. Thus there exist accurate calculations for the spontaneous magnetization[7] and the specific heat[7,8] for two-dimensional Ising models. There has, however, to our knowledge, been essentially no practical development of a way of using these methods to compute space and time dependent correlation functions over the complete range of temperature and spatial separations. In this paper we carry out a formal analysis that allows us to relate a correlation function defined for the original lattice model to a correlation function defined for a coarse grained or renormalized lattice. We then restrict ourselves to a rather straightforward approximate analysis of these formal results. One can learn a great deal in this simple analysis about the general structure of perturbation theories as carried out in the context of the RSRG. One finds, for example,[9] that a straightforward cumulant expansion of the type developed by Niemeijer and van Leeuwen[6] leads to algebraically decaying correlation functions for all temperatures in the disordered phase! The development of expansion techniques which lead to the correct exponential decay of correlation functions at large distances led us to a number or rather general insights concerning the appropriate couplings to be used in treating uncoupled cells as a zeroth-order theory, and also into the fundamental structure of the recursion relations. These insights have led to a substantial improvement in the quantitative accuracy of our theory.

The main practical results of this paper are: (1) The derivation of a simple set of recursion relations relating time and space dependent correlation functions defined on lattices with different lattice spacing; and (2) the solution of these recursion relations for various physical quantities like the magnetization, the susceptibility, the single spin time autocorrelation function, and finally time and space dependent correlation functions. We can easily extract various critical properties from these quantities. We obtain excellent results for the static critical indices $\nu=1.011(=1$ exactly), $\beta=0.122(=0.125$ exactly), and the critical coupling $\tanh K_c=0.4150(=\sqrt{2}-1$ exactly). The dynamic critical index z we obtain is 1.79 which is within the range of known possible[10] and approximate values.[11] The various critical indices are found to agree with the known scaling laws. It seems equally important to us, with an eye to eventual comparison with experiments, that we are able to calculate various physical quantities over a wide range of temperatures, times, and distances. The calculated magnetization and susceptibilities are in excellent agreement (see Sec. VI, Figs. 4 and 5) with known results over a wide range of temperatures. Most surprising is that our simple theory reproduces the maximum in the static structure factor at temperatures above the critical temperature pointed out by Fisher and Burford[12] (see Sec. VI, Fig. 6).

We do not believe that this paper is the last word on perturbation theory calculations within the RSDRG. Indeed we know of more systematic ways of proceeding. These very systemative procedures are rather technically involved and to some degree hide some rather important general features associated with a quantitative application of the RSDRG. Consequently this more sophisticated analysis will be presented elsewhere. In this paper we will follow a more pragmatic and straightforward approach.

In Secs. II and III we describe the form of the KI model that we have studied and briefly review the RSDRG formalism. In Sec. IV we introduce the formal structure necessary to relate correlation functions on the original lattice and those defined on the coarse grained lattice. We also look at these interrelationships using a very simple approximation scheme. Some of the insights gained in this section are then used in Sec. V to develop a perturbation theory expansion for explicit implementation of the RSDRG that we believe will lead to quantitative results at low order. In Sec. VI we explicitly discuss the result of a direct calculation of various physical quantities.

II. RSDRG AND THE KINETIC ISING MODEL

A. Kinetic Ising model on a square lattice

Let us consider a system of N Ising spins set on a two-dimensional square lattice. Let i label the lattice sites and let σ stand for a given spin configuration $\sigma\equiv\{\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots,\sigma_N\}$. The equilibrium probability distribution for this system is assumed to be given by

$$P[\sigma]=\frac{e^{H[\sigma]}}{Z}\ , \tag{2.1}$$

where $H[\sigma]$ is the Ising Hamiltonian (multiplied by $-\beta$) and Z is the partition function. The Hamiltonian can be written as

$$H[\sigma]=\frac{K}{2}\sum_{i=1}^{N}\sum_{a=1}^{4}\sigma_i\sigma_{i+\delta_a}\ , \tag{2.2}$$

where K is the nearest-neighbor coupling constant and the δ's are basis vectors connecting a spin at site i with its four nearest neighbors (see Fig. 1).

We assume that the dynamics of our system is driven by a spin-flip operator (SFO) $\tilde{D}_\sigma$ such that the time dependent spin-spin correlation function is given by

$$C_{ij}(t)=\sum_{\sigma}P[\sigma]\sigma_j e^{\tilde{D}_\sigma t}\sigma_i\ . \tag{2.3}$$

The various properties of the operator $\tilde{D}_\sigma$ are discussed in Sec. II of Ref. 3. In this paper we consider a one-spin flip operator of the form given by

$$D[\sigma|\sigma']=-\frac{\alpha}{2}\sum_i\Lambda^{[i]}_{\sigma,\sigma'}W_i[\sigma]\sigma_i\sigma_i'\ , \tag{2.4}$$

where α is an inverse spin flip time; $\Lambda^{[i]}_{\sigma,\sigma'}$ sets $\sigma_j=\sigma_j'$ except at lattice site i, and $W_i[\sigma]$ is essentially the spin-flip probability. The form of $W_i[\sigma]$ is restricted by stationarity and detailed balance conditions [see Eqs. (2.16) and (2.19) of Ref. 3] and demands of locality. In this paper we consider a $W_i[\sigma]$ of the form

$$W_i[\sigma]=e^{-\sigma_i E_i[\sigma]}V_i[\sigma]\ , \tag{2.5}$$

where

$$E_i[\sigma]=K\sum_a\sigma_{i+\delta_a} \tag{2.6}$$

is the coupling of the spin at site i to its environment. The factor $V_i[\sigma]$ in Eq. (2.5) is required to be independent of the spin at site i. We will also assume that $V_i[\sigma]$ is a symmetric function under interchange

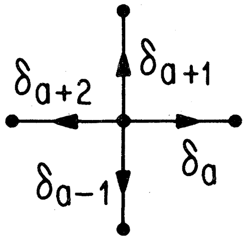

FIG. 1. Basis vectors on the original σ lattice.

of the four nearest neighbors about the spin at site i. The spin configurations formed from the four nearest neighbors of the spin at site i that are symmetric under interchange are

$$\sigma_i^s = \sum_a \sigma_{i+\delta_a} , \tag{2.7a}$$

$$\sigma_i^n = \sum_a \sigma_{i+\delta_a} \sigma_{i+\delta_{a+1}} , \tag{2.7b}$$

$$\sigma_i^{nn} = \frac{1}{2} \sum_a \sigma_{i+\delta_a} \sigma_{i+\delta_{a+2}} , \tag{2.7c}$$

$$\sigma_i^T = \sum_a \sigma_{i+\delta_{a-1}} \sigma_{i+\delta_a} \sigma_{i+\delta_{a+1}} , \tag{2.7d}$$

$$\sigma_i^c = \sigma_{i+\delta_{a-1}} \sigma_{i+\delta_a} \sigma_{i+\delta_{a+1}} \sigma_{i+\delta_{a+2}} . \tag{2.7e}$$

1, σ^s, σ^n, σ^{nn}, σ^T, and σ_i^c form a closed set under multiplication. This means that any function of these symmetric functions can be expressed as a linear combination of this complete set. Then since $V_i[\sigma]$ must be an even function under flipping of all the spins we have

$$V_i[\sigma] = V_0(1 + b_1\sigma_i^n + b_2\sigma_i^{nn} + b_3\sigma_i^c) . \tag{2.8}$$

The "dynamic" parameters V_0, b_1, b_2, and b_3 are free up to the constraint that system is locally stable. It is straightforward to expand the exponential factor in Eq. (2.5) in terms of the set of functions defined by Eq. (2.7). It is then a simple matter to multiply this result times $V_i[\sigma]$ to show that $W_i[\sigma]$ is generally of the form

$$W_i[\sigma] = 1 + A_1\sigma_i\sigma_i^s + A_2\sigma_i^n + A_2'\sigma_i^{nn} + A_3\sigma_i\sigma_i^T + A_4\sigma_i^c , \tag{2.9}$$

where the constants A_i can be expressed in terms of the b's and K. Note that any overall constant can be included in α.

Under our first-order RSDRG analysis we will find that we generate a coarse grained W_i of the same form as Eq. (2.9) above, but with

$$A_2' = A_3 = A_4 = 0 .$$

An easy calculation reveals that the spin-flip probability

$$W_i[\sigma] = 1 + A_2\sigma_i\sigma_i^s + A_2\sigma_i^n \tag{2.10}$$

satisfies the stationarity condition if we choose

$$A_1 = -\frac{1}{2}\tanh 2K \tag{2.11}$$

and

$$A_2 = A_1^2 .$$

This particular form for $W_i[\sigma]$, which we call the "minimal coupling" form, is very convenient since it involves only next-nearest-neighbor couplings (note that the general $W_i[\sigma]$ involves third and next-nearest-neighbor couplings). In our detailed calculations later in the paper we assume that we start with the minimal coupling operator.

III. REAL-SPACE DYNAMIC RENORMALIZATION GROUP

A. Basic formalism

Let us briefly recall some of the fundamental results for the RSDRG. The basic idea is to introduce a coarse graining transformation $T[\mu|\sigma]$ which projects a set of spins σ onto a new set of Ising spins μ defined on a lattice with a lattice constant s times larger than that for the original lattice. The equilibrium probability distribution describing the new set of spins is related to the old distribution by

$$P[\mu] \equiv \frac{e^{H[\mu]}}{Z} = \sum_\sigma P[\sigma]T[\mu|\sigma] , \tag{3.1}$$

where $H[\mu]$ is the new Hamiltonian governing the coarse grained spins. In order that the partition function be invariant under this transformation we require that the mapping function satisfy

$$\sum_\mu T[\mu|\sigma] = 1 . \tag{3.2}$$

In treating dynamic problems we want to have a procedure for determining the new spin-flip operator governing the time evolution of the coarse grained spins. In order to avoid very unpleasant non-Markovian effects and in order to obtain a new spin-flip operator of qualitatively the same type as we had before renormalization we introduced the eigenvalue condition[3]

$$\tilde{D}_\sigma T[\mu|\sigma] = D_\mu T[\mu|\sigma] , \tag{3.3}$$

which leads to a self-consistent determination of the new spin-flip operator $\tilde{D}_\mu$ and the mapping function $T[\mu|\sigma]$. Note that D_μ is the adjoint of $\tilde{D}_\mu$. This equation must be supplemented by the normalization condition

$$\sum_\sigma P[\sigma]T[\mu|\sigma]T[\mu'|\sigma] = \delta_{\mu,\mu'}P[\mu] . \tag{3.4}$$

The implications and benefits of the eigenvalue equation formulation of this problem are discussed in detail in Ref. 3.

In the problem we investigate here we want to introduce block spin variables associated with cells partitioning the original system. We will use the simple division shown in Fig. 2 where we put four spins in each cell and locate a block spin (specified by an × in Fig. 2) at the center of a cell. Let us assume that the center of the ith cell is labeled by a vector $\vec{r}_i$ and each spin in that cell by basis vectors $\vec{b}_a$ ($a = 1, 2, 3, 4$) as shown in Fig. 2. Then each spin is

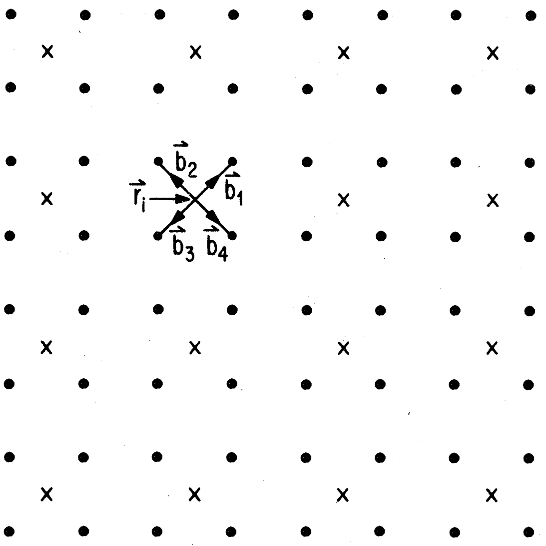

FIG. 2. Basis vectors connecting block spins (denoted by ×) to σ spins (denoted by dots).

labeled by the set (i,a). The nearest neighbors of the spin (i,a) in the same cell are given by $(i,a+1)$ and $(i,a-1)$. The nearest-neighbor cells are labeled by the set of basis vectors δ'_a shown in Fig. 3. The nearest neighbors of (i,a) outside of cell i are given by $(i+\delta'_a,a-1)$ and $(i+\delta'_{a-1},a+1)$ (see Fig. 3). We will use the abbreviated notation $\sigma_{i,a}$ for a spin in cell i at site $\vec{b}_a$.

B. Zeroth-order theory

We want to solve Eq. (3.3) approximately in a perturbation theory expansion where we treat the case of uncoupled cells as our zeroth-order approximation. Ignoring any coupling between cells we have a zeroth-order probability distribution

$$P_0[\sigma]=e^{H_0[\sigma]}/Z_0\ , \tag{3.5}$$

$$Z_0=\sum_{\sigma} e^{H_0[\sigma]}\ , \tag{3.6}$$

where

$$H_0\sum_i H_0^i\ , \tag{3.7}$$

$$H_0^i=\sum_a K_0\sigma_{i,a}\sigma_{i,a+1}\ . \tag{3.8}$$

We allow for the possibility that the intercell coupling for a system of uncoupled cells may be different than that for the coupled system $(K_0\neq K)$. The cell averages that will enter into our analysis are

$$r=\sum_{\sigma} P_0[\sigma]\sigma_{i,a}\sigma_{i,a+1}\equiv\langle\sigma_{i,a}\sigma_{i,a+1}\rangle_0=\frac{1-Y_0^2}{1+6Y_0+Y_0^2}\ , \tag{3.9}$$

$$s=\langle\sigma_{i,a}\sigma_{i,a+2}\rangle_0=\frac{(1-Y_0^2)}{1+6Y_0+Y_0^2}\ , \tag{3.10}$$

where $Y_0=e^{-4K_0}$.

We assume that the appropriate "cell" SFO is of the form

$$\tilde{D}_0[\sigma|\sigma']=\sum_i(\Lambda^{[i]}_{\sigma,\sigma'})\tilde{D}_0^i[\sigma|\sigma']\ , \tag{3.11}$$

where $\Lambda^{[i]}_{\sigma,\sigma'}$ sets $\sigma=\sigma'$ at all sites except those in the ith cell, and

$$\tilde{D}_0^i[\sigma|\sigma']=-\frac{\alpha_0}{2}\sum_a\Lambda^{[i,a]}_{\sigma,\sigma'}\sigma_{i,a}\sigma'_{i,a}W_0^{[i,a]}[\sigma]\ , \tag{3.12}$$

where

$$W_0^{[i,a]}[\sigma]=1+J_1\sigma_{i,a}(\sigma_{i,a+1}+\sigma_{i,a-1})+J_2\sigma_{i,a+1}\sigma_{i,a-1}\ . \tag{3.13}$$

The condition that P_0 be invariant under time translations, $D_\sigma^0 P_0[\sigma]=0$, is satisfied if

$$J_1=-\frac{u_0}{1+u_0^2}(1+J_2)\ , \tag{3.14}$$

where

$$u_0\equiv\tanh K_0\ . \tag{3.15}$$

In this most naive decomposition of $\tilde{D}_\sigma$ into intercell and intracell contributions one would directly identify

$$\alpha=\alpha_0\ , \tag{3.16a}$$

$$\alpha_0 J_1=\alpha A_1\ , \tag{3.16b}$$

$$\alpha_0 J_2=\alpha A_2\ , \tag{3.16c}$$

$$K_0=K\ . \tag{3.16d}$$

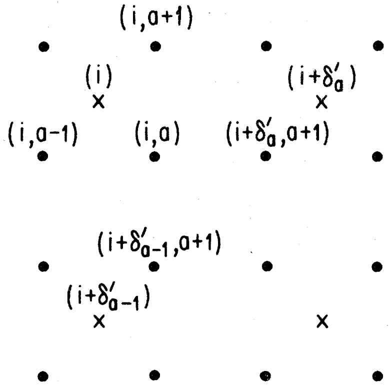

FIG. 3. Specification of nearest-neighbor block and σ spins with respect to a σ spin at site (i,a).

This identification implies simply cutting any bonds in Eq. (2.9) connecting different cells. A main point we intend to make in this paper is that this naive identification of $\tilde{D}_\sigma^0$ is not sufficient for developing a quantitative theory. It will be necessary to develop more sophisticated methods for determining the parameters α_0, J_1, and J_2.

Note that in general $W_0^{[i,a]}[\sigma]$ could contain the additional couplings $J_3\sigma_a\sigma_{a+2}$, $J_4\sigma_{a+2}(\sigma_{a+1}+\sigma_{a-1})$, and $J_5\sigma_a\sigma_{a+1}\sigma_{a+2}\sigma_{a-1}$. We will not consider this more general form for $W_0^{[i,a]}[\sigma]$ in this paper in view of the form of Eq. (2.10).

The solution of the zeroth-order eigenvalue problem

$$\tilde{D}_\sigma^0 T_0[\mu|\sigma] = D_\mu^0 T_0[\mu|\sigma] \tag{3.17}$$

has been dicussed in Ref. 3. We easily find that T_0 is of the form

$$T_0[\mu|\sigma] = \prod_i T_0^{[i]}[\mu|\sigma] \ . \tag{3.18}$$

To find $T_0^{[i]}$ it suffices to solve the eigenvalue equation

$$\tilde{D}^0 \psi_i^{(n)}(\sigma) = -\lambda_n \psi_i^{(n)}(\sigma) \tag{3.19}$$

and we then can choose

$$T_0^{[i]}[\mu|\sigma] = \tfrac{1}{2}[1+\mu_i\psi_i^{(1)}(\sigma)] \ , \tag{3.20}$$

where $\psi_i^{(1)}$ is the odd eigenfunction $\psi_i^{(n)}$ corresponding to the smallest nonzero eigenvalue $\lambda_1=\lambda$. The associated coarse grained SFO is given by

$$\tilde{D}_0[\mu|\mu'] = -\frac{\lambda}{2}\sum_i \Lambda_{\mu,\mu'}^{[i]}\mu_i\mu_i' \ . \tag{3.21}$$

We solve the eigenvalue problem Eq. (3.19) in the Appendix and show that

$$\psi_i^{(1)}(\sigma) = N(\sigma_i^s + f\,\sigma_i^T) \ , \tag{3.22}$$

where N and f are functions of J_1 and J_2 and given by Eqs. (A15) and (A16). It is easy to show that in this approximation we can evaluate the sum in Eq. (3.4) to obtain

$$P_0[\mu] = \frac{1}{2^{N'}} \ , \tag{3.23}$$

where $N'=N/4$ gives the number of cells and

$$\sum_\sigma P_0[\sigma]T_0[\mu|\sigma]T_0[\mu'|\sigma] = \delta_{\mu,\mu'}P_0[\mu] \tag{3.24}$$

so that the normalization condition given by Eq. (3.4) is satisfied.

The form for $T_0[\mu|\sigma]$ given by Eqs. (3.18), (3.19), and (3.22) is identical to that used in a number of RSRG calculations. We know from the work of Kadanoff and Houghton,[13] and Barber[14] that a good "choice" for the parameter f (near the critical temperature) is $f \sim -\frac{1}{3}$ for the square lattice. In particular, this choice for f leads to good first-order results for the thermal eigenvalues.

The choice $f=-\frac{1}{3}$ for the square lattice is analogous to the majority rule choice for triangular lattice. Thus with $f=-\frac{1}{3}$ and $N=\frac{3}{8}$ one has

$$\begin{aligned} T^0[\mu|\sigma] &= \delta_{\mu,\,\mathrm{sgn}(\sigma^s)}, \quad \sigma^s \neq 0 \ , \\ T^0[\mu|\sigma] &= \tfrac{1}{2}, \quad \sigma^s = 0 \ , \end{aligned} \tag{3.25}$$

where σ^s is the sum of the four spins in a cell. Unlike the majority rule for the triangular lattice this is not a possible T^0 within the constraints of the RSDRG because it does not satisfy the normalization condition given by Eq. (3.24). If one fixed $f=-\frac{1}{3}$ then the normalization condition (3.24) requires that

$$N = \frac{3}{2[2(5+4r-s)]^{1/2}} \tag{3.26}$$

which ranges between $3/2\sqrt{10}$ at high temperatures and $\frac{3}{8}$ at low temperatures.

We see in the case of the RSDRG that N and f are fixed once we specify the parameters J_1 and J_2 characterizing the cell SFO.

C. First-order RSDRG calculation

As explained in Ref. 3 in carrying out higher-order calculations, rather than solving Eqs. (3.3), (3.2), and (3.4) directly, in practice it is easier to first solve an auxiliary eigenvalue equation

$$\tilde{D}_\sigma \bar{T}[\mu|\sigma] = \bar{D}_\mu \bar{T}[\mu|\sigma] \ , \tag{3.27a}$$

where $\bar{T}[\mu|\sigma]$ satisfied the normalization conditions

$$\sum_\sigma P_0[\sigma]T_0[\mu|\sigma]\bar{T}[\mu'|\sigma] = \delta_{\mu,\mu'}P_0[\mu'] \ . \tag{3.27b}$$

We then perform a rotation in μ space

$$T[\mu|\sigma] = \sum_{\mu'} S[\mu|\mu']\bar{T}[\mu'|\sigma] \tag{3.28}$$

such that $T[\mu|\sigma]$ does satisfy Eq. (3.4). We must simultaneously rotate the eigenvalue $\bar{D}_\mu$:

$$D[\mu|\mu'] = \sum_{\bar{\mu},\bar{\mu}'} S[\mu|\bar{\mu}]\bar{D}[\bar{\mu}|\bar{\mu}']S^{-1}[\bar{\mu}'|\mu'] \ . \quad (3.29)$$

After performing the rotation, we have then constructed the full solution to Eqs. (3.3), (3.2), and (3.4). Quite generally, we can assume that $S[\mu|\mu']$ is symmetric. If we write

$$\tilde{D}_\sigma = \tilde{D}_\sigma^0 = \epsilon\tilde{D}_\sigma^I \ , \quad (3.30)$$

where ϵ is an ordering parameter we set equal to one at the end of the calculation, then the first order in ϵ contribution to $\bar{D}[\mu|\mu']$ is given by

$$\bar{D}^{(1)}[\mu|\mu']P_0[\mu'] = \sum_\sigma P_0[\sigma]T_0[\mu'|\sigma]\tilde{D}_\sigma^I T_0[\mu|\sigma] \quad (3.31)$$

$$= \frac{\lambda^{(1)}}{2}\sum_i \Lambda_{\mu,\mu'}^{[i]}\mu_i\mu_i' P_0[\mu'] - \frac{\alpha}{2}\sum_i\sum_a \Lambda_{\mu,\mu'}^{[i]}(R_1\mu_i\mu'_{i+\delta_a'} + R_2\mu_i\mu_i'\mu'_{i+\delta_a'}\mu'_{i+\delta_{a-1}'})P_0[\mu'] \ , \quad (3.32)$$

where

$$\lambda^{(1)} = 2\alpha(W + 2A_1W_1 + A_2W_2) - \lambda \ , \quad (3.33)$$

$$R_1 = \nu(qA_1 + \tilde{q}A_2) \ , \quad (3.34)$$

$$R_2 = \frac{\nu^2}{2}WA_2 \ , \quad (3.35)$$

and

$$\nu = \langle\sigma_a\psi^{(1)}\rangle_0 = N[1 + 2r + s + 2f(r+s)] \ , \quad (3.36)$$

$$q = \langle\sigma_a\Delta\psi_a^{(1)}\rangle_0 = 2N[1 + f(s+2r)] \ , \quad (3.37)$$

$$\tilde{q} = \langle\sigma_{a\pm1}\Delta\psi_a^{(1)}\rangle_0 = 2N[r + f(r+2s)] \ , \quad (3.38)$$

$$W = \langle\psi^{(1)}\Delta\psi_a^{(1)}\rangle_0 = 2N[\nu + 3f\nu_3] \ , \quad (3.39)$$

$$W_1 = \langle\psi_{(1)}\Delta\psi_a^{(1)}\sigma_a\sigma_{a\pm1}\rangle_0 = 2N[\nu(1+2f) + f\nu_3] \ , \quad (3.40)$$

$$W_2 = \langle\psi^{(1)}\Delta\psi_a^{(1)}\sigma_{a+1}\sigma_{a-1}\rangle_0 = 2N[\nu_3(1+2f) + \nu f] \ , \quad (3.41)$$

$$\begin{aligned}\nu_3 &= \langle\psi^{(1)}\sigma_a\sigma_{a+1}\sigma_{a+2}\rangle_0 \\ &= N[2(r+s) + f(2r+s+1)] \ , \end{aligned}\quad (3.42)$$

and

$$\Delta\psi_a^{(1)} = \psi^{(1)}(\sigma) - \psi^{(1)}(\sigma;-\sigma_a) \ , \quad (3.43)$$

where $\langle\ \rangle_0$ indicates an average over $P_0[\sigma]$ and r and s are defined by Eqs. (3.9) and (3.10).

A rather involved calculation leads to the result that the rotation $S[\mu|\mu']$, evaluated to $O(\epsilon)$, leads to no change in $D[\mu|\mu']$ and we obtain

$$D[\mu|\mu'] = \bar{D}[\mu|\mu'] \ . \quad (3.44)$$

Notice then that

$$D[\mu|\mu'] = D^0[\mu|\mu'] + \epsilon D^{(1)}[\mu|\mu'] \quad (3.45)$$

is of the minimal coupling form given by Eq. (2.10) with new parameters

$$A_1' = \frac{\epsilon R_1\alpha}{\lambda} + O(\epsilon^2) \ , \quad (3.46)$$

$$A_2' = \frac{\epsilon R_2\alpha}{\lambda} + O(\epsilon^2) \ , \quad (3.47)$$

$$\alpha' = 2\alpha(W + 2A_1W_1 + A_2W_2) = \tilde{\lambda} \ , \quad (3.48a)$$

$$\tilde{\lambda} = \Delta\alpha \ . \quad (3.48b)$$

We easily see however $A_2' \neq (A_1')^2$ as required if the Hamiltonian is given by Eq. (2.2).

What is happening here is that new static interactions are being generated by the renormalization group (RG). The symmetry of the problem dictates that we must generalize our initial Hamiltonian to include a four-spin coupling:

$$H = \frac{K'}{2}\sum_i \mu_i\mu^s(i) + \frac{L'}{4}\sum_i \mu_i\mu^T(i) \ . \quad (3.49)$$

The stationarity condition with this Hamiltonian gives rise to

$$A_1' = -\tfrac{1}{2}\tanh 2(K' - L') \ , \quad (3.50)$$

$$A_2' = \frac{1}{4}\left[\frac{2\tanh 2(K'-L')}{\tanh 4(K'+L')} - 1\right] \ , \quad (3.51)$$

which can be inverted to obtain

$$K' = \frac{1}{8}\left[\ln\left(\frac{1-2A_1'}{1+2A_1'}\right) + \tfrac{1}{2}\ln\left(\frac{1+4A_2'-4A_1'}{1+4A_2'+4A_1'}\right)\right] \ , \quad (3.52)$$

$$L' = \frac{1}{8}\left[-\ln\left(\frac{1-2A_1'}{1+2A_1'}\right) + \tfrac{1}{2}\ln\left(\frac{1+4A_2'-4A_1'}{1+4A_2'+4A_1'}\right)\right] \ . \quad (3.53)$$

We can then use the equation

$$D_\mu^0 P_1[\mu] = -D_\mu^{(1)}P_0[\mu] \ , \quad (3.54)$$

valid to $O(\epsilon)$, to compute the first-order contribution to $P[\mu]$ as

$$P_1[\mu]=P_0[\mu]\sum_i\sum_a \tfrac{1}{2}K'\mu_i\mu_{i+\delta_a} \ , \tag{3.55}$$

where the renormalized coupling is given by

$$K'=\frac{-\epsilon\alpha R_1}{\lambda}+O(\epsilon^2) \tag{3.56}$$

and

$$L'=O(\epsilon^2) \ . \tag{3.57}$$

If we insert Eqs. (3.46) and (3.47) into Eqs. (3.52) and (3.53) and expand to lowest order in ϵ we regain Eqs. (3.56) and (3.57).

In summary, to $O(\epsilon)$, we have the recursion relations (3.48) and (3.56) and the renormalized $\tilde{D}_\mu$ is of the minimal coupling form. It is quite important to realize that these recursion relations depend on the intracell parameters α_0, J_1, and J_2 which we must now determine.

IV. CORRELATION FUNCTIONS AND THE RSDRG

A. General development

In this section we discuss how to compute dynamic correlation functions within the context of the RSDRG. It is this aspect of our formalism which we believe is of particular interest. We are able to calculate arbitrary space and time dependent correlation functions over the entire temperature range. Close to the critical point, the correlation functions calculated show the usual critical divergences with exponents that satisfy the proper static and dynamic scaling laws and coincide with the exponents deduced directly from the recursion relations

We are interested in calculating a general time dependent function of the form

$$C_{AB}(t)=\sum_\sigma P[\sigma]B(\sigma)e^{\tilde{D}_\sigma t}A(\sigma) \ . \tag{4.1}$$

Here, $A(\sigma)$ and $B(\sigma)$ are arbitrary operators depending on the σ spins. Alternatively we can look at the Laplace transform

$$C_{AB}(z)=\sum_\sigma P[\sigma]B(\sigma)R(z)A(\sigma) \ , \tag{4.2}$$

where $R(z)$ is the resolvent operator

$$R(z)=(z-i\tilde{D}_\sigma)^{-1} \ . \tag{4.3}$$

We wish to express the correlation function (4.1) in terms of correlation functions of the new spins μ. For a given transformation $T[\mu|\sigma]$, we define the "coarse grained variables" $A(\mu)$ associated with $A(\sigma)$ by

$$A(\mu)P[\mu]=\sum_\sigma P[\sigma]T[\mu|\sigma]A(\sigma) \ . \tag{4.4}$$

Notice that the average of $A(\mu)$ in μ space equals the average of $A(\sigma)$ in σ space, i.e.,

$$\sum_\mu A(\mu)P[\mu]=\sum_\sigma A(\sigma)P[\sigma] \ . \tag{4.5}$$

Next, we introduce the projection operator $\mathcal{P}$ in σ space, defined by

$$\mathcal{P}A(\sigma)=\sum_\mu T[\mu|\sigma]P^{-1}[\mu]\sum_{\bar{\sigma}}P[\bar{\sigma}]T[\mu|\bar{\sigma}]A(\bar{\sigma})=\sum_\mu T[\mu|\sigma]A(\mu) \ . \tag{4.6}$$

It is easily seen that $\mathcal{P}$ is indeed a projection operator, since $\mathcal{P}^2=\mathcal{P}$. Notice that this requires enforcement of the normalization condition given by Eq. (3.2). The projection operator selects that part of a given variable $A(\sigma)$ which will be mapped onto the "coarse grained" variable $A(\mu)$. If the RG transformation is so defined that it maps the slower degrees of freedom in σ space onto the new variables in μ space, the projection operator will select the slow part of a given variable in σ space. As an illustrative example take for $A(\sigma)$ a single spin $\sigma_{i,a}$ and compute $\mathcal{P}A(\sigma)$ to lowest order. It is easily seen that

$$\mathcal{P}\sigma_{i,a}=\nu\psi_i^{(1)}(\sigma) \ , \tag{4.7}$$

where $\psi_i^{(1)}(\sigma)$ is the slowest eigenfunction in cell i, and $\nu=\langle\psi_i^{(1)}(\sigma)\sigma_{i,a}\rangle_0$. The associated coarse grained variable in μ space is simply

$$A(\mu)=\nu\mu_i \ . \tag{4.8}$$

We now introduce also the complement of p in the usual way

$$\mathcal{Q}=1-\mathcal{P} \ . \tag{4.9}$$

and rewrite the correlation function (3.1) as

$$\begin{aligned}C_{AB}(t)&=\sum_\sigma P[\sigma][(\mathcal{P}+\mathcal{Q})B(\sigma)]e^{\tilde{D}_\sigma t}(\mathcal{P}+\mathcal{Q})A(\sigma)\\&=\sum_\sigma P[\sigma][\mathcal{P}B(\sigma)]e^{\tilde{D}_\sigma t}\mathcal{P}A(\sigma)+\sum_\sigma P[\sigma][\mathcal{Q}B(\sigma)]e^{\tilde{D}_\sigma t}\mathcal{Q}A(\sigma)=C'_{AB}(t)+\bar{C}_{AB}(t) \ .\end{aligned} \tag{4.10}$$

The additional "cross terms" are readily seen to vanish. Now, the basic idea is to treat iteratively the long range (in space and time) part of $C_{AB}(t)$ (i.e., C'_{AB}) but to treat perturbatively the short-range part, $\bar{C}_{AB}(t)$. $C'_{AB}(t)$ can be written as

$$C'_{AB}=\sum_{\sigma}P[\sigma][\mathcal{P}B(\sigma)]e^{\tilde{D}_{\sigma}t}[\mathcal{P}A(\sigma)]=\sum_{\sigma}\sum_{\mu}\sum_{\mu'}P[\sigma]T[\mu|\sigma]B(\mu)e^{\tilde{D}_{\sigma}t}T[\mu'|\sigma]A(\mu')=\sum_{\mu}P[\mu]B(\mu)e^{\tilde{D}_{\mu}t}A(\mu)\ , \tag{4.11}$$

where we have used

$$\sum_{\sigma}P[\sigma]T[\mu|\sigma]e^{\tilde{D}_{\sigma}t}T[\mu'|\sigma]=e^{\tilde{D}_{\mu}t}\delta_{\mu\mu'}P[\mu]\ . \tag{4.12}$$

The important point is that C' is the same correlation function as C, but for the coarse grained variables and the coarse grained D_{μ}, while $\bar{C}$ represents a correlation function for the short-range parts which can be assumed to be amenable to perturbative treatment. Equation (4.10) is then a recursion relation for the correlation function. At each step C'_{AB} is reexpressed in terms of correlation functions of functions of the new spin variable and $\bar{C}_{AB}$ which is evaluated explicitly within perturbation theory. These calculations will be discussed in Sec. VI.

B. Spin-spin correlation function

We now specialize to the case of interest, where $A(\sigma)$ and $B(\sigma)$ are spin variables. We want to compute

$$C_{i,a;j,a'}(t)=\sum_{\sigma}P[\sigma]\delta\sigma_{i,a}e^{\tilde{D}_{\sigma}t}\delta\sigma_{j,a'} \tag{4.13}$$

with

$$\delta\sigma_{i,a}=\sigma_{i,a}-\langle\sigma_{i,a}\rangle\ . \tag{4.14}$$

To lowest order in perturbation theory, the projection of a spin variable is given by Eq. (4.7), and the associated coarse grained variable by Eq. (4.8). For the short-ranged part we have

$$\mathcal{Q}\sigma_{i,a}=(1-\mathcal{P})\sigma_{i,a}=\sigma_{i,a}-\nu\psi_i^{(1)}(\sigma)\ . \tag{4.15}$$

Expanding $\sigma_{i,a}$ in the complete set of cell eigenfunctions of the cell-dynamical operator we obtain

$$\mathcal{Q}\sigma_{i,a}=\sum_{n}(1-\delta_{1,n})\nu_n\psi_i^{(n)}(\sigma) \tag{4.16}$$

with

$$\nu_n=\langle\sigma_{i,a}\psi_i^{(1)}(\sigma)\rangle_0\ . \tag{4.17}$$

Note that only the odd-parity eigenfunctions contribute to Eq. (4.16), and that $\nu_1=\nu$. Using the orthonormality condition

$$\langle\psi_i^{(n)}(\sigma)\psi_i^{(m)}(\sigma)\rangle_0=\delta_{mn} \tag{4.18}$$

we obtain, combining Eqs. (4.10), (4.11), and (4.12) the recursion relation

$$C_{i,a;j,a'}(t)=\nu^2C'_{i,j}(t')+\delta_{i,j}\sum_{n}(1-\delta_{n,1})\nu_n^*(a)\nu_n(a')e^{-\lambda_n t}\ , \tag{4.19}$$

where $C'_{i,j}(t')$ is the spin-spin correlation function for the μ spins evaluated at $t'=\Delta t$ and with dynamics governed by the SFO $D'_{\mu}=D_{\mu}/\Delta$. Thus we have rescaled time so that D_{σ} and D'_{μ} have the same time scale α^{-1}. Note that Eq. (4.14) satisfies the equal time condition for the same site correlation function

$$C_{i,a;i,a}(t=0)=\langle\sigma_{i,a}\sigma_{i,a}\rangle=1\ , \tag{4.20}$$

since

$$\sum_{n}|\nu_n(a)|^2=1\ . \tag{4.21}$$

This is due to the completeness of the cell eigenfunctions.

Equation (4.19), or its Laplace transform

$$C_{i,a;j,a'}(z)=(\nu^2/\Delta)C'_{i,j}(z')+\delta_{i,j}\sum_{n}(1-\delta_{n,1})\frac{\nu_n^*(a)\nu_n(a')}{z+i\lambda_n}\ , \tag{4.22a}$$

$$z'=z/\Delta\ , \tag{4.22b}$$

are the basic equations of our approach. From them, together with the recursion relation for K and α, discussed in the next section, we can obtain arbitrary static and dynamic correlation functions by iteration. For computing static correlation functions, we simply set $t=0$ in Eq. (4.19). Defining

$$\chi_{i,a;j,a'}=\langle\delta\sigma_{i,a}\delta\sigma_{j,a'}\rangle\ , \tag{4.23}$$

we have from Eq. (3.19)

$$\chi_{i,a;j,a'}=\nu^2\chi'_{i,j}+\delta_{i,j}\sum_{n}(1-\delta_{n,1})\nu_n^*(a)\nu_n(a')\ . \tag{4.24}$$

We can rewrite Eq. (4.24) by noting that

$$\sum_n \nu_n^*(a)\nu_n(a') = \langle \sigma_a \sigma_{a'} \rangle_0 = \delta_{a,a'} + r(\delta_{a,a'+1} + \delta_{a,a'-1}) + s\delta_{a,a'+2} \ , \tag{4.25}$$

where r and s are given by Eqs. (3.9) and (3.10). These expressions for single-cell averages follow directly from a probability distribution of the form given by Eqs. (2.1) and (2.2). Replacing in Eq. (4.24) we obtain

$$\chi_{i,a:j,a'} = \nu^2 \chi'_{i,j} + \delta_{i,j}[\delta_{a,a'} + r(\delta_{a,a'+1} + \delta_{a,a'-1}) + s\delta_{a,a'+2} - \nu^2] \tag{4.26}$$

and in particular, for the static susceptibility

$$\chi = \frac{1}{N} \sum_{\substack{i,j \\ a,a'}} \chi_{i,a:j,a'} \ , \tag{4.27}$$

we obtain the simple recursion relation

$$\chi = 4\nu^2(\chi' - 1) + 1 + 2r + s \ . \tag{4.28}$$

C. Correlation functions and recursion relations

We now come to a very important point: the form of the recursion relation $K'(K)$ determines the asymptotic behavior of the correlation functions. In particular, the fact that away from the critical point the correlation functions must decay exponentially with distance places serious restrictions on the form of the recursion relation for the parameter K.

To see how this comes about consider the correlation function for two spins separated by a distance R along the x or y axis. Let us first discuss the high-temperature phase. We have from Eq. (4.24)

$$\langle \sigma_0 \sigma_R \rangle = \nu^2 \langle \mu_0 \mu_{R/s} \rangle \ , \tag{4.29}$$

where s is the length rescaling parameter ($s = 2$ in our case). Taking $R = s^n$, we obtain after iterating Eq. (4.29) n times a nearest-neighbor correlation function

$$\langle \sigma_0 \sigma_R \rangle \approx \left\{ \prod_{i=0}^{n-1} \nu^2(K_i) \right\} \langle \mu_0 \mu_1 \rangle_n \ . \tag{4.30}$$

Here, K_i is the coupling after i iterations. The nearest-neighbor correlation function $\langle \mu_0 \mu_1 \rangle_n$ is evaluated with the coupling K_n. At high temperatures, we have the lowest-order contribution in a high-temperature expansion

$$\langle \mu_0 \mu_1 \rangle_n = K_n \ .$$

In general, $\nu(K)$ will go a nonzero constant in the high-temperature limit so that the first factor in Eq. (4.30) depends algebraically on R:

$$\prod_{i=0}^{n-1} \nu^2(K_i) \approx [\nu^2(0)]^n \sim \frac{1}{R^{2|\ln\nu|/\ln s}} \ . \tag{4.31}$$

To obtain exponential decay with distance we need therefore:

$$K_n \sim K_0^R = (K)^{s^n} \tag{4.32}$$

so that the recursion relation at high temperatures should be

$$K_{n+1} = aK_n^s \ . \tag{4.33}$$

Recall that the usual cumulant expansions give K_{n+1} proportional to K_n at high temperatures so that a different expansion procedure has to be used to obtain the correct limiting form [Eq. (4.33)].

In a similar way, by using the low-temperature expansion result

$$\langle \delta\mu_0 \delta\mu_1 \rangle_n \propto e^{-4K_n} \ , \tag{4.34}$$

we can show that to obtain exponential decay for $\langle \delta\sigma_0 \delta\sigma_R \rangle$ at low temperatures, we need

$$K_n \sim RK = s^n K_0 \tag{4.35}$$

so that the limiting form at low temperatures should be

$$K_{n+1} = sK_n \ . \tag{4.36}$$

We conclude therefore that acceptable recursion relations must interpolate smoothly between the behaviors (4.33) and (4.36).

V. SINGLE-CELL PROBLEM

An important step in our analysis is the choice of the zeroth-order operator $\tilde{D}_\sigma^0$, given $\tilde{D}_\sigma$. In Refs. 3 and 4 we treated the interaction between cells in the Hamiltonian as a perturbation. Since the Hamiltonian, in that case, appears in an exponential within the SFO, the SFO depends on the perturbation expansion parameters at all orders. The decomposition Eq. (3.30), which we use here, avoids these problems.

We pointed out in Sec. III that in order to develop a quantitative theory the "bond cutting" determination [Eq. (3.16)] of the cell parameters α_0, K_0, J_2 is inadequate. Our task in the present section is twofold: first we must find the eigenvalues and eigenfunctions of D^0, in terms of α_0, J_1, and J_2; and second, we must choose or determine the parameters α_0, K_0, J_2 in terms of α and K. The first is done, for the odd eigenfunctions, in the Appendix, in which we make extensive use of the methods developed in Ref. 4.

The second question is more complicated. A detailed discussion of a systematic way of determining the cell parameters in perturbation theory will be given elsewhere.[15] For the purpose of this paper, which, as stated in the Introduction is to show how the correlation functions are calculated, we will simplify matters by choosing the cell parameters K_0, α, J_2 [note that J_1 follows from Eq. (3.14)], in the manner described below. We shall be guided by the principle that the cell parameters should be chosen in such a way that the short-distance correlations are well described. While doing so we must also consider the recursion relation for K and its implications for the asymptotic behavior of the correlation functions. As we have seen in Eqs. (4.33) and (4.36) their asymptotic behavior requires (for $s=2$)

$$K'=aK^2 \quad (K \ll 1) \ , \tag{5.1a}$$

$$K'=2K \quad (K \gg 1) \ . \tag{5.1b}$$

A standard cumulant approximation on a square lattice satisfies Eq. (5.1b) but not Eq. (5.1a), so, one does not find the correct exponential decay in the disordered phase. In our analysis $K'(K)$ is given by Eq. (3.56). That relation depends on α_0, as yet unspecified. Our strategy in this paper shall be to find another equation for $K'(K)$, satisfying Eq. (5.1) and then determine α_0 by using Eq. (3.56). We will obtain the new relation from the correlation function equations discussed in the last section. As the temperature changes, the recursion relation must interpolate between the two known limits. A natural variable to introduce is

$$\phi \equiv e^{2K}\tanh K \ . \tag{5.2}$$

Then, the recursion relation

$$\phi'=\phi^2 \tag{5.3}$$

gives a proper interpolation. This gives us a large hint on how to proceed. If we define $\epsilon(n)$ as the two-spin static correlation function for spins separated by n sites along the x or y directions, we have, (for $n>1$) from Sec. IV, $\epsilon(2n)=\nu^2\epsilon'(n)$. The ratio

$$\frac{\epsilon(4)}{\epsilon(2)}=\frac{\epsilon'(2)}{\epsilon'(2)} \tag{5.4}$$

gives a relation between K and K'. In one dimension this definition of $K'(K)$ coincides with the result of a decimation transformation. In our case, one cannot explicitly solve for $K'(K)$. We instead have used the high-temperature results for $\epsilon(n)$ to determine the coefficients $\{a_i\}$ in the expansion

$$\phi'=\phi^2+\sum_{i=0}^{\infty} a_i h^{i+3} \ , \tag{5.5}$$

where $h=e^{-4K}\tanh K$. The results of this analysis give $a_0=-4$, $a_1=-42$, $a_2=-396$, $a_3=-3750$, and the fixed point properties $\tanh K_c=0.4150$ and exponent $\nu^{-1}=y_T=1.011$. These compare quite well with the exact results $\tanh K_c=0.4142$; $\nu^{-1}=1$. Thus, it is seen that the corrections, to $\phi'=\phi^2$ in Eq. (5.5) are small near the critical point. If one defines $f(u)\equiv(\phi^2-\phi')/\phi^2$ ($u\equiv\tanh K$) one finds that $f(u)$ has a maximum of 0.195 for $u\approx 0.10$ and decreases rapidly away from its maximum: $f(u) \leq 0.02$ for $u>0.32$. The relatively larger value of f occurs away from any fixed points: one has $f(u)>0.1$ only for $0.03<u<0.2$, and $f(u_c)=0.004$. Therefore these corrections have only a small effect on the iteratively computed values of the correlation functions. If we look at the effect of the correction terms on the critical indices (the method of analysis is presented in the next section) we find, for example,[5] $\beta=0.1215$, $z=1.796$ with the full recursion relation (5.5) while $\beta=0.1219$, $z=1.791$ for $\phi'=\phi^2$. That is, the correction terms given by $f(u)\phi^2$ near the transition [as reflected in the dependence of β, z, and T_c on $f(u)$] are of order 0.3%. Therefore we will, for simplicity, use the recursion relation $\phi'=\phi^2$ in our numerical work. We believe, however, that the general form of Eq. (5.5) and its range of validity merit further study.

Having done this, the condition that the recursion relation Eq. (3.56) coincides with that obtained from Eq. (5.5), as discussed above, determines the function $\alpha_0(K,K_0,J_2)$. This procedure nicely decouples statics and dynamics.

Next, we specify K_0 and J_2. The nearest-neighbor recursion relation

$$\epsilon(1)=\frac{r}{2}+\frac{\nu^2}{2}\epsilon'(1) \tag{5.6}$$

gives the behavior of K_0 at high temperatures

$$K_0=2K \tag{5.7}$$

to lowest order in K [r is defined in Eq. (3.9)]. A similar analysis shows that Eq. (5.7) gives also the low-temperature behavior of K_0. Consider next the parameter J_2. This is analogous to the parameter δ of the second model in Ref. 4. In analogy with that example, it is reasonable to require that $T_0[\mu|\sigma]$ reduces to the majority rule at low temperatures. This implies $f=-\frac{1}{3}$, $J_2=1$ as $K\rightarrow\infty$. As $K\rightarrow 0$, we expect that J_2 goes as K_0^2. We have used the simple interpolation formulas

$$K_0=2K-(\tanh^2 K)/8q \ , \tag{5.8}$$

$$J_2=\tanh^2(2pK_0) \ , \tag{5.9}$$

and determined p and q simultaneously by demanding that $\epsilon(1)$ calculated from Eq. (5.6) yields the exact result $\epsilon_c(1)=\sqrt{2}/2$ at $T=T_c$, and that α_0 be as close to α as possible at T_c. This gives $p = 0.434$ and $q = 0.38$. The operator $\tilde{D}_0$, and the recursion relations are then fully specified.

While the above determinations are not free of criticism, we should point out once more that our objective here is the evaluation of the correlation functions, for which purpose the standard cumulant expansion is wrong. Rather than choosing our zeroth-order SFO by arbitrary bond cutting, it is clearly better to choose its parameters from the short-range recursion relations such as Eqs. (5.4) and (5.6) which would clearly be botched up by bond cutting. We do not claim the details of our treatment to be definitive, but only that they incorporate a number of necessary physical principles which must go into the calculation of the static and dynamic correlation functions.

Results for these correlation functions are presented in the next section.

VI. RESULTS AND DISCUSSION

The spin-spin correlation functions can now be calculated by iteration of the first-order recursion relations given in Sec. IV B. We use, at each iteration, the cell operator parameters obtained according to the prescriptions developed in Sec. V. That is: given the initial value of K, we obtain K_0, α_0, and J_1 as functions of K. The new coupling $K'(K)$ follows from the recursion relation Eq. (5.5) or Eq. (3.56). The new local parameters α_0', K_0', J_1', can then be obtained. The odd eigenvalues of $\tilde{D}^0$ are calculated at each step.

Let us consider first time-independent quantities. Consider, for example, the magnetization per spin, for which the recursion relation is readily found to be

$$M = \nu M' \tag{6.1}$$

Iteration of this recursion relation produces

$$M = \nu'\nu'', \ldots, \nu^{(n-1)} M^{(n)} \ . \tag{6.2}$$

It is easy to see that $\nu = 1$ at $T = 0$ and $\nu = \frac{1}{2}$ at $T \rightarrow \infty$. Since we always have $K' < K$ at $T > T_c$ and $K' > K$ at $T < T_c$ we immediately find that $M = 0$ at all $T > T_c$ and $M \rightarrow 1$ as $T \rightarrow 0$. In detail, the curve for $M(T)$ in Fig. 4 is obtained. It is practically identical to the known exact result[16] at all temperatures. Near T_c, the exponent β is found to be $\beta = 0.122$. This result for $M(T)$ gives a strong indication of the soundness of our approximations. Further confirmation may be found below.

In Fig. 5 we plot the magnetic susceptibility χ, obtained by iterating Eq. (4.28). Our results are in excellent agreement with the most reliable series results[12,17] at all temperatures above T_c, and are only somewhat different below T_c. The critical exponent

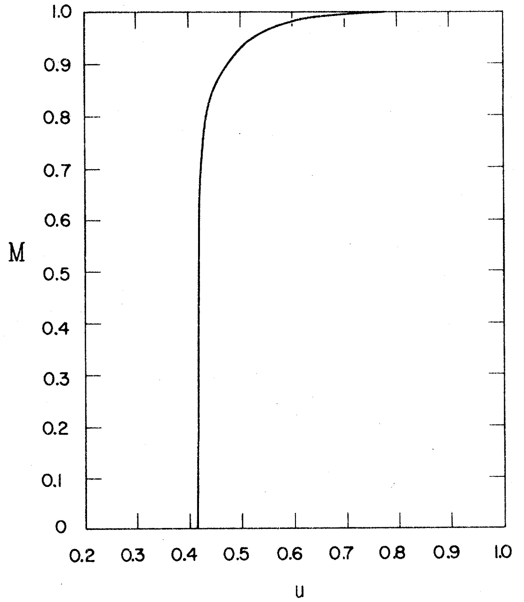

FIG. 4. The spontaneous magnetization obtained in this work. [$u = \tanh(K)$.] It is virtually identical to the exact result, see Ref. 16.

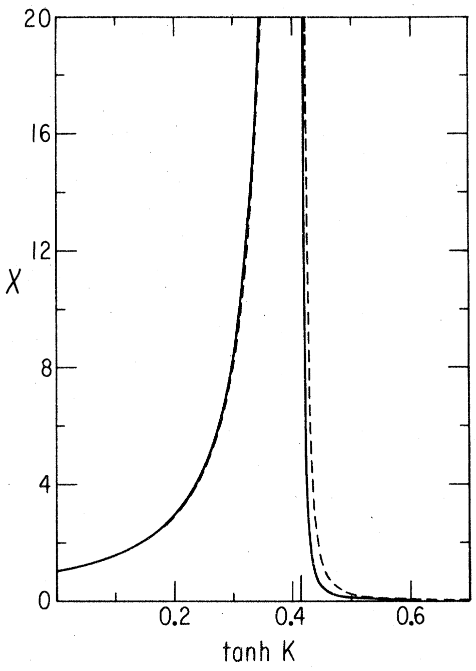

FIG. 5. The magnetic susceptibility. Solid curve is the present result; dashed curve, series expansion results (Refs. 12 and 17).

$\gamma = 1.756$ is within 0.4% of the exact result 1.75. It should be emphasized that the values of these exponents are not built in by our procedure for determining the zeroth-order parameters.

As a final static result, we present results (Fig. 6) for the correlation function $C(q)$ for several values of $|\vec{q}|$ in the temperature range around T_c. The recursion relation for $C(q)$ is given by the spatial Fourier transform of Eq. (4.26). We compare our results with those of Ref. 17. It can be seen that the plots are extremely similar. We also note that our results are obtained with considerably less computational effort than used in Ref. 17. We obtain a maximum at $q \neq 0$ for $T > T_c$, corresponding to deviations from the Ornstein-Zernike theory. At $T = T_c$, we have extracted the exponent η given by

$$\eta = -\ln\nu^2(T_c)/\ln 2 = 0.244 \ . \tag{6.3}$$

It is excellent concordance with the exact result $\eta = \frac{1}{4}$.

These static results illustrate the importance of satisfying the conditions [Eq. (5.1)] and of the proper choice of the static cell parameter K_0 [see Eq. (5.8)]. It is well known that the standard cumulant expansion,[6] in which $K_0 = K$ and Eq. (5.1) is not satisfied, yields very poor results for $M(T)$. The fact that our static results are good is due to the careful treatment of the static parameters, and also, in the cases of χ and $C(q)$ to the retention of the inhomogeneous terms which are not important in the critical region but necessary elsewhere. Of course, dynamic considerations, such as those leading to the eigenvalue Eq. (3.3) do not come into play here. Indeed, if our purpose was to study only static properties we would write, instead of Eq. (5.5), an equivalent equation for the parameter f (which, together with K_0 fully determines ν, r, and s) and then define the transformation $T[\mu|\sigma]$ via Eqs. (3.20) and (3.22) without any reference to a dynamical operator. Equations (6.1), (4.26), and (4.28) are, thus, uncoupled from the dynamics, as they should be.

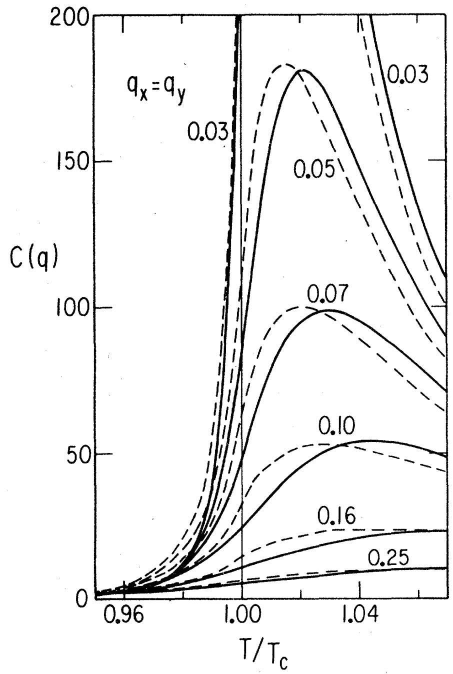

FIG. 6. Static correlation function $C(q)(q_x = q_y)$ vs temperature for various wave numbers $q_x a$. Solid lines: our results. Dashed lines: from Ref. 17.

We now proceed to the dynamics. First, an analysis of the recursion relation for α [Eq. (3.48)] yields the dynamic exponent $z = 1.79$. There is a rather large spread of values for z quoted in the literature.[11] It is not clear why this uncertainty in the value of z exists. Our result is larger than the rigorous lower bound $z = 1.75$.[10]

In Fig. 7 we exhibit our results for the "same site" correlation function $C_{ii}(t) = \langle \sigma_i \sigma_i(t) \rangle$. At high temperatures, this function decays quickly to zero with time. As the temperature approaches T_c from above, the decay becomes very slow, and, below T_c, the asymptotic value is M^2. Not only is this the expected behavior at all temperatures, but near T_c we have a further check: $C_{ii}(\omega)$ is sharply peaked at $\omega = 0$ for $K \sim K_c$. We have

$$C_{ii}(\omega = 0) \sim |K - K_c|^{-n} \ , \tag{6.4}$$

$$C_{ii}(\omega, K = K_c) \sim \omega^{-\mu} \ . \tag{6.5}$$

From our calculated $C_{ii}(\omega)$ we extract $n = 1.54$, and $\mu = 0.86$. These results are in agreement with the scaling relations[18]

$$n = \nu(z + 2 - d - \eta) \ , \tag{6.6}$$

$$\mu = \frac{1}{z}(z + 2 - d - \eta) \ , \tag{6.7}$$

when we substitute the values of z and η previously

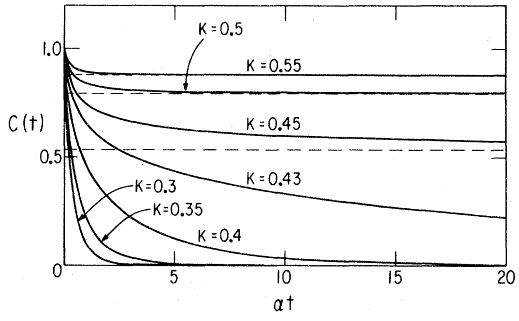

FIG. 7. Present results for the same-site correlation function $\langle \sigma_i \sigma_i(t) \rangle$ vs αt for several values of K. The dashed horizontal lines correspond to values of M^2.

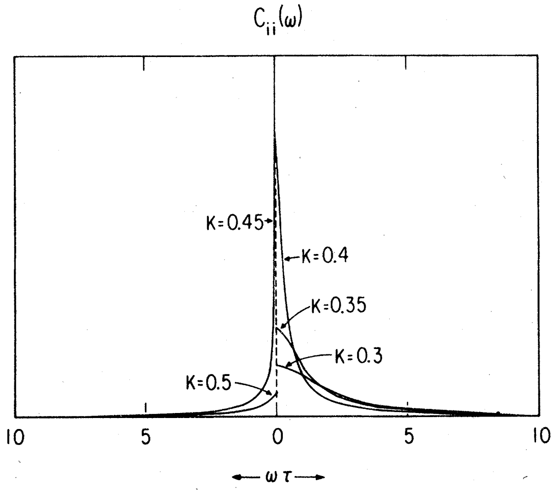

FIG. 8. The temporal Fourier transform of the same-site correlation function at various temperatures ($\tau = \alpha^{-1}$).

found from the direct analysis of the recursion relations. This is an important consistency check. The function $C_{ii}(\omega)$ is plotted in Fig. 8 at several temperatures.

Finally (Fig. 9) we present the full correlation function, $C(q,\omega)$, for different values of q and different temperatures. In particular one can see in Fig. 9 the effect on $C(q,\omega)$ of lowering the temperature from above to below T_c, at constant q, as well as the effect of changing q at constant temperature. Results can readily be obtained at any values of K, q, and ω by the use of the recursion Eq. (4.22) or its spatial Fourier transform.

In the next four figures we consider the behavior of $C(q,\omega)$ in the neighborhood of T_c. In Fig. 10 we have plotted $C(q,\omega=0)$ as a function of q, at $K=K_c$ and $K=K_c \pm 10^{-3}$. As before, we find a peak at finite q for $K \neq K_c$. At $K=K_c$ we have $C(q,0) \propto q^{-3.55}$ (see solid line). This exponent equals our calculated value of $2+z-\eta$, in agreement with scaling. In Fig. 11 we display $C(q,\omega=0)$ as a function of $|K-K_c|$ for several wave vectors. Again, $C(q=0,\omega=0) \propto (K-K_c)^{-(2-\eta)}$. In Fig. 12 we have $C(q,\omega)$ vs ω at and near T_c for several values of q. The proper scaling behavior at $q=0$ and $T=T_c$ is exhibited: $C(0,\omega) \propto \omega^{-(2+z-\eta)/z}$. The value of the exponent (slope of the solid lines in Fig. 12) is 1.98, in perfect concordance with the value obtained from the exponents. In Figs. 10 to 12 q is in units of $1/a$.

In the critical scaling region it is conventional[19] and convenient to write the dynamic structure factor in the dynamic scaling form

$$C(q,\omega) = \frac{C(q)}{\omega(q)} f_x(\bar{\nu}) \ , \qquad (6.8)$$

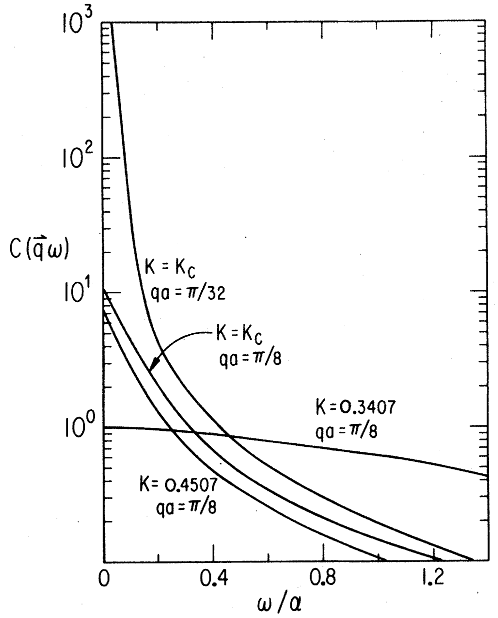

FIG. 9. The correlation function $C(q,\omega)$ vs ω for the values $q_x = q$, $q_y = 0$ and K indicated.

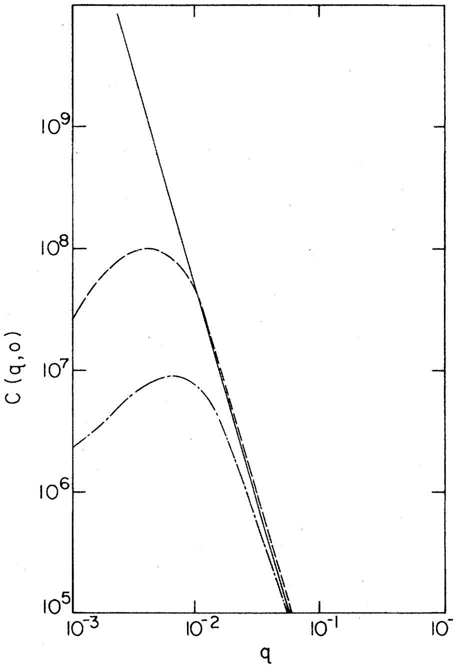

FIG. 10. The correlation function $C(q,\omega)$ vs q ($q=q_x, q_y=0$) for $\omega=0$ and several values of K. The solid line is for $K=K_c$, $\omega=0$; the dashed line: $K-K_c=-1.0\times10^{-3}$, $\omega=0$; the dot-dashed line: $K-K_c=+1.0\times10^{-3}$, $\omega=0$.

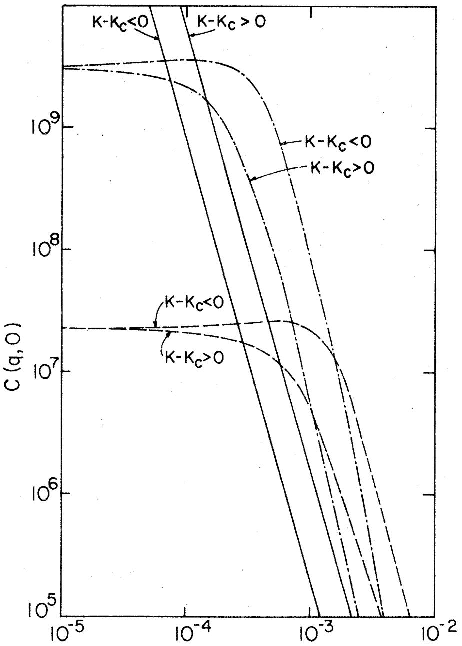

FIG. 11. The correlation function $C(q,\omega)$ vs $|K-K_c|$ for $\omega=0$ and several values of q ($q=q_x, q_y=0$). The two solid lines are for $q=0$, $\omega=0$; the dot-dashed lines are for $q=\pi/2^{10}$, $\omega=0$, and the dashed lines are for $q=\pi/2^8$, $\omega=0$.

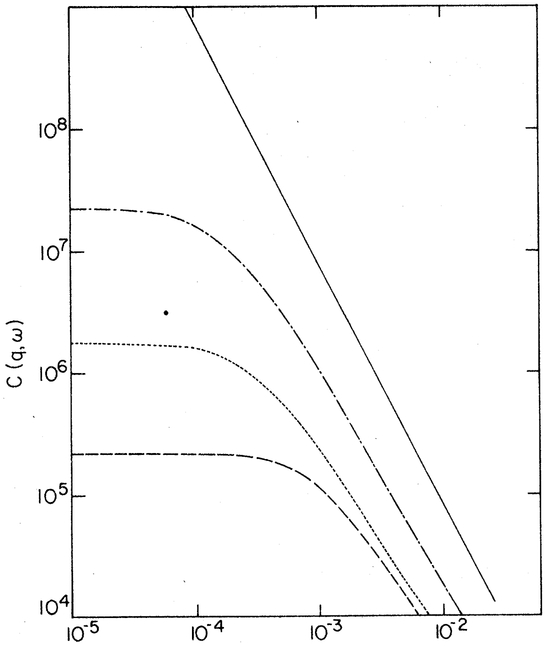

FIG. 12. The correlation function $C(q,\omega)$ vs ω for several values of q and K ($q=q_x, q_y=0$). The solid line is for $q=0$, $K=K_c$; the dot-dashed line is for $q=\pi/2^8$, $K=K_c$, the dotted line: $q=0$, $K-K_c=1\times10^{-3}$; the dashed line: $q=0$, $K-K_c=-1\times10^{-3}$.

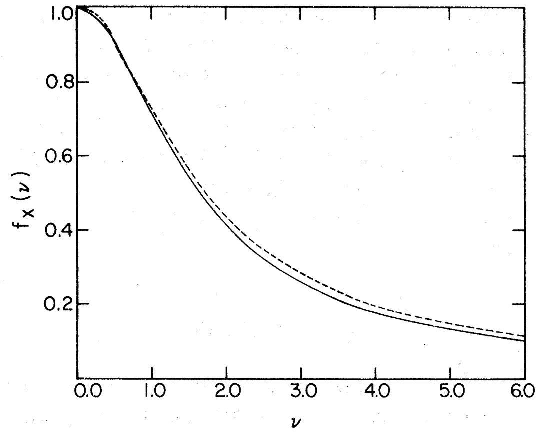

FIG. 13. The shape function $f_x(\nu)$ vs ν for $x=\infty$ (solid line) and $x=0$ (dashed line).

where $\omega(q)$ is the characteristic frequency, $f_x(\bar{\nu})$ is the shape function, $x=q\xi$, and $\bar{\nu}=\omega/\omega(q)$. We have determined $f_x(\bar{\nu})$ in the limits $x\to\infty$ (critical regime) and $x=0$ (hydrodynamic regime). In the case $x\to\infty$ we have $\omega_c=Aq^z$ and determine A by the normalization condition, $f_\infty(0)=1$. In the case $x=0$ we have $\omega_c=\bar{A}|K-K_c|^z$ and determine $\bar{A}$ by the condition (see Ref. 19)

$$\int_0^1 d\bar{\nu} f_\infty(\bar{\nu}) = \int_0^1 d\bar{\nu} f_0(\bar{\nu}) \ . \tag{6.9}$$

We see from Fig. 13 that $f_x(\bar{\nu})$ is essentially Lorentzian in both limits.

We conclude, therefore, that it is possible to use real-space renormalization-group methods to calculate the time dependent correlation functions of the kinetic Ising model, and in particular, the correlation functions and thermodynamics of the ordinary Ising model. The key ingredients are the decomposition of the initial variables (Sec. IV A) by means of projection operators which allows us to write the exact Eqs. (4.10) and (4.11); a careful treatment of the effective couplings in an isolated cell; and development of methods for carrying out the RSRG that eliminate unphysical memory effects and lead to the appropriate spatial decay at large distances. Our results are in excellent agreement with previously known results, when these exist, and satisfy all expected scaling relations.

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APPENDIX

In this Appendix we work out the eigenfunctions and eigenvalues of the cell operator given by Eqs. (3.11) and (3.13). The general procedure is essentially the same as in the triangular case. For the square lattice, we have $2^4=16$ independent eigenfunctions, half of them even, half odd. We will solve here for the odd ones. The even eigenfunctions can be obtained in a similar way, but they are not needed here.

Operating with $\tilde{D}^0[\sigma|\sigma']$ on all odd-cell spin combinations we readily obtain

$$\tilde{D}^0[\sigma|\bar{\sigma}]\bar{\sigma}_a=-\alpha_0[\sigma_a+J_1(\sigma_{a+1}+\sigma_{a-1})+J_2\sigma_a^{(3)}] \ , \tag{A1}$$

$$\begin{aligned}\tilde{D}^0[\sigma|\bar{\sigma}]\bar{\sigma}_a^{(3)}=-\alpha_0[3\sigma_a^{(3)}+J_1(2\sigma_{a+1}+2\sigma_{a-1}\\ +\sigma_{a+1}^{(3)}+\sigma_{a-1}^{(3)})\\ +J_2(2\sigma_{a+2}^{(3)}+\sigma_a)] \ ,\end{aligned} \tag{A2}$$

where a sum over barred configurations is understood and the cell index i has been omitted. The quantity $\sigma_a^{(3)}$ is defined by $\sigma_a^{(3)}=\sigma_{a-1}\sigma_a\sigma_{a+1}$.

The above equations have a "hopping" structure. If we think of the four spins as located on a circle it is clear that we can diagonalize our operator through Fourier transformation. That is, the eight odd eigenfunctions must be linear combinations of the eight functions

$$\phi_1(q)=\sum_a e^{iqa}\sigma_a \ , \tag{A3}$$

$$\phi_3(q)=\sum_a e^{iqa}\sigma_a^{(3)} \ , \tag{A4}$$

where the index ϕ is:

$$q=\frac{n\pi}{2a}, \quad n=0,1,2,3 \ . \tag{A5}$$

Equations (A1) and (A2) yield

$$\tilde{D}^0[\sigma|\bar{\sigma}]\bar{\phi}_1(q)=-[a(q)\phi_1(q)+b(q)\phi_3(q)] \ , \tag{A6}$$

$$\tilde{D}^0[\sigma|\bar{\sigma}]\bar{\phi}_3(q)=-[c(q)\phi_1(q)+d(q)\phi_3(q)] \ , \tag{A7}$$

where

$$a(q)=\alpha_0(1+2J_1\cos qa) \ , \tag{A8a}$$

$$b(q)=\alpha_0 J_2 \ , \tag{A8b}$$

$$c(q)=\alpha_0(4J_1\cos qa+J_2) \ , \tag{A8c}$$

$$d(q)=\alpha_0(3+2J_1\cos qa+2J_2e^{-2iqa}) \ . \tag{A8d}$$

We now write the eigenfunctions in the form

$$\psi(q)=B\phi_1(q)+D\phi_3(q) \ . \tag{A9}$$

The eigenvalue equation

$$\tilde{D}^0[\sigma|\bar{\sigma}]\psi(\bar{\sigma})=-\lambda\psi(\sigma) \tag{A10}$$

is easily solved by the use of Eqs. (A6) and (A7). The eight eigenvalues are

$$\lambda_\eta(q)=\tfrac{1}{2}(a(q)+d(q)+\eta\{[a(q)+d(q)]^2-4[a(q)d(q)-b(q)c(q)]\}^{1/2}), \quad \eta=\pm 1 \tag{A11}$$

and B and D are related by

$$D(q)=\frac{b(q)B(q)}{\lambda-d(q)} \ . \tag{A12}$$

Using also the normalization condition

$$\langle|\psi(q)|^2\rangle_0=1 \ , \tag{A13}$$

we see that the eight odd eigenfunctions can be characterized by the label q,η and written as

$$\psi_\eta(q)=N_\eta(q)[\phi_1(q)+f_\eta(q)\phi_3(q)] \ , \tag{A14}$$

$$f_\eta(q)=\frac{b(q)}{\lambda_\eta(q)-d(q)} \ , \tag{A15}$$

$$N_\eta(q)=\tfrac{1}{2}\{[1+f_\eta^2(q)][1+2r\cos q+s\cos 2q]+2f_\eta(q)[s(1+\cos 2q)+2r\cos q]\}^{-1/2} \ , \tag{A16}$$

with

$$r \equiv \langle \sigma_a \sigma_{a+1} \rangle_0 \;, \tag{A17a}$$

$$s \equiv \langle \sigma_a \sigma_{a+2} \rangle_0 \;. \tag{A17b}$$

We find in practice, as we would expect, that the slowest odd mode corresponds to the uniform ($q=0$) case, with $\eta=-1$. This is the function $\psi_-(0)$ called $\psi^{(1)}$ in the text, and the corresponding eigenvalue $\lambda_-(0)$ is called λ in the text.

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