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Singular thermodynamic properties in random magnetic chains

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The nearest-neighbor random-exchange Heisenberg antiferromagnetic Heisenberg chain ($s=\frac{1}{2}$) is studied at low temperatures via an approximate renormalization-group method. This entails renormalization of the random-exchange coupling constant J and of the probability law for J, $P_0(J)$. After n iterations we find that the renormalized probability function $P_n(J^{(n)})$, a function of the renormalized coupling $J^{(n)}$, develops singular behavior for small $J^{(n)}$, independent of the initial form of $P_0(J)$. This happens both for $P_0(J)$ that diverge or go to zero as $J\rightarrow 0$. The singular form of $P_n(J^{(n)})$ is such that it leads to a specific heat and susceptibility that behave like $C\sim T^{1-\alpha_C(T)}$ and $\chi\sim T^{-\alpha_\chi(T)}$. $\alpha_C(T)$ and $\alpha_\chi(T)$ are exponents weakly dependent on temperature (T) that go to one as $T\rightarrow 0$ and satisfy $\alpha_\chi(T)>\alpha_C(T)$, for arbitrary initial probability laws. The classical n-vector models, and in particular the classical Heisenberg model are also studied using a renormalization-group approach and it is found that their behavior is different than that of the quantum model: singular behavior in the thermodynamic properties is only obtained if the starting $P_0(J)$ is singular. An explanation of these results in terms of the scaling theory of localization is suggested. The relevance of our results to the experimental findings on the magnetic properties of tetracyanoquinodimethanide complexes is also discussed.

I. INTRODUCTION

The organic charge transfer salts of tetracyanoquinodimethanide (TCNQ) have been the subject of extensive study in the past few years. Among their most remarkable properties is the one-dimensionality of their electronic and magnetic behavior. Examples of these substances are acridinium di-tetracyanoquinodimethanide [Ad-$(TCNQ)_2$] and quinolinium $(TCNQ)_2$ [Qn$(TCNQ)_2$]. In this paper we will concentrate on their static magnetic properties and will have in mind Qn$(TCNQ)_2$, which is the most studied compound in this respect. It is found experimentally that the magnetic susceptibility grows as

$$\chi\sim T^{-\alpha_\chi} \tag{1.1}$$

for temperature below 20 K down to 10 mK.[1,2] The measured values of α_χ range from 0.72 to 0.84, depending on sample preparation. Measurements of the magnetic contributions to the specific heat lead to[3]

$$C\sim T^{1-\alpha_C}\ , \tag{1.2}$$

with T between 0.07 and 5 K and $\alpha_C=\alpha_\chi$ within experimental accuracy. The high-field magnetization as a function of magnetic field behaves as

$$M\sim h^{1-\alpha_M} \tag{1.3}$$

for temperatures down to 30 mK and magnetic fields h bigger than ~ 5 kG.[4] Again, $\alpha_M=\alpha_\chi$ for a given sample.

It is currently accepted[2] that the magnetic behavior of Qn$(TCNQ)_2$ can be described by a random-exchange Heisenberg-antiferromagnetic-chain model (RHAF), defined by the Hamiltonian

$$H=\sum_i J_i\vec{S}_i\cdot\vec{S}_{i+i}\ . \tag{1.4}$$

Here J_i is a positive random variable and $\{\vec{S}_i\}$ are the usual Pauli spin-$\frac{1}{2}$ operators. This interpretation was first suggested by Bulaevskii *et al.*[1] To calculate the thermodynamic properties of the RHAF they started by transforming H into its fermion representation: the XY part of H leads to a free spinless Fermi-gas problem whereas the z component gives the interactions between the fermions. This canonical transformation is exact but the solution to the interacting fermion problem is not known. Bulaevskii *et al.*[1] made the assumption that the low-lying excitations in the interacting problem could be described by

Fermi quasiparticles, with an unknown density of states $\rho(\epsilon)$. In order to explain their experimental results, they were led to assume a singular density of states with the form

$$\rho(\epsilon) \sim 1/|\epsilon|^{\alpha_B} ,$$

with α_B a constant. From these two assumptions they obtain results like in Eqs. (1.1)–(1.3) with $\alpha_B = \alpha_X = \alpha_C = \alpha_M$, in agreement with the experimental findings. However from the theoretical point of view it is not clear how the fully interacting Heisenberg Hamiltonian leads to a free-fermion problem. Also, it is not clear how to derive the singular density of states that was proposed to fit the experimental results.

Some progress in understanding this problem from a more microscopic point of view was achieved by Theordorou and Cohen (TC).[5,6] Starting from a Hubbard Hamiltonian with random on-site energies and small transfer matrix elements, they were able to show that: (i) The molecular orbitals in TCNQ go over to localized states because of disorder. This leads to localized spin states distributed randomly along the chain. (ii) At low temperature, the spins interact via an antiferromagnetic Heisenberg random exchange coupling J_i, obeying the probability law

$$P_0(J_i) \sim 1/J_i^{\alpha} , \tag{1.5}$$

with $\alpha < 1$ depending on the microscopic parameters of the underlying Hubbard model, and $0 \leq J_i \leq J_0$ (J_0 is a cutoff). For $\alpha > 0$, the probability distribution (1.5) has a singularity at the origin. The result (1.5) is in fact not dependent on the model but only on the facts that the spins are located randomly along the chain and the exchange constant decays exponentially with distance. This last point is related to the localization of the spin moments in the chain.

Equations (1.4) and (1.5) define the problem completely. To evaluate the thermodynamic properties of this model we should calculate the averaged free energy

$$\frac{\langle F \rangle}{k_B T} = -\lim_{N\to\infty} \frac{1}{N} \int \prod_i dJ_i \, P(J_i) \ln(\mathrm{Tr}_{\{\vec{S}_i\}} e^{-H/k_B T}) \tag{1.6}$$

where N is the number of sites in the chain and k_B is the Boltzmann constant (taken equal to 1 from now on). The evaluation of the free energy is highly nontrivial in the periodic case, let alone in the random case. Faced with this difficulty, TC resorted to an approximate cluster argument to compute the thermodynamic properties of the RHAF. Within that approximation (which we will discuss in Sec. II) they obtain for a $P_0(J_i)$ as given in Eq. (1.5) thermodynamic properties as given by Eqs. (1.1) and (1.2), with $\alpha_X = \alpha_c = \alpha$. Therefore, they conclude that singularities in the thermodynamic properties will only be obtained if the underlying probability distribution for the couplings $P_0(J_i)$ is singular at the origin [$\alpha > 0$ in Eq. (1.5)].

To further support this conclusion, TC considered the behavior of classical n-vector models, with n the number of components of the vector spin $\vec{S}^{(n)}$ ($n = 3$, 2, and 1 correspond to the classical Heisenberg, XY, and Ising models, respectively). In the absence of a magnetic field the disordered n-vector model can be solved exactly using the transfer-matrix technique for an arbitrary probability distribution $P_0(J_i)$, and here again one obtains $\alpha_X = \alpha$ (see Sec. IV for a discussion of this point). We shall find however from our renormalization-group analysis that the quantum and classical disordered Heisenberg models are in different universality classes: for the classical model, a nonsingular $P_0(J)$ gives nonsingular behavior, while in the quantum case any arbitrary $P_0(J)$ yields singular thermodynamic properties.

In a more recent paper, Clark and Tippie[7] (CT) studied the problem defined by Eqs. (1.4) and (1.5) within the approximation of taking the exchange constant between every other spin equal to zero. In this case the evaluation of $\langle F \rangle$ can be done exactly to zero. In this case the evaluation of $\langle F \rangle$ can be done exactly even in the presence of a magnetic field. Their results can be expressed as in Eqs. (1.1)–(1.3) with $\alpha_X = \alpha_C = \alpha_M = \alpha$. Thus, their results coincide with the TC results in that a singular $P_0(J)$ is needed to yield singular thermodynamic properties. However, as we will see in Sec. II, the reason is that both approaches are essentially equivalent in that the thermodynamic properties are determined by the probability of a single coupling being less than the temperature. In the fully coupled chain we find, however, that correlations between different couplings are important and lead to different results.

In order to study the relevance of the CT results for the fully coupled chain we studied, as a first step, the stability of these results when taking the exchange coupling J equal to zero every four spins. As we show in Sec. II, the results change and the thermodynamic behavior becomes more singular at low temperatures. Due to this fact, we are led to consider the fully interacting random problem in its own right. Of course, our analysis is approximate. The method is based on the idea of thinning of degrees of freedom, or renormalization-group (RG) method, properly extended to deal with quantum random problems at low temperatures. This RG method was introduced by Drell *et al.*[8] to calculate ground-state properties of quantum Hamiltonians and it has been shown to be useful in treating various quantum problems.[9,10] The method has been extended to calculate approximately finite temperature properties[11] of the periodic problem and to deal with random problems.[12] In the random problem, the procedure entails renormalizing simultaneously J_i and $P_0(J_i)$, with $P_0(J_i)$ the initial probability law for J_i. Thus, we find a renormalized $J_i^{(n)}$ and a renormalized $P_n(J_i^{(n)})$ at every stage of the rescaling. The form of the renormalized $P_n(J_i^{(n)})$ is directly related to the recursion relation for $J_i^{(n)}$, and $\langle F \rangle$ is in turn related to the form of $P_n(J_i^{(n)})$. We find that it does not matter what the initial $P_0(J)$ is because after the first few iterations of the RG the new P's tend to be more and more singular towards $J=0$. This in turn leads to a χ and a C that can be written as in Eqs. (1.1) and (1.2) but with $\alpha_c = \alpha_c(T)$, $\alpha_\chi = \alpha_\chi(T)$, and $\alpha_\chi \neq \alpha_C$. This temperature dependence in α_χ and α_C turns out to be rather weak ($\sim \ln 1/T$) and possibly difficult to check experimentally. Perhaps easier to check may be the result that $\alpha_\chi(T)$ is always bigger than $\alpha_C(T)$. The exponents $\alpha_\chi(T)$ and $\alpha_C(T)$ tend slowly to one as $T \rightarrow 0$, meaning that the effective coupling between spins becomes weaker and weaker as T goes down leading to a paramagneticlike susceptibility. Of course, experimentally the interaction between chains may become more relevant at lower temperatures and this result may not be seen. The relevance of interchain coupling has not been studied and it is still not understood why at the lowest tempertaures studied experimentally so far no signature of two- or three-dimensional effects have been reported.

The paper is organized as follows: In Sec. II, we undertake a numerical evaluation of $\langle F \rangle$ for the two and four-spin-cell problems. A theoretical understanding of the results arises when recognizing that the main contributions to $\langle F \rangle$ come from calculating the probability of having an energy gap between the ground state and first excited states that is smaller than the temperature. Here it is found that in the four-spin-cell analysis the exponents α_χ and α_C acquire a $\ln T$ correction to their two-spin-cell values. In Sec. III we carry out the RG evaluation of $\langle F \rangle$ as applied to the RHAF. We take cells with three spins. It is important to take an odd number of sites per cell to make sure that we map a spin-$\frac{1}{2}$ problem into a spin-$\frac{1}{2}$ problem. First we find the recursion relation for J_i. Next we calculate the renormalized probability law $P_i(J')$ from the recursion relation. Here is where we note that the form of $P_n(J^{(n)})$ after a few renormalizations is nearly independent of the initial form of $P_0(J)$. We illustrate this fact by looking at the probability laws given by (i) $P_0(J) \sim 1/J^\alpha$ $(\alpha > 0)$, (ii) $P_0(J)=1$, and (iii) $P_0(J)=2J$. They are (i) singular, (ii) flat, and (iii) $P_0(J) \rightarrow 0$ as $J \rightarrow 0$. We find that in all cases $P_n(J^{(n)})$ after a few iterations becomes strongly peaked near the origin. This fact leads to a χ and C of the form given in Eqs. (1.1) and (1.2). The evaluation of $\langle F \rangle$ requires taking into account higher excited states within the cells. This is done within the approximation of treating the higher cell states as decoupled. This approximation has been shown to give good answers for the periodic case,[11] and will therefore be used here too. We end Sec. III with plots of $\ln\chi$ vs $\ln T$, $\ln C$ vs $\ln T$, $\alpha_\chi(T)$ vs $\ln T$, and $\alpha_C(T)$ vs $\ln T$, plus some discussion about the accuracy of the results.

Similar results to the ones given in Sec. III were reported in a parallel study by Ma *et al.*[13] They used a decimation procedure that eliminates strongly interacting spins on each iteration. This leads then to a problem with renormalized P's and renormalized cutoffs for J. We mention along the paper what we see as advantages and disadvantages of their approach as compared to ours. However, we stress that the two methods lead to qualitatively analogous results for the RHAF, thus strengthening our belief in their correctness. Further, recent finite cell calculations of Soos *et al.*[14] show also qualitatively similar results at higher temperatures.

In Sec. IV we study the classical random n-vector models, and in particular the classical Heisenberg chain $(n=3)$. In the absence of a magnetic field this problem can be solved exactly for arbitrary $P_0(J)$. It is immediately apparent that these models are not in the same universality class as the RHAF. In order to shed further light into this difference we study the renormalization properties of $P_0(J)$, by using a RG method of the decimation variety. There we find that if $P_0(J) \sim 1/J^\alpha$, $P_1(J_i) \sim 1/J^\alpha$. That is, the renormalized $P(J)$ remains of the same form as the original one. Thus, if we start with a nonsingular $P_0(J)$ we do not expect that the renormalized $P_1(J')$ will develop singularities, contrary to what happens in the RHAF.

It should be mentioned that for the quantum XY model, exact results[15,16] show that one obtains singular behavior for the thermodynamic quantities for arbitrary probability distributions for the couplings, as we find here for the Heisenberg model. In another paper,[11] we study the quantum anisotropic Heisenberg model and in particular the XY limit by the same RG technique and find good agreement with the known exact results in the XY limit. This gives us confidence in the accuracy of the technique and in the results for the Heisenberg case. One may conclude therefore that quantum random models with a continuous symmetry are in a different universality class than their classical counterparts. The quantum fluctuations seem to be all important in determining the low-temperature thermodynamic properties for these models. In Sec. V we suggest a possible explanation for the results found in this paper in terms of a scaling theory of localization.[17,18]

Part of the results presented in this paper were reported briefly elsewhere.[12]

II. TWO- AND FOUR-SPIN-CELL ANALYSIS

As a first step in our study of the RHAF we will try to understand the relevance of correlations between different coupling strengths. This is motivated because of the apparent success by the TC and CT approximations when compared with experiments. In this section we calculate numerically the averaged free energy $\langle F \rangle$ defined in Eq. (1.6) within an approximation that divides the chain into decoupled sets of spins. The simplest case is that considered by CT in which they consider two-site spin cells. In this case the calculation can be done analytically for any $P_0(J_i)$. In particular when taking

$$P_0(J_i) = (1-\alpha)/J_i^{\alpha} \ , \tag{2.1}$$

with $0 < \alpha < 1$ and the cutoff $J_0 = 1$, CT obtain, at low temperatures, the forms for χ, C, and M given Eqs. (1.1)–(1.3) with $\alpha_\chi = \alpha_C = \alpha_M = \alpha$.

Our numerical method consists in generating a large number of sets of spins with random couplings chosen to obey $P_0(J_i)$, and then the direct evaluation of $\langle F \rangle$ by computing the exact cell eigenvalues. As a check to the method we calculated the susceptibility and the specific heat for a chain of 1×10^4 two-spin sets, with $\alpha = 0.3$ and 0.6, respectively. The number of cells is determined by the requirement of getting good statistics in the results. Of course we recover the results obtained by CT.

We can understand the two-spin set calculation by the following simple argument, that will prove to be useful when we consider a set with four spins. In the two-spin set we have two energy levels, the singlet (ground state) and the triplet (three degenerate excited states). When we put the ground-state energy equal to zero, the energy of the triplet is J_i, with J_i the coupling between the spins at i and $i+1$. The free energy per pair is given as

$$F(J_i) = -T \ln(1 + 3e^{-J_i/T}) \ . \tag{2.2}$$

Here we have taken $k_B = 1$. The total averaged free energy $\langle F \rangle$ is obtained straightforwardly from Eq. (2.2) because of the independence of $P(J_i)$ for different J_i's. Thus we see that the averaged free energy per pair is essentially determined by the probability of having $J_i < T$, since

$$F(J_i) \sim \begin{cases} 0 & \text{if } J_i >> T \\ -T \ln 4 & \text{if } J_i << T. \end{cases} \tag{2.3}$$

Using the probability law given in Eq. (2.1) we get

$$P(J < T) = (1-\alpha) \int_0^T \frac{dJ}{J^{\alpha}} \sim T^{1-\alpha} \ , \tag{2.4}$$

leading to an averaged free energy

$$\langle F \rangle \sim -T^{2-\alpha} \tag{2.5}$$

that gives a specific heat $C = -T\partial^2 \bar{F}/\partial T^2 \sim T^{1-\alpha}$. A similar argument with a magnetic field present leads to $\chi \sim T^{-\alpha}$. Note that the form of $P_0(J)$ is crucial to obtain these results. If we take for example $P_0(J) = 1$ or $P(J) = 2J$ the corresponding susceptibilities go like $\chi \sim$ const, $\chi \sim T$ when T goes to zero, respectively. Thus, it is essential to take $P_0(J) \sim 1/J^{\alpha}$, with $0 < \alpha < 1$ to get a singular χ within this approximation. As we show later in this paper, this happens to be true also for the classical n-vector models but not for the fully coupled quantum Heisenberg model.

The same type of conclusions were reached by TC using a cluster argument. For a given temperature T, they define a cluster by a set of spins coupled by "strong couplings" $J_i > T$, separated from the rest by "weak couplings" $J_i < T$. Next, they consider only the contribution to the low-temperature thermodynamic properties due to the degenerate ground state of the clusters with an odd number of spins. These odd clusters will contribute to χ with a Curie-like susceptibility $\chi_C \sim 1/T$. The chain susceptibility is then equal to

$$\chi \sim N_{\text{odd}}(T)\chi \ , \tag{2.6}$$

where $N_{\text{odd}}(T)$ is the number of odd cluster, a function of temperature. From Eqs. (2.1) and (2.4) it follows that $N_{\text{odd}} \sim NT^{1-\alpha}$, so that $\chi \sim T^{-\alpha}$. A similar argument gives $C \sim T^{1-\alpha}$ for the specific heat. Of course, the treatment is approximate in that the internal excitations in the clusters are completely neglected. The point to be made here is that the CT and TC arguments are equivalent in that they determine the low-temperature properties of the system in terms of the probability of having *only one* coupling being smaller than the temperature. The question then arises about how relevant is it to consider more couplings in the problem. After all the RHAF is a many body problem and we would expect that correlations between spins should be of relevance.

To shed some light into this question next we take cells of four spins. The Hamiltonian for the cell is (see Fig. 1)

$$H_4 = J_1 \vec{S}_1 \cdot \vec{S}_2 + J_2 \vec{S}_2 \cdot \vec{S}_3 + J_3 \vec{S}_3 \cdot \vec{S}_4 \ . \tag{2.7}$$

In this case we take 1×10^4 four-spin cell with $\alpha = 0.3$ and 0.6 and calculate the susceptibility and the specific heat. The results are shown in Figs. 2 and 3 over a range of three orders of magnitude in temperature. The circles are the result of the numerical calculation, with the vertical bars being the standard deviations. The dashed lines are the results that would be obtained by taking two-spin cells ($\alpha_\chi = \alpha_C = \alpha$). It is immediately apparent that our numerical results deviate considerably from these, in particular the trend is

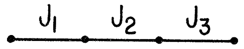

FIG. 1. Four-spin cell.

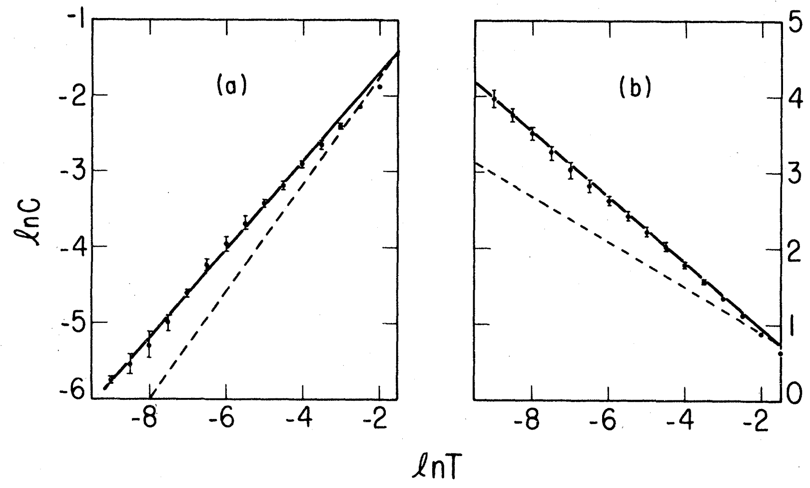

FIG. 2. (a) Specific heat and (b) susceptibility vs temperature for a four-spin cell with $P(J)$ given by Eq. (2.1) with $\alpha=0.3$. The circles are the numerical calculation and the full lines the theoretical estimates (see text). The dashed lines are the corresponding two-spin-cell results.

towards more singular behavior. The full lines were obtained from the theoretical calculation described below, and they correspond to

$$\alpha_\chi=\alpha_C=\begin{cases}0.43, & \alpha=0.3\\ 0.71, & \alpha=0.6.\end{cases} \tag{2.8}$$

We can understand the four-spin-cell results by using an argument analogous to the one given above for the two-spin cell. As in that case here the thermodynamic properties will be determined by the probability of having the energy gap between the ground state and the first excited state being smaller than T. In this case, however, there are several possible ways in which this can happen.

We will obtain a small energy gap if at least one of the couplings is small. If only J_2 is small however (see Fig. 1), spins 1–4 will be frozen in singlet states and the energy gap will still be large (of order of the minimum between J_1 and J_3). We therefore need one of the couplings at the boundaries to be small, and choose this to be J_3. The gap between the

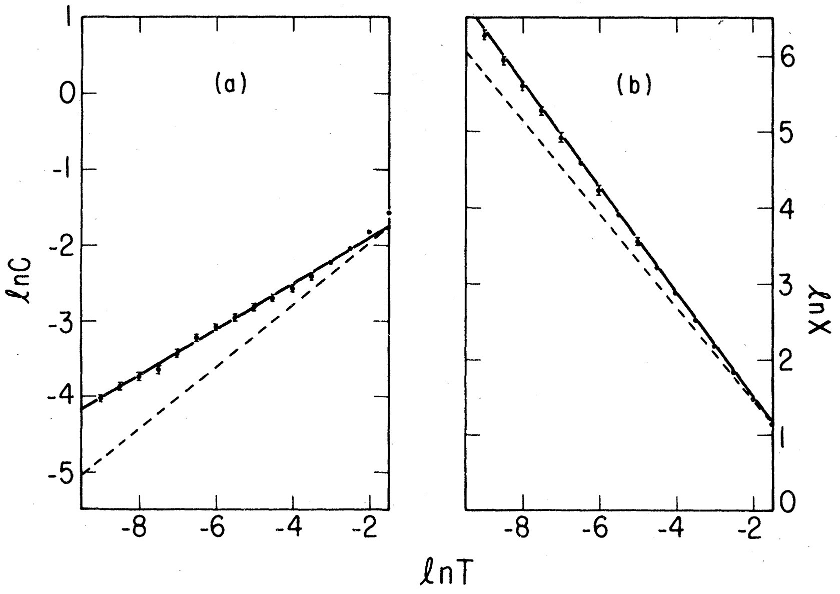

FIG. 3. Same as Fig. 2 with $\alpha=0.6$.

ground state and the first excited states can then be expressed as (to first order in J_3)

$$\Delta E = A(J_1,J_2)J_3 \ . \tag{2.9}$$

$A(J_1,J_2)$ can be evaluated explicitly by solving the three-spin-cell problem. The calculation of ΔE as given in Eq. (2.9) will also be needed in Sec. IV, thus we shall describe it here for future reference.

In the three-spin-cell problem we have eight-independent states that are formed with the vectors $\lvert +++\rangle$, $\lvert ++-\rangle$, $\lvert +-+\rangle$, $\lvert -++\rangle$, $\lvert +--\rangle$, $\lvert -+-\rangle$, $\lvert --+\rangle$, and $\lvert ---\rangle$. The eigenvalues and eigenstates of the three-spin-cell Hamiltonian are found in the usual way and are listed in Table I. Note that the ground state is degenerate. This degeneracy is lifted in first order of perturbation theory. In the four-spin-cell case we have 16 states. Without having to enumerate them we note that the ground state is not degenerate with $S_z=0$. The first excited states are a triplet with $S_z=\pm 1, 0$. Thus in order to calculate ΔE it is sufficient to consider the $S_z=0$ states. The perturbation is given by

$$V = J_3 \vec{S}_3 \cdot \vec{S}_4 \ , \tag{2.10}$$

which to lowest order in perturbation theory leads to

$$A(J_1,J_2) = \frac{8}{3}\frac{b}{3+b^2}(3+b) \ , \tag{2.11}$$

with b defined as

$$b = \frac{J_1+J_2+2(J_1^2+J_2^2-J_1J_2)^{1/2}}{J_1-J_2} \ . \tag{2.12}$$

In order to understand the implications of the previous result we consider the three limiting cases (see Fig. 4)

$$\Delta E = \begin{cases} \frac{1}{2}J_1J_3/J_2, \ \ \text{(i)}\ J_1 \ll J_2 & (2.13) \\ \frac{2}{3}J_3, \ \ \text{(ii)}\ J_1 = J_2 & (2.14) \\ J_3, \ \ \text{(iii)}\ J_1 \gg J_2 . & (2.15) \end{cases}$$

TABLE I. Eigenstates and eigenvalues of the three-spin-cell problem. $\lvert +\rangle_c$ and $\lvert -\rangle_c$ are the ground states for the cell. ΔE_i denotes the difference between the higher-lying cell-state energies and the ground-state energy.

State	Energy
$\lvert +\rangle_c = \frac{1}{(1+\frac{1}{3}b^2)^{1/2}}\left(\frac{\lvert ++-\rangle - \lvert -++\rangle}{\sqrt{2}} - \frac{b}{\sqrt{3}}\frac{\lvert ++-\rangle - 2\lvert +-+\rangle + \lvert -++\rangle}{\sqrt{6}}\right)$	$E_c = \frac{1}{4}[-(J_1+J_2) - 2(J_1^2+J_2^2-J_1J_2)^{1/2}]$
$\lvert -\rangle_c = \frac{1}{(1+\frac{1}{3}b^2)^{1/2}}\left(\frac{\lvert --+\rangle - \lvert +--\rangle}{\sqrt{2}} - \frac{b}{\sqrt{3}}\frac{\lvert --+\rangle - 2\lvert -+-\rangle + \lvert +--\rangle}{\sqrt{6}}\right)$	
$\lvert +,1\rangle = \frac{1}{(1+\frac{1}{3}c^2)^{1/2}}\left(\frac{\lvert ++-\rangle - \lvert -++\rangle}{\sqrt{2}} - \frac{c}{\sqrt{3}}\frac{\lvert ++-\rangle - 2\lvert +-+\rangle + \lvert -++\rangle}{\sqrt{6}}\right)$	$E_1 = -\frac{1}{4}[(J_1+J_2) + 2(J_1^2+J_2^2-J_1J_2)^{1/2}]$
$\lvert -,1\rangle = \frac{1}{(1+\frac{1}{3}c^2)^{1/2}}\left(\frac{\lvert --+\rangle - \lvert +--\rangle}{\sqrt{2}} - \frac{c}{\sqrt{3}}\frac{\lvert --+\rangle - 2\lvert -+-\rangle + \lvert +--\rangle}{\sqrt{6}}\right)$	$\Delta E_1 = \frac{1}{2}(J_1^2+J_2^2-J_1J_2)^{1/2}$
$\lvert +,2\rangle = \frac{\lvert ++-\rangle + \lvert +-+\rangle + \lvert -++\rangle}{\sqrt{3}}$	$E_2 = \frac{1}{4}(J_1+J_2)$
$\lvert -,2\rangle = \frac{\lvert --+\rangle + \lvert -+-\rangle + \lvert +--\rangle}{\sqrt{3}}$	$\Delta E_2 = \frac{1}{4}[J_1+J_2+(J_1^2+J_2^2-J_1J_2)^{1/2}]$
$\lvert +,3\rangle = \lvert +++\rangle$	$E_3 = \frac{1}{4}(J_1+J_2)$
$\lvert -,3\rangle = \lvert ---\rangle$	$\Delta E_3 = \Delta E_2 = \frac{1}{2}[J_1+J_2+(J_1^2+J_2^2-J_1J_2)^{1/2}]$

$$b = \frac{J_1+J_2+2(J_1^2+J_2^2-J_1J_2)^{1/2}}{J_1-J_2}; \quad c = \frac{J_1+J_2-2(J_1^2+J_2^2-J_1J_2)^{1/2}}{J_1-J_2}$$

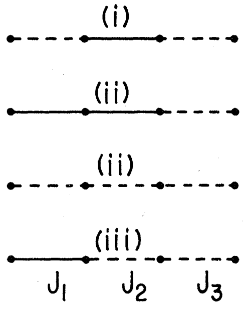

FIG. 4. Different cases that give rise to a small energy gap in the four-spin cell. A dashed line represents a weak coupling, a full line a strong coupling.

Within the region defined by (ii) and (iii), ($J_2 \lesssim J_1$), the probability of having a small energy gap is $P(\Delta E < T) \sim T^{1-\alpha}$, as in the two-spin cells, since the gap is proportional to only one coupling (J_3).

The case (i) however leads to a more interesting conclusion. The probability of $\Delta E < T$ in this case is

$$P\left(\frac{J_3 J_1}{2J_2} < T\right) \sim \int_0^a \frac{dJ_1}{J_1^\alpha} \int_0^a \frac{dJ_3}{J_3^\alpha} \Theta\left(T - \frac{\frac{1}{2} J_1 J_3}{J_2}\right)$$

$$\sim T^{1-\alpha}\left[1 - (1-\alpha)\ln\left(2T\frac{\langle J_2 \rangle}{a}\right)\right] , \quad (2.16)$$

Here a is some cutoff above which Eq. (2.13) is not valid. We have substituted J_2 by its average because it does not change ΔE significantly. The important feature of this result is that although $P(\Delta E < T)$ is still proportional to $T^{1-\alpha}$ there is a logarithmic correction. As we shall see in the following section this fact will turn out to be of central importance for the fully coupled chain problem.

Of course we can always write $P(\Delta E < T) \sim T^{1-\alpha'}$. From Eq. (2.16) we see that

$$\alpha' = \alpha + \frac{1-\alpha}{1-(1-\alpha)\ln 2(\langle J_2 \rangle / a)T} . \quad (2.17)$$

Note that replacing J_2 by $\langle J_2 \rangle$ Eq. (2.16) gives approximately the energy gap of the case (i), such that we can take $a = 1$ in Eqs. (2.16) and (2.17). By a reasoning similar to the one used in the two-spin-cell case, we can conclude that $\alpha_\chi = \alpha_C = \alpha'$. (In fact, there are corrections to α_C proportional to $\partial\alpha'/\partial T$ which we neglect here.) The value of α' given by Eq. (2.17) is always bigger than α and weakly temperature dependent. There is however no significant change in α' over the three orders of magnitude in T studied numerically and our error bars are too large to clearly detect it. The full lines in Figs. 2 and 3 correspond to the prediction of Eq. (2.17) for an intermediate temperature ($\ln T = -5$). Clearly, the fit to the numerical results is quite good. In Fig. 3(b), where the statistics is best, one can even appreciate the small change in α' with temperature predicted by Eq. (2.17).

The main conclusion from these results is that the many-body effects in this problem are very important, and that the tendency when considering larger sets of spins is towards more singular behavior. We could imagine going to larger and larger cells numerically but it would become very time consuming and the theoretical analysis, in the spirit followed in this section, becomes increasingly more difficult. Even when considering larger cells we would always wonder about extrapolating the finite cell results to the infinite-chain properties. Rather than doing that in Sec. III we undertake the study of the RHAF using a renormalization group (RG) prescription that deals with the whole chain.

III. RANDOM HEISENBERG ANTIFERROMAGNETIC CHAIN. RENORMALIZATION GROUP APPROACH

We have learned from Sec. II that significant changes occur when we incorporate more than two spins in the analysis. At the same time we found that a good approximation to the numerical results comes from considering the probability of having small energy gaps. We shall see that this fact will continue being useful when looking at the chain as a whole in the low-temperature approximation.

As mentioned in the Introduction our method is based on a RG method introduced by Drell *et al.* in the same spirit as that used by Wilson in his analysis of the Kondo problem. This method has been mostly used by different authors to calculate ground-state properties, and an extension to finite temperatures has been reported by one of us (J.H.) elsewhere.[11] We use this finite temperature method extended appropriately to deal with random problems.

In Sec. III A we derive the recursion relations. In Sec. III B we study the recursion properties for the probability law $P(J)$. There we show that, due to the form of the recursion relations found in *A*, the form of the renormalized $P(J)$ is to a large extent independent from the intial form of $P(J)$. These universal results are interpreted in Sec. V in terms of the scaling theory of localization.[17,18] From the results of Sec. III B it follows that the thermodynamic functions, which are directly related to the set of P_n's are universal too. Similar universal behavior for P_n was found by Ma *et al.* too.[13] At the end of this section we compare our approach to that of Ma *et al.*

A. RG recursion relations

The main idea of the RG method for quantum systems on a lattice can be traced back to the original Kadanoff block-spin transformation, appropriately extended to quantum models. This starts by dividing the chain into cells of spins. In our case the number of spins per cell is chosen to be on-odd number (3) such that we map spin-$\frac{1}{2}$ Hamiltonians into spin-$\frac{1}{2}$ Hamiltonians. The intracell Hamiltonian is diagonalized exactly and a subset of the eigenstates, usually the low-lying states, are used to construct the renormalized Hamiltonian cell operator. For the three-spin-cell Hamiltonian the eigenvalues and eigenstates are listed in Table I. As usual we keep the two lowest-lying states ($|+\rangle_c$ and $|-\rangle_c$) to write

$$H_p=\sum_p J_p \vec{S}_p\cdot\vec{S}_{p+1}+\sum_p E_c^p \quad . \tag{3.1}$$

Here $\vec{S}_p$ are the cell spin operators for cell p. The lattice spacing for the cell lattice is 3 and the constant term in Eq. (3.1) represents the contribution to the ground-state energy associated with the $\vec{S}_p$ states.

J_p represents the renormalized coupling constant of the cell Hamiltonian. For the case at hand (see Fig. 5 for definition of the labels)

$$J_3'=J_3A\,(J_1,J_2)A\,(J_5,J_4) \tag{3.2}$$

to lowest order in J_3. The function A is the same as found in Sec. II [Eqs. (2.11) and (2.12)]. From the form A we see that the explicit form of the recursion relation Eq. (3.2) is a rather complicated function of (J_1,J_2,J_4,J_5). However, from what we learned in the four-spin-cell analysis, we can look at the limiting forms of A that will give a small coupling J_3'. From Eqs. (2.13)–(2.15) we see that $A\,(J_1,J_2)$ goes to zero as $J_1/2J_2$ if $J_1<<J_2$ while $A\,(J_1,J_2)$ remains of $O(1)$ if $J_2\lesssim J_1$. This leads us to approximate the recursion relation (3.2) by

$$\frac{J'}{J_0'}=\frac{J_3J_1J_5}{J_0^3} \quad . \tag{3.3}$$

J_0' is the new cutoff of the probability distribution, which is related to "a typical value" of J_2 and J_4. We will defer the question of the precise choice of J_0' to the end of Sec. III B where it can be addressed more clearly.

The approximate recursion relation (3.3) is a good approximation for all values of the J's except when $J_1\sim J_2<<J_0$ or $J_5\sim J_4<<J_0$. As we shall see when we calculate the thermodynamic functions numerically, using Eq. (3.2) leads us to results that differ from those obtained from Eq. (3.3) by about 10%. The importance of using the recursion relation given in Eq. (3.3) is that we are able to follow the analysis analytically and this will provide further insight in the physics of the problem.

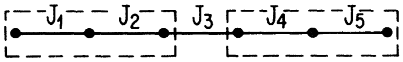

FIG. 5. Three-spin cells involved in the renormalization-group transformation.

B. Renormalized probability laws

It is clear that when changing the problem from one lattice with one lattice spacing to a new problem with a new lattice spacing we should find the probability law for the new coupling constants. In order to find out how $P_0(J)$ does change when going to the rescaled problem we ought to solve the integral equation

$$P_1(J_3')=\int\prod_{i=1}^{5}dJ_i\,P_0(J_i)\,\delta(J_3'-J_3A\,(J_i,J_2)A\,(J_5,J_4)) \quad . \tag{3.4}$$

Solving this integral equation analytically using the explicit form of the function A looks rather complicated. Instead we use the simpler recursion relation given in Eq. (3.3), to have

$$P_1\left(\frac{J_3'}{J_0'}\right)=\int\prod_{i=1,3,5}dJ_i\,P_0\left(\frac{J_i}{J_0}\right)\delta\left(\frac{J_3'}{J_0'}-\frac{J_3J_1J_5}{J_0^3}\right) \quad . \tag{3.5}$$

As we mentioned before, this approximation will lead to qualitatively the same results than using Eq. (3.4) explicitly. At this point we have to specify $P_0(J_i)$ to solve Eq. (3.5). For the particular case of a power-law distribution

$$P_0(J)=(1-\alpha)/J^\alpha \tag{3.6}$$

with $0\leqslant J\leqslant 1$, the integral (3.5) can be done analytically and we obtain

$$P_1(J')=\tfrac{1}{2}(1-\alpha)\ln^2 J'/J'^\alpha \tag{3.7}$$

and after iterating this procedure n times we get

$$P_n(J^{(n)})=\frac{(1-\alpha)^{3^n}}{(3^n-1)!}\,\frac{\ln^{3^n-1}J^{(n)}}{(J^{(n)})\alpha} \quad . \tag{3.8}$$

In order to see more graphically what happens to $P_n(J)$ as a function of n we have plotted in Fig. 6(a) the probability function for different values of n when $\alpha=0.6$. Note that after $n=2$ $P(J)$ becomes strongly peaked at $J=0$ and has very long tails. One could think that this behavior only happens because we start with a singular $P_0(J)$ and the RG only amplifies this after a few iterations. In Fig. 6(b) we have taken $\alpha=0$ (flat probability function) and see that essentially the same thing happens, except that

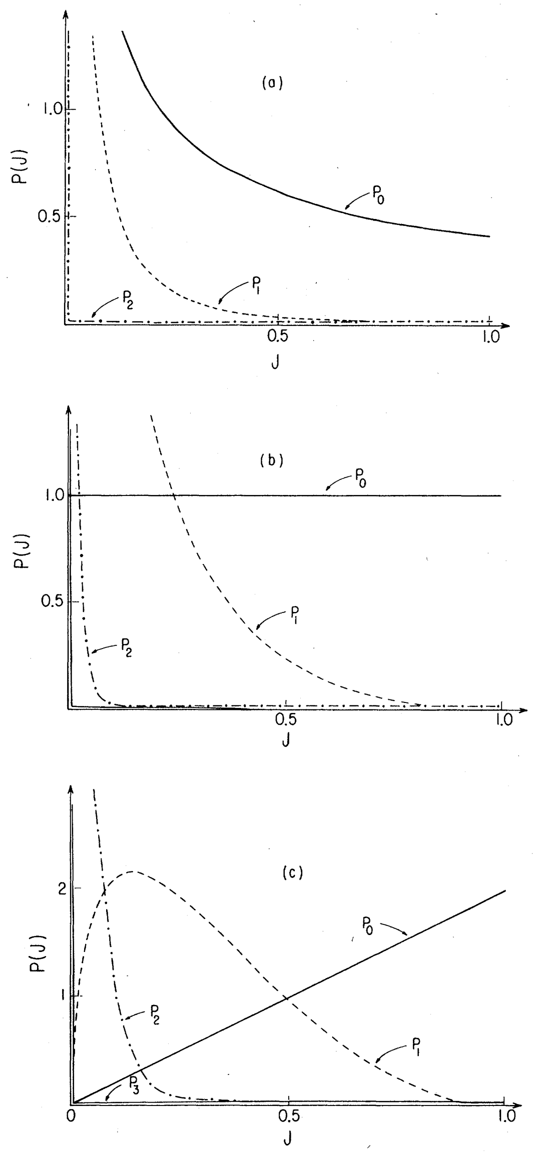

FIG. 6. Evolution of the probability distribution $P_n(J)$ under iterations. The initial form for $P_0(J)$ is the power law given by Eq. (2.1), with (a) $\alpha=0.6$; (b) $\alpha=0$; (c) $\alpha=-1$.

slower than in the previous case (strongly peaked at $J=0$ for $n\geq 3$).

Note that we derived Eq. (3.8) for arbitrary α and in particular we can take $\alpha=-1$ that corresponds to a triangular distribution

$$P_0(J)=2J \ . \qquad (3.9)$$

This probability distribution is completely different from one with $\alpha>0$ in Eq. (3.6). Instead of having most of the couplings being zero, here we have that most of the J's are strong. We have plotted in Fig. 6(c) the renormalized $P_n(J^{(n)})$ associated with Eq. (3.9). Again we see that after a few iterations $P_n(J^{(n)})$ becomes singular around $J=0$. This happens in a rather different way than in the previous cases. $P_n(J^{(n)})$ has in fact a maximum. For $n=1$, $P_{\max}(J)=2.165$ for $J=0.135$. When $n=2$, $P_{\max}(J)=66$ at $J=9\times10^{-4}$, and for $n=3$, $P_{\max}(J)=1\times10^{27}$ at $J=1.4\times10^{-11}$. Again a singular $P(J)$ centered around $J=0$. As we shall see in Sec. III C, the fact that P_n becomes singular around $J=0$ determines that the thermodynamic properties become singular at low temperatures. This is so because the averaged free energy will be written as a series in terms of the P_n's.

We turn now to the renormalized cutoff for $J^{(n)}$. In the microscopic problem we had $0\leq J\leq J_0$, and for simplicity defined our units of energy such that $J_0=1$. The question is to find the value of $J_0^{(n)}$ when $0\leq J^{(n)}\leq J_0^{(n)}$. This question is important in the analysis of Ma *et al.* where J_0 gets renormalized with their rescaling method. In our case we have implicitly taken $J_0^{(n)}=1$ for all n in the derivation of $P_n(J^{(n)})$. In order to see how this comes about, note that the maximum value that the function A, defined in Eq. (2.11), can take is one, thence

$$J_3'\leq J_3 \ ,$$

which implies that $J_0'\leq J_0$. In general $J_3^{(n)}\leq J_3^{(n-1)}$ and $J_0^{(n)}\leq J_0^{(n-1)}$. Because of the singular character of $P_n(J^{(n)})$ favoring values of $J^{(n)}$ close to zero the actual value of $J_0^{(n)}$ is in fact irrelevant in our analysis.

C. Thermodynamic properties

In this subsection we consider the extension of the previous analysis to finite temperatures. It is clear that we will not be able to calculate the thermodynamic properties if we keep only the two lowest-lying cell states: we have to take into account the higher-lying cell states as well. For calculating low-temperature properties, however, it is reasonable to treat the higher cell states in a more approximate way by taking them to be decoupled between different cells. Later one could include corrections via perturbation theory. This procedure has been used to treat the thermodynamic properties of the periodic Heisenberg antiferromagnetic chains and reasonable results were obtained.[11]

The method is rather simple and can be described as follows: we start by rewriting the Hamiltonian as

$$H=H_{ll}+H_{lh}+H_{hh} \ , \qquad (3.10)$$

where l stands for the set of states in our Hilbert space formed by only the low-lying cell states, and h for the remaining states. H_{ll} represents the projection of the Hamiltonian onto the subspace spanned by the "low states," H_{hh} the projection onto the subspace of h states and H_{lh} the cross terms. The zero temperature approach consists in neglecting H_{lh} and H_{hh} entirely, and taking as renormalized Hamiltonian

$$H'=H_{ll}\ . \tag{3.11}$$

To extend this approach to finite temperatures, we will keep H_{hh} from Eq. (3.10) but still neglect H_{lh}. This approximation will be best if the low and high cell states are well separated in energy, and can always be corrected by perturbation theory in the intercell coupling (H_{lh} is first order in the intercell coupling). The partition function is now

$$Z=\operatorname{Tr}\exp[-(H_{ll}+H_{hh})/T]\ . \tag{3.12}$$

The two pieces of the Hamiltonian now commute, since they are defined on different subspaces of our Hilbert space, so that we can separate the above exponential into a product of two exponentials. Let us define the partition function for a chain of k cells where we trace only over the low cell states:

$$Z_l(k)=\operatorname{Tr}_l\exp(-H_{ll}/T) \tag{3.13}$$

and similarly, for tracing only over high cell states

$$Z_h(k)=\operatorname{Tr}_h\exp(-H_{hh}/T)\ . \tag{3.14}$$

The full partition function then reads

$$\begin{aligned}Z=Z_l(N')\Bigg[1&+\sum_{k_1}\frac{Z_l(k_1)Z_h(1)Z_l(N'-k_1-1)}{Z_l(N')}+\sum_{k_1<k_2+1}\frac{Z_l(k_1)Z_h(1)Z_l(k_2-k_1-1)Z_h(1)Z_l(N'-k_2)}{Z(N')}\\&+\sum_{k_1}\frac{Z_l(k_1)Z_h(2)Z_l(N'-k_1-2)}{Z_l(N')}+\cdots+\sum_{k_1}\frac{Z_h(k_1)Z_l(1)Z_h(N'-k_1-1)}{Z_l(N')}+\frac{Z_h(N')}{Z_l(N')}\Bigg]\ ,\end{aligned} \tag{3.15}$$

where N' is the number of cells in our system. We have taken $Z_l(N')$ as a common factor, which is what we can evaluate more accurately by the zero-temperature RG method. For the terms inside the bracket, one has to do some further approximation. As the simplest approximation, we will treat everything inside the brackets as decoupled. We have then

$$\begin{aligned}Z&=Z_l(N')\left[1+N'\frac{Z_h(1)}{Z_l(1)}+\tfrac{1}{2}N'(N'-1)\left[\frac{Z_h(1)}{Z_l(1)}\right]^2+\cdots\right]\\&=Z_l(N')\left[1+\frac{Z_h(1)}{Z_l(1)}\right]^{N'}\ .\end{aligned} \tag{3.16}$$

The free energy per site obeys then the recursion relation

$$f=\frac{f'}{s}-\frac{T}{s}\ln\left[1+\frac{Z_h(1)}{Z_l(1)}\right]\ , \tag{3.17}$$

where s is the number of sites per cell, and f' is the free energy for the "renormalized system." The partition functions for the high- and low-lying states for the three-spin-cell case are given by

$$\frac{Z_h}{Z_l}=e^{-\Delta E_1/T}+e^{-\Delta E_2/T}+\frac{\cosh(3\tilde{h}/T)}{\cosh(\tilde{h}/T)}e^{-\Delta E_2/T}\ , \tag{3.18}$$

where the energy gaps ΔE_1 and ΔE_2 are given in Table I, and h is a small magnetic field applied. As usual we iterate Eq. (3.17) n times to get the recursion relation

$$f_n=\tfrac{1}{3}f_{n+1}-\tfrac{1}{3}T\ln\left[1+\frac{Z_h(n)}{Z_l(n)}\right]\ . \tag{3.19}$$

Here the n index in Z_h and Z_l denote the partition functions in terms of the $J^{(n)}$ variables. In the $n\to\infty$ limit, Eq. (3.19) becomes

$$F=-\tfrac{1}{3}\sum_{n=0}^{\infty}\frac{T}{3^n}\ln\left[1+\frac{Z_h(n)}{Z_l(n)}\right]\ . \tag{3.20}$$

The physical free-energy results from averaging F to give

$$\langle F\rangle=-\tfrac{1}{3}T\sum_{n=0}^{\infty}\frac{1}{3^n}\left\langle\ln\left[1+\frac{Z_h(n)}{Z_l(n)}\right]\right\rangle_n\ , \tag{3.21}$$

with the configurational average $\langle\ \rangle_n$ defined by

$$\langle A\rangle_n=\left[\int dJ^{(n)}P_n(J^{(n)})\right]A(J^{(n)})\ . \tag{3.22}$$

At low temperatures Eq. (3.21) can be written as (for $h=0$)

$$\langle F\rangle=-\tfrac{1}{3}T\sum_{n=0}^{\infty}\frac{1}{3^n}[P_n(\Delta E_1(n)\text{ or }\Delta E_2(n)<T)]\ . \tag{3.23}$$

Equivalently $\langle F\rangle$ can be generally reexpressed as

$$\langle F\rangle=-T^{2-\alpha(T)}\ , \tag{3.24}$$

where $\alpha(T)$ is essentially related to the specific-heat exponent as defined in Eq. (1.2).

From Eq. (3.23) it is now clear why the thermodynamic properties of the model are universal and inherently connected to the form of the renormalized probability law. In particular we note that the first term in the series in Eq. (3.21) at low temperatures is smaller than the second and third, whereas the contributions of the higher order terms in n are small on account of the 3^{-n} factor. Thence we see that the main contributions to $\langle F \rangle$ come from intermediate values of n. The number of terms that are important to keep in the series depends on the value of the temperature.

We have evaluated the thermodynamic derivatives of the sum given in Eq. (3.21) and computed the thermodynamic quantities by direct integration. In order to get smooth behavior for C and χ as well as the exponents $\alpha_C(T)$ and $\alpha_\chi(T)$ defined as

$$\frac{d\ln C}{d\ln T} = 1 - \alpha_C(T) , \tag{3.25}$$

$$\frac{d\ln\chi}{d\ln T} = -\alpha_\chi(T) , \tag{3.26}$$

we have analytically continued the result (3.23) to continuous values of the lattice spacing $3 \to s$ and then taken the limit $s \to 1$. This procedure is occasionally used in RG calculations. The results are given in Figs. 7 and 8 for various initial probability distributions.

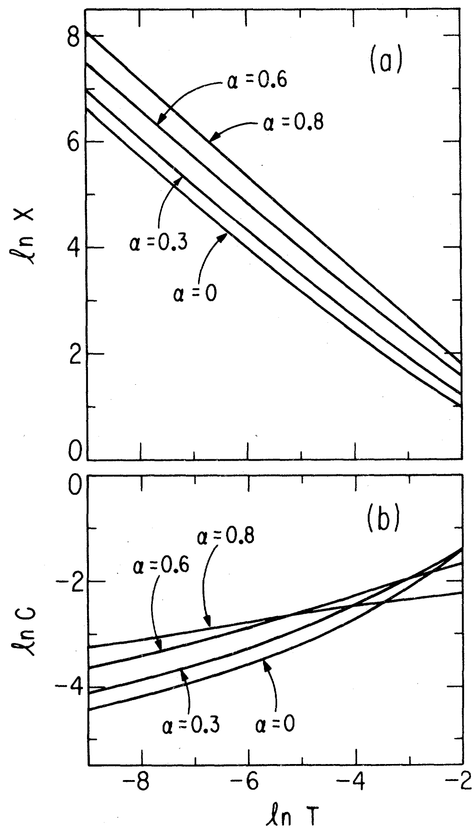

FIG. 7. (a) Susceptibility and (b) specific heat vs temperature for different values of α. The temperature ranges from 0.14 to 0.12×10^{-3} in units of J_0, the cutoff in the probability distribution.

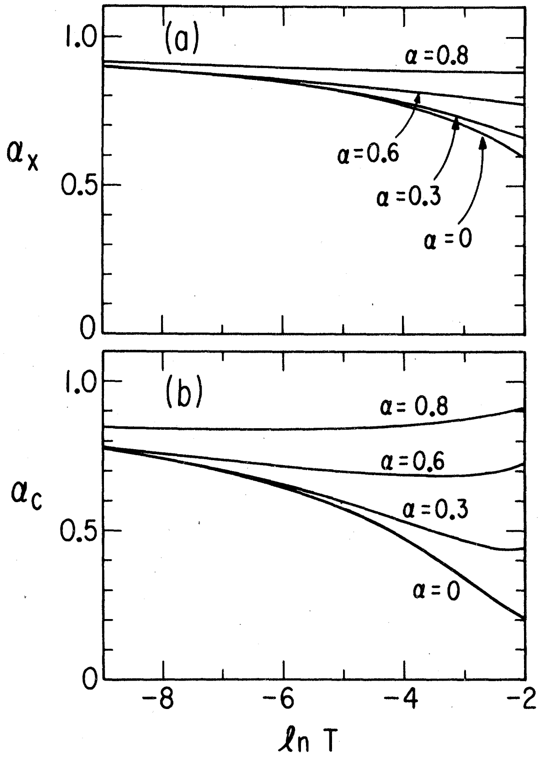

FIG. 8. Exponents for the susceptibility (a) and specific heat (b), as defined by Eqs. (1.1) and (1.2) vs temperature for different values of α.

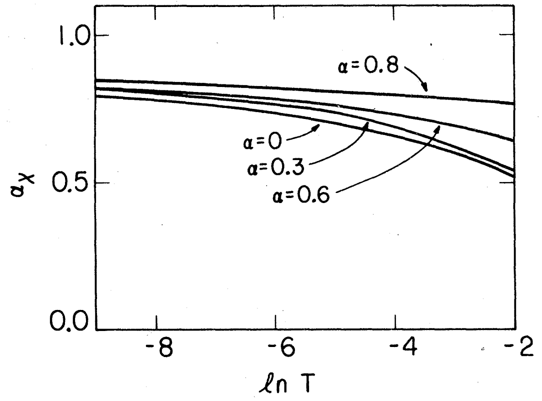

FIG. 9. Exponents for the susceptibility vs temperature from the numerical renormalization-group calculation for different values of α.

The thermodynamic properties have also been calculated numerically, using the full recursion relation given in Eq. (3.2) and the results for $\alpha_\chi(T)$ are shown in Fig. 9. As already mentioned, they differ from the analytic results by about 10%.

IV. CLASSICAL RANDOM MODELS

To complete our discussion of isotropic random magnetic chains in this section we shall consider the classical n-vector models defined by the Hamiltonian

$$H_c=\sum_i J_i \vec{S}_i\cdot\vec{S}_{i+1}+\vec{h}\sum_i \vec{S}_i \ . \tag{4.1}$$

Here the spins $\vec{S}_i$ are classical n-component unitary vectors, with J_i chosen according to the probability law $P_0(J_i)$. In the absence of a field these models can be solved exactly for arbitrary n and arbitrary $P_0(J_i)$. To see why this is the case we write the partition function as

$$Z[\{J_i\}]=\mathrm{Tr}_{\{\vec{S}_i\}}\prod_{i=1}^{N-1} T_i(\vec{S}_i,\ \vec{S}_{i+1},J_i) \ , \tag{4.2}$$

where the $\mathrm{Tr}_{\{\vec{S}_i\}}$ means the usual integration in the $n-1$ dimensional sphere in spin space, and T_i is the usual transfer matrix

$$T_i=\exp[\beta J_i \vec{S}_i\cdot\vec{S}_{i+1}+\tfrac{1}{2}h(\vec{S}_i+\vec{S}_{i+1})] \tag{4.3}$$

for the n-vector model. In the case of $h=0$ the T_i's can be diagonalized at once leading to[19]

$$Z[\{J_i\}]=(1+\delta_{1,n})^N\prod_{i=1}^{N-1}(\tfrac{1}{2}\beta J_i)^{1-n/2}\Gamma(\tfrac{1}{2}n) \times_i I_{n/2-1}(\beta J_i) \ , \tag{4.4}$$

with I_ν the modified Bessel function of the first kind of order ν. In the limit $N\rightarrow\infty$ the configurational free energy is

$$\beta F[\{J_i\}]=-\sum_{i=1}^{N-1}[(1-\tfrac{1}{2}n)\ln\beta J_i+\ln I_{n/2-1}(\beta J_i)] \ . \tag{4.5}$$

Here we have not written the temperature-independent constants. In the case where there is no correlation between the random variables

$$P(J_1,J_2,\ldots,J_n)=\prod_{i=1}^{N}P_0(J_i) \tag{4.6}$$

and taking J_0 as a cutoff for $P_0(J_i)$, the averaged free energy is simply written as

$$\beta\langle F\rangle_J=-\int_0^{J_0}dJ\,P_0(J)[(1-\tfrac{1}{2}n)\ln\beta J+\ln I_{n/2-1}(\beta J)] \ . \tag{4.7}$$

Note that this result depends intrinsically on the fact that the magnetic field is zero and the random independence of the J's. With the change of variables

$$\beta J=x \ ,$$

Eq. (4.7) reads

$$\langle F\rangle_J=-\beta^{\alpha-2}\int_0^{\beta J_0}dx\,P_0(x)[\ln I_{n/2-1}(x)+(1-\tfrac{1}{2}n)\ln x] \ . \tag{4.8}$$

Here we have assumed that $P_0(J)=(1-\alpha)/J^\alpha$. In the low-temperature limit, the region of interest in this paper, we can take $\beta J_0\rightarrow\infty$ and the specific heat corresponding to $\langle F\rangle$ goes like

$$C\sim T^{1-\alpha} \ , \tag{4.9}$$

which is the same result obtained in the two-spin-cell approximation in Sec. II. We can obtain the equivalent result for the susceptibility of the antiferromagnetic problem at low temperatures by noting that the spin-spin correlation function of the random problem can be expressed as[19]

$$\langle\vec{S}_l\cdot\vec{S}_m\rangle=\prod_{i=1}^{m-1}\frac{I_{n/2}(J_i)}{I_{n/2-1}(J_i)} \ . \tag{4.10}$$

This result arises from a similar argument to obtain $\langle F\rangle$ using the transfer matrix T_i. Taking the configurational average of Eq. (4.10) with respect to $P_0(J_i)$, and using the fluctuation dissipation theorem

$$N\chi=\beta\sum_{l,m=1}^{N}\langle\langle\vec{S}_l\cdot\vec{S}_m\rangle\rangle_{\mathrm{av}} \ , \tag{4.11}$$

where $\langle\cdots\rangle_{\mathrm{av}}$ stands for the average over J_i, the low-temperature limit of the antiferromagnetic case leads to

$$\chi\sim T^{-\alpha} \ . \tag{4.12}$$

Here again we used $P_0(J)=(1-\alpha)/J^\alpha$. It is clear that the results of Eqs. (4.9) and (4.12) depend strongly on the form of $P_0(J)$. This fact led TC to suggest that this result was general and extended to the quantum cases as well. As we have shown in Sec. III this is not the case, and the classical and quantum random problems belong to a different universality class. This is physically clear when we note that the results of Eqs. (4.9) and (4.12) correspond to the one bond approximation of Sec. II, thus implying that the quantum many body interference effects are of crucial importance for this problem.

We have pointed out in Sec. III, that the universal results found for the RHAF are intrinsically related to the scaling properties of the probability law. To further our understanding of the differences between the classical and quantum problems we consider next the scaling properties of $P_0(J_i)$ in the classical case.

For simplicity we will restrict our analysis to the classical planar and Heisenberg models in the limit when $\beta \rightarrow \infty$. In this limit the behavior of both problems is qualitatively the same and thus what applies to one applies to the other. We use an RG prescription known as decimation. It consists of taking the trace over spin variables of every other spin in the chain. Explicitly,

$$\mathrm{Tr}_{\{\vec{S}_2\}} \exp(\beta J_1 \vec{S}_1 \cdot \vec{S}_2 + \beta J_2 \vec{S}_2 \cdot \vec{S}_3) \cong \exp(\beta J_3 \vec{S}_1 \cdot \vec{S}_3) \quad . \tag{4.13}$$

At sufficiently low temperatures the spins have small fluctuations in their orientation and then the trace in Eq. (4.13) becomes an integral over Gaussian integrals leading to

$$\frac{1}{J_3} = \frac{1}{J_1} + \frac{1}{J_2} \quad . \tag{4.14}$$

This result has been obtained previously for the homogeneous problem[20] and has also been used in the spin-glass problem.[21] The RG equation for the probability function $P(J)$ in this case is given by

$$P_1(J_3) = \int dJ_1 dJ_2 P_0(J_1) P_0(J_2)\, \delta\left[J_3 - \frac{J_1 J_2}{J_1 + J_2}\right] \quad . \tag{4.15}$$

This equation is solved in the Appendix, where it is shown that, when taking $P_0(J) = (1-\alpha)/J^{\alpha}$,

$$P_1(J_3, J_0') = P_0(J_3, \tfrac{1}{2} J_0) + g(J_3) \quad . \tag{4.16}$$

Here the function $g(J_3)$ has at most a weaker singularity than $P_0(J)$. Therefore in the limit when $J_3 << 1$ the behavior of $P_1(J_3)$ is essentially dominated by the first term in Eq. (4.16). The singularities in $g(J_3)$ appear when $\alpha > 0.5$ but they are always weaker than those in $P_0(J_3)$. The cutoff in $P_0(J)$ is reduced by a factor of 2, but the behavior near $J = 0$ remains unchanged.

The essential point of the result given in Eq. (4.16) is that in contrast to the quantum case, the microscopic probability law is basically scale invariant, and therefore the form of $P_0(J)$ is crucial in determining the form of the thermodynamic functions.

From the discussions of Secs. III and IV it is clear that the isotropic random quantum problems and their classical counterparts are in a different universality class at very low temperatures, where quantum fluctuations are important. *Note, however, that this applies to the antiferromagnetic problem where zero-point fluctuations are important. This explains the differences between the results found in this paper and the results found in the quantum Heisenberg ferromagnetic chain.*[22] There we do not have zero-point fluctuations, and the model is in some sense classical.

V. CONNECTIONS WITH THE SCALING THEORY OF LOCALIZATION

As we have mentioned before the results presented in Sec. III are qualitatively equivalent to the ones obtained by Ma *et al.*[13] and briefly reported by us elsewhere.[12] In both papers, however, a physical explanation of the results was not given. It is the purpose of this section to suggest such an explanation in terms of the scaling theory of localization[18] as recently formulated by Anderson *et al.*[17] We will assume no knowledge of the results derived in Sec. III and will reobtain them under rather general and straightforward conditions.

One of the interesting points of the Anderson *et al.* paper is their remarking on the relevance of the properties of the probability distribution for the random variables in a disordered problem. After all, we are talking of a statistical problem and questions like well-behaved moments and the central limit theorem should be addressed to make sure that indeed the theory is statistically well defined. It turns out that the theory is statistically sound when it is expressed in terms of the scaling variable Y.[17] In terms of this variable, that we shall define below, the probability function $T(Y)$ satisfies the central limit theorem, thus leading to a good statistical theory for Y.

As usual, the scaling procedure consists in adding "blocks" of material A_i with the total system being $A = \bigcup_{i=1}^{N} A_i$. In each block A_i we define a physical quantity $x_i \in A_i$. This quantity can be the exchange constant within the block in a magnetic system or the resistance in an electronic problem. The scaling variable Y_i is a function of x_i that satisfies

$$Y_N\left[\bigcup_{i=1}^{N} A_i\right] = \sum_{i=1}^{N} Y_i(x_i) \quad . \tag{5.1}$$

This additivity condition is of course satisfied by the average

$$\left\langle Y_N\left[\bigcup_{i=1}^{N} A_i\right]\right\rangle = \sum_{i=1}^{N} \langle Y_i(x_i) \rangle \tag{5.2}$$

taken with respect to the probability law for Y_N. However this is not necessarily the case for higher order moments of Y, as in the electronic problem studies in Ref. 17. However, if the random variables $\{Y_i\}$ are randomly independent among each other all the higher order moments satisfy an equation similar to Eq. (5.2). This question of dependence or independence of the random variables is of great importance in the proof of the central limit theorem for $T(\{Y_i\})$. When the dependence is weak, as is the case studied in this paper and in Ref. 17, the scaling equation given in Eq. (5.1) is also satisfied for the moments. Note that Eq. (5.1) for two blocks reads

$$Y = Y_1 + Y_2 \quad ,$$

which solution is

$$Y(x)=N\alpha+c \ , \tag{5.3}$$

where N is the size of the block $A=A_1 \bigcup A_2$, α is a random variable if Y is, and c is a constant that we take equal to zero. It is clear from the definition given in Eq. (5.1) that the order in which each of the A_i's are added to obtain A is irrelevant. Next let us assume that the variables $\{Y_i\}$ are positive, independent, and equally distributed exponential random functions; i.e.,

$$T_1(Y_1(x_i))=\lambda \exp[-\lambda Y_1(x_i)] \ . \tag{5.4}$$

Here $\lambda > 0$ is a scale parameter and $0 < Y < \infty$. We shall see that this is indeed the appropriate form for the RHAF problem and for other problems as well. Now we ask what is the form of the renormalized probability law $P_N(Y_N)$ corresponding to $A=\sum_i A_i$? The answer follows from using the RG as applied to the probability law and the fact that Y is the scaling variable. Namely, to obtain the probability law for the block $A_{N+1}=A_N \bigcup A_1$, with $A_N=\bigcup_{i=1}^{N} A_i$, we solve

$$\begin{aligned} T_{N+1}(Y_{N+1}) &= \int dY_N\, dY_1\, T_N(Y_N)T_1(Y_1) \\ &\qquad \times \delta(Y_{N+1}-(Y_N+Y_1)) \\ &= \int dY_1 T_N(Y_N-Y_1)T_1(Y_1) \ . \end{aligned} \tag{5.5}$$

It is easy to see that with $P_1(Y_1)$ of the form given in Eq. (5.4) the solution to Eq. (5.5) is

$$T_N(Y_N)=\lambda\frac{(\lambda Y_N)^{N-1}}{\Gamma(N)}e^{-\lambda Y_N} \ . \tag{5.6}$$

Here $Y_N=\sum_{i=1}^{N} Y(x_i)$ and Γ is the usual Γ function. It is evident that we could have added two blocks of size N and N' and the resulting T would have the form (5.6) with $N+N'$ instead of N.

The probability function given in Eq. (5.6) appears often in the theory of Markov processes and is known as the Γ distribution. It plays an important role in the theory of random variables bounded at one side.

It follows from Eq. (5.6) that the mean goes like

$$\langle Y\rangle = N/\lambda \ , \tag{5.7}$$

whereas the standard deviation

$$\sigma = N^{1/2}/\lambda \ , \tag{5.8}$$

are such that the fluctuations γ,

$$\gamma=\frac{\langle Y\rangle}{\sigma}=\frac{1}{\sqrt{N}} \ ,$$

are statistically small in the required way. Then it is easy to show that in the limit when $N\rightarrow\infty$ the distribution T_N tends to a Gaussian distribution thus obeying the central limit theorem.

Note that none of this is apparently satisfied by the probability law derived in Sec. III that leads to

$$\langle(\Delta J^{(n)})^2\rangle/\langle J^{(n)}\rangle^2 >> 1 \tag{5.9}$$

as n is large for any α. Thus, as pointed out by Anderson *et al.* in many cases the probability function of a random problem is defined in terms of a variable for which the previous statements valid for Y do not hold. We can now make contact with the results given in Sec. III. From the form of the approximate recursion relation given in Eq. (3.3) it is apparent that

$$+Y=+|\ln J| \ , \tag{5.10}$$

and for the power-law form (3.6) for $P_0(J)$ we obtain for Y an exponential distribution of the form (5.4), with

$$\lambda=(1-\alpha) \ . \tag{5.11}$$

From the general arguments given before we see that the renormalized T_N [Eq. (5.6)] takes exactly the form given in Eq. (3.8), if we take $N\rightarrow S^n-1$, with S the spin cell size in the RG analysis. From this identification we find immediately from Eq. (5.7) that the β function is given by

$$\beta=\frac{d\langle Y\rangle}{d\ln N}=\langle Y\rangle \ . \tag{5.12}$$

This equation for β is of exactly the same form as the scaling equation written by Abrahams *et al.*[18] in the localized region if we identify $\langle Y\rangle\rightarrow -\ln g$ with g the conductance. This rather straightforward derivation of β is nevertheless illuminating. We notice that this result has arisen first, based in the neglecting of correlations between the different Y's. This approximation is valid in our case as well as in the Anderson *et al.* analysis because the correlations are weak. Second, we see that Eq. (5.12) follows from the behavior of the mean of Y which is appropriate for a good statistical theory. Integrating Eq. (5.12) leads to

$$J_s=e^{\langle Y\rangle_N}=e^{-N/\lambda} \tag{5.13}$$

with J_s the scale exchange constant defined in parallel to the scale resistance of Ref. 17. The result given in Eq. (5.13) is often identified with localization.[5] From these statements we conclude that the universal results derived by Ma *et al.* and us may be interpreted in terms of the scaling theory of localization.

By contrast, we found in the classical case the recursion relations

$$\frac{1}{J}=\frac{1}{J_1}+\frac{1}{J_2} \tag{5.14}$$

that imply that the scaling variable in this case is

$$Y_c=1/J \ . \tag{5.15}$$

We can repeat again the analysis we presented above for Y_c starting with $P_c(Y_c)$ given as in Eq. (5.4). In this case we obtain for the "scaling-exchange constant"

$$J_s \sim 1/N \ . \tag{5.16}$$

This result is interesting if we note that the form of J_s is reminiscent of a $J_s \sim 1/N^d$ behavior, where d is the dimensionality. When $d = 3$ this has essentially the form of the Rudaman-Kittel-Kasuya-Yosida (RKKY) interaction valid in the extended regime. Therefore, we see that the behavior in the $s = \frac{1}{2}$ quantum case gives a localized behavior for J_s whereas the $s = \infty$ case leads to an extendedlike result. Of course, we would like to calculate the crossover from localized to extended as a function of s. We leave this as an open question.

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APPENDIX: DERIVATION OF Eq. (4.16)

The recursion relation for the probability law in this case reads [Eq. (4.15)]

$$P(J_3) = \int dJ_1\, dJ_2\, P_0(J_1) P_0(J_2)\, \delta\left(J_3 - \frac{J_1 J_2}{J_1 + J_2}\right) \ . \tag{A1}$$

Note that the recursion relation given in Eq. (4.14) is similar to the addition of classical resistance in parallel. Also $J_3 < \min(J_1, J_2)$, thus leading to smaller values for the renormalized couplings in every iteration of Eq. (4.14). Again we take the cutoff for $P_0(J)$ to be $J_0 = 1$. From the J_1 integration in Eq. (A1) we get

$$P(J_3) = \int_{J_3/(1-J_3)}^{1} dJ_2 \left(1 - \frac{J_3}{J_2}\right)^{-2} P_0\left(\frac{J_3 J_2}{J_2 - J_3}\right) P_0(J_2) \ . \tag{A2}$$

Within the spirit of this paper we take again $P_0 = (1-\alpha)/J^{\alpha}$, leading to

$$P(J_3) = \frac{(1-\alpha)^2}{J_3^{\alpha}} \int_{J_3/(1-J_3)}^{1} dJ_2\, J_2^{-\alpha} \left(1 - \frac{J_3}{J_2}\right)^{\alpha-2} \ . \tag{A3}$$

With the change of variable

$$t = J_3/J_2 \ , \tag{A4}$$

Eq. (A3) reads

$$P(J_3) = (1-\alpha)^2 \int_{J_3}^{1-J_3} J_3^{1-2\alpha}\, dt\, t^{\alpha-2}(1-t)^{\alpha-2} \ . \tag{A5}$$

Note that $0 < J_3 < \frac{1}{2}$ now, so that the cutoff is reduced by $\frac{1}{2}$. Using the binomial expansion

$$(1-t)^{\alpha-2} = \sum_{k=0}^{\infty} \binom{k-\alpha+1}{k} (-1)^k t^k \ ,$$

the integration in Eq. (A5) can be evaluated term by term giving

$$P(J_3) = (1-\alpha)^2 \sum_{k=0}^{\infty} \binom{k-\alpha+1}{k} (-1)^k \frac{1}{\alpha - 1 + k} \times [-J_3^{-\alpha+k} + (1-J_3)^{\alpha-1+k} J_3^{1-2\alpha}] \ . \tag{A6}$$

Neglecting J_3^2 over the J_3 terms ($J_3 < 1$), Eq. (A6) reads

$$P(J_3) = (1-\alpha)^2 \sum_{k=0}^{\infty} \binom{k-\alpha+1}{k} (-1)^k \frac{1}{\alpha - 1 + k} \times (-J_3^{-\alpha+k} + J_3^{1-2\alpha}) \ . \tag{A7}$$

It is clear that the most singular term is the first term with $k = 0$ (since $\alpha > 2\alpha - 1$ for $\alpha < 1$) so that

$$P(J_3) = \frac{1-\alpha}{J_3^{\alpha}} + \cdots \ , \tag{A8}$$

where the ellipsis represents less singular terms ($0 < J_3 < \frac{1}{2}$).

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