Marginality, universality, and expansion techniques for critical lines in two dimensions

Adrianus M. M. Pruisken* and Leo P. Kadanoff
The James Franck Institute, University of Chicago, Chicago, Illinois 60637
(Received 22 January 1980)

Many $d=2$ phase transition problems have lines of critical points on which critical behavior varies continuously as a function of a single parameter λ. In treating these problems, it is important to know how λ depends upon the other parameters in the Hamiltonian. This paper is concerned with the developments and application of techniques for getting perturbation expansions of λ in other system parameters. Two examples are described in detail: the Gaussian model and its near relative the low-temperature planar model. The Gaussian model serves as a kind of base problem in which the effects of marginality may be fully explored. We show explicitly how the planar model may be mapped into a Gaussian model and extend the Kosterlitz-Thouless analysis to calculate the marginal parameter to fourth order in y_0.

I. INTRODUCTION

Such two-dimensional critical phenomena problems as the Ashkin-Teller model,[1] the eight-vertex model,[2] the planar or XY Model,[3–7] and models of surface roughening[8–11] all have a line of critical points in which the critical properties vary continuously as a function of a single parameter, which we will call λ. For each model, λ has a different physical meaning. It is important to know how in each case the critical indices vary with λ and with variation of other parameters in the critical point Hamiltonian $\mathcal{H}^*(\lambda)$. This knowledge would be a useful advance in extending critical phenomena methodology, even if each of these problems were separate and distinct. But, this kind of knowledge becomes even more useful when we recognize that in many of their aspects all of the problems mentioned above are examples of a simple soluble model, the Gaussian model.[12,13] The latter has a critical line defined by a simple parameter K.

A major step in treating all the models mentioned above—and many other related problems—is to "map" these problems onto the Gaussian model. Universality implies that there exists a function $K(\lambda)$ which relates the Gaussian model K value to the value of λ. Under this mapping the problems have identical critical properties, i.e., for both problems the asymptotic behavior of the correlation functions are the same and, consequently, the sets of critical indices are identical.

Evidence for this universality is given in the work of Luther and Peschel[14] and Bander and Richardson.[15] Kadanoff and Brown[16] have built upon the findings of these authors and derive presumably exact mapping functions for the eight-vertex and Ashkin-Teller models onto the Gaussian model, using the known results for these models. Another, rather different example of universality is contained in the theory of Berezinskii[3] and Kosterlitz and Thouless[4] for the planar model, which also forms a major subject of study in the present paper. According to these authors and others, the low-temperature planar or XY model exhibits continuously varying critical behavior, which can be understood in terms of the very same Gaussian model.

In this paper we construct a formalism for perturbation expansion for critical points in the presence of marginality.[17,18] As such, this paper may be considered as an extension of the work of Kadanoff and Wegner.[18] The existence of varying criticality is fundamentally related to the presence of a marginal operator in the theory. (A marginal operator is a scalar under scale transformations, i.e., has scaling index $x=d$.) Indeed, the very notion of a marginal operator can be used as an important tool in studying critical lines. For instance, in case of the eight-vertex model, we only have detailed knowledge about the correlation functions for one point on the critical line, namely, the point which corresponds to two decoupled Ising spin systems. However, by an expansion in the marginal parameter, one can obtain the critical properties of the Baxter line outside the decoupling point.[14,16] The first-order calculation of this type was done in Ref. 18.

It turns out that the evaluation of the second- and higher-order terms in an expansion in a marginal operator is rather subtle. Reference 19 for example, contains an error in its speculations about the second-order expansion, produced by a mishandling of the expansion.

Fortunately, however, there is an example for which we can examine the effect of marginal operators quite precisely, namely, the Gaussian model.[12,13] For this model the correlation functions are exactly known all along the fixed line.

In Sec. II we describe the Gaussian model and its

critical-point Hamiltonian, $\mathcal{H}^*(K)$. We construct a set of marginal operators $\mathcal{U}_M$ which all have the property that a Hamiltonian $[\mathcal{H}^*(K)+\lambda\mathcal{U}_M]$ has identical critical properties to one with Hamiltonian $\mathcal{H}^*(K(\lambda))$. However, the mapping function, $K(\lambda)$, varies as the detailed structure of $\mathcal{U}_M$ is changed. The various marginal operators have identical long-wavelength properties. This difference in mapping functions is entirely produced by irrelevant[20] terms in the $\mathcal{U}_M$.

In this way we have an example of universality which can be fully explored: The different marginal operators can be viewed as being the generators of critical lines, each line being in the same universality class.

The calculations in this chapter are all explicit and exact. For most problems of interest, however, we cannot do exact calculations. We mentioned already above that, in the usual case, we have only exact information about the correlation structure for special, simple situations, such as the decoupling limit of the eight-vertex or Ashkin-Teller models, or the Gaussian model in Berezinskii's and Kosterlitz and Thouless's theory for the planar model. Hence, in Sec. III we develop a perturbation theory which is based upon universality for critical lines. We start out by introducing a criterion for universality, which states the conditions under which two critical problems with marginality exhibit continuously varying critical behavior which lies in the same universality class. Our exact calculations for the Gaussian model then serve as a direct check upon these ideas. The next step is to use this formalism in a perturbative context. Provided the condition for universality is satisfied we are led instantly to a way of calculating the mapping function $K(\lambda)$ in a power series in λ. This calculation explicitly makes use of our exact results for the Gaussian model.

In Sec. IV, finally, we focus on the Villain[6] form of the planar model. We report a detailed analysis of perturbation arguments of a type originally employed by Kosterlitz and Thouless.[4] These arguments are set forward in a manner similar to that of Jose *et al.*[7] and they tend to show that the planar model lies in the same universality class as the Gaussian model. As a result, the critical indices describing the sum (x_+) and difference (x_-) of excitations with vorticity $+1$ and -1 are equal and are related to the lowest spin-wave critical index η by

$$2x_+ = 2x_- = \eta^{-1} \quad . \tag{1.1}$$

Moreover, we can write these indices in terms of the Gaussian model K_{eff} as

$$\eta = 1/2\pi K_{\text{eff}} \quad . \tag{1.2}$$

Finally, we use the techniques developed in this paper to expand K_{eff} to fourth order in y_0, one higher than the order previously calculated.[4,7]

II. GAUSSIAN MODEL

A. Generating functions

Several recent papers[12,13] have developed a generating function formalism for treating Gaussian correlation functions in two dimensions. Here, we bring together the results in a compact formalism which will serve as the basis for the remainder of this paper. Appendix B gives detailed examples of how this approach may be used to calculate correlation functions.

The basic generating function for Gaussian correlation functions can be written in terms of a four component vector field $W(\vec{r})$ and its transpose, $W^\dagger(\vec{r})$. The four components of the latter are u_+, u_-, v_+, and v_-. The generating function is

$$Q[W] = \exp\left[\tfrac{1}{2}\int d\vec{r}\, d\vec{r}\,' W^\dagger(\vec{r}) M^T(\vec{r}-\vec{r}\,') W(\vec{r}\,')\right] \quad . \tag{2.1}$$

This generating function is related in a rather intricate manner to the previously defined[12,13] Gaussian model correlation functions. There are two kinds of operators which were discussed in the earlier work: "charged operators" $\mathcal{O}_{n,m}(\vec{r})$ and uncharged operators $\mathcal{F}_{ij}(\vec{r})$. The charged operators have spin-wave quantum number n and vortex quantum number m. The allowed range of n and m depends somewhat upon the particular application of the Gaussian model, but in this paper we shall assume that both n and m are positive or negative integers or zero. The uncharged operator $\mathcal{F}_{ij}$ is also described by two integers i and j, but i and j must be non-negative. For us, the only really interesting $\mathcal{F}$ is $\mathcal{F}_{11}(\vec{r})$ which is a marginal operator for the Gaussian model.

To begin the present work, we state a set of rules for calculating Gaussian model correlation functions from the generating function $Q[W]$. These rules will then give us a set of conditions upon Q which turn out to be conditions upon the long distance behavior of $M^T(\vec{r}-\vec{r}\,')$. The rules are essentially restatements in a more compact notation of the data in Eqs. (4.6)–(4.11) of Ref. 13.

As in this reference, we use the quantities $u_+(\vec{r})$ and $u_-(\vec{r})$ to generate correlations of the $\mathcal{F}_{ij}$'s. In particular, $\mathcal{F}_{ij}(\vec{r})$ is represented by the following differential operator, which is to be applied to $Q[W]$

$$\mathcal{F}_{ij}(\vec{r}) \sim \left[\frac{\delta}{\delta u_+(\vec{r})}\right]^i \left[\frac{\delta}{\delta u_-(\vec{r})}\right]^j \quad . \tag{2.2}$$

The most important of these is the marginal operator which serves to generate motion along the critical line which is

$$\mathcal{F}_{11}(\vec{r}) = 2\frac{\delta}{\delta u_+(\vec{r})}\frac{\delta}{\delta u_-(\vec{r})} \quad . \tag{2.3}$$

The charged operators are also representable by differential operators to be applied to Q, namely,

$$\mathcal{O}_{n,m}(\vec{r}) = \exp\left[(n+m)\frac{\delta}{\delta v_+(\vec{r})} + (n-m)\frac{\delta}{\delta v_-(\vec{r})}\right] . \tag{2.4}$$

The specific meaning of expression (2.4) is that $\mathcal{O}_{n,m}(\vec{r})$ operating on a functional of v gives

$$\mathcal{O}_{n,m}(\vec{r}_0)P[v] = P[v'] , \tag{2.5a}$$

with

$$v'_\pm(\vec{r}) = v_\pm(\vec{r}) + \delta(\vec{r}-\vec{r}_0)(n \pm m) . \tag{2.5b}$$

To calculate any correlation function, then, we apply the appropriate differential operators to $Q[W]$ and finally set $W(\vec{r}) = 0$. The result of this operation is the desired correlation function.

Given these definitions, the Gaussian model correlation functions are fully defined if we state a value for the four by four matrix $M^T(\vec{r}-\vec{r}')$. In doing this we must make sure that we set the same answers for large $|\vec{r}-\vec{r}'|$ as they appear in Ref. 13. This identification determines the long-distance part of $M^T(\vec{r}-\vec{r}')$, but leaves the short-distance part at our disposal. Let us simply state the result of the comparison. (Some examples of correlation function calculations are given in Appendix B.)

First we separate out an angular factor proportional to $\Theta(\vec{r}-\vec{r}')$, where for large $\vec{r} = (x,y)$, $\Theta(\vec{r}) = \tan^{-1} y/x$. This angular factor describes correlations between spin waves and vortices and appears in the combination

$$W^\dagger(\vec{r})M^T(\vec{r}-\vec{r}')W(\vec{r}') = \tfrac{1}{2}[v_+(\vec{r}) + v_-(\vec{r})]\Theta(\vec{r}-\vec{r}')[v_+(\vec{r}') - v_-(\vec{r}')] + W^\dagger(\vec{r})M(\vec{r}-\vec{r}')W(\vec{r}') . \tag{2.6}$$

To write this in a convenient manner we visualize combining u_+ and u_- into a spinor u and also v_+ and v_- into a spinor v. Thence, each of the matrices can be written in block form, for example,

$$M = \begin{pmatrix} M_{uu} & M_{uv} \\ M_{vu} & M_{vv} \end{pmatrix} . \tag{2.7}$$

Within each of the four blocks, we rewrite submatrices like M_{vv} in terms of the unit matrix 1 and the standard two by two Pauli spin matrices τ_1 and τ_3. Then, for large separations, we find

$$\begin{aligned} M_{vv}(\vec{r}-\vec{r}') &= \tfrac{1}{2}(2\pi K)^{-\tau_1}\ln|\vec{r}-\vec{r}'| , \\ M_{uv}(\vec{r}-\vec{r}') &= \tfrac{1}{2}\frac{x-x'-\tau_3(y-y')}{|\vec{r}-\vec{r}'|^2}(2\pi K)^{-\tau_1/2} , \\ M_{vu}(\vec{r}-\vec{r}') &= -\tfrac{1}{2}(2\pi K)^{-\tau_1/2}\frac{x-x'-\tau_3(y-y')}{|\vec{r}-\vec{r}'|^2} , \\ M_{uu}(\vec{r}-\vec{r}') &= \tfrac{1}{2}\frac{[x-x'-\tau_3(y-y')]^2}{|\vec{r}-\vec{r}'|^2} . \end{aligned} \tag{2.8}$$

These expressions then give all the known facts about asymptotic Gaussian model correlations.

For calculational purposes, it is more convenient to employ a Fourier-transform representation of Eqs. (2.8), defining

$$M(\vec{r}-\vec{r}') = \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\cdot(\vec{r}-\vec{r}')} M(q) . \tag{2.9}$$

After one Fourier-transforms expression Eqs. (2.8), one sees that the resulting $M(q)$ can be written as a product of three matrices, $ab\bar{a}$, times a scalar factor, $-2\pi/q^2$. The matrices a and $\bar{a}$ are expressed in terms of a complex Fourier-transform variable, $\bar{q} = q_x - i\tau_3 q_y$ as

$$\begin{aligned} a &= \begin{pmatrix} i\bar{q}/2 & 0 \\ 0 & (2\pi K)^{-\tau_1/2} \end{pmatrix} , \\ \bar{a} &= \begin{pmatrix} -i\bar{q}/2 & 0 \\ 0 & (2\pi K)^{-\tau_1/2} \end{pmatrix} , \end{aligned} \tag{2.10}$$

while, b is simply

$$b = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} . \tag{2.11}$$

To give a unique meaning to $M(\vec{r}-\vec{r}')$ and to define its behavior fully for small $|\vec{r}-\vec{r}'|$, we also include in $M(q)$ a scalar cutoff factor $B(q)$ which obeys

$$B(q) = \begin{cases} 1 & \text{for small } q \\ O(1/q^2) & \text{for large } q \end{cases} . \tag{2.12}$$

Then, all the data in Eq. (2.8) are reproduced by choosing

$$M(q) = -(2\pi/q^2)Bab\bar{a} . \tag{2.13}$$

Later on we shall see that some marginal operators generate a change in the detailed form of M. In particular, b will be slightly modified. However, the most important part of b, its v-v matrix element, will remain unmodified for small q. This matrix element fully determines the most significant correlations in the model, those among the charged operators $\mathcal{O}_{n,m}$. From Appendix B we find, for example, that the vortex-vortex correlation function is

$$\langle \mathcal{O}_{0,m}(\vec{r}_1)\mathcal{O}_{0,-m}(\vec{r}_2)\rangle = \exp[-2\pi K m^2 L(\vec{r}_1 - \vec{r}_2)] . \tag{2.14}$$

Here $L(r)$ is a cutoff version of the logarithm

$$L(r) = -\int \frac{d^2q}{(2\pi)^2} \frac{B(q)}{q^2} (e^{i\vec{q}\cdot\vec{r}} - 1) \ , \qquad (2.15)$$

which is equal to $\ln(r)$ for large r and finite for small r. The appearance of the K in Eq. (2.14) shows that the vortex critical index $x_m = \pi K m^2$ is proportional to K. Hence, the K that appears in the expressions for a and $\bar{a}$ is directly the marginal parameter of the Gaussian model.

For large $|\vec{r}_1 - \vec{r}_2|$, Eq. (2.14) reduces to the known form of Gaussian model vortex-vortex correlations. In fact, detailed calculations show that all charged operator correlation functions obtained from the generating function exactly match those of the Gaussian model.

B. Marginal operators

The $\mathcal{O}_{n,m}(\vec{r})$ correlations are determined fully by the b_{vv} term in $b(q)$. The remaining portions of b are important because they describe correlations between marginal operators and among the marginal operators and the $\mathcal{O}_{m,n}(\vec{r})$. For example, the $\mathfrak{F}_{11}(\vec{r})$ defined by Eq. (2.3) has a correlation function which can be formed as

$$\langle \mathfrak{F}_{11}(\vec{r}) \mathfrak{F}_{11}(\vec{r}') \rangle = 2 \frac{\delta}{\delta u_+(\vec{r})} \frac{\delta}{\delta u_-(\vec{r})} \times 2 \frac{\delta}{\delta u_+(\vec{r}')} \frac{\delta}{\delta u_-(\vec{r}')} Q[W] \Big|_{W=0} \ ,$$

which may then be calculated in the limit of large separations as

$$\langle \mathfrak{F}_{11}(\vec{r}) \mathfrak{F}_{11}(\vec{r}') \rangle = |\vec{r} - \vec{r}'|^{-4} \ .$$

This r^{-4} falloff is exactly the behavior expected of a marginal operator. Hence, if we construct any operator $\mathcal{U}_M$, which is proportional to the spacial integral of $\mathfrak{F}_{11}(\vec{r})$ plus other quantities with smaller critical indices, we should expect this operator to generate motion along the critical line—in effect to change the value of K.

We shall consider a very specific form of the marginal operator defined by

$$\mathcal{U}_M = \int d\vec{r}\, d\vec{r}' \left[\frac{\delta}{\delta W(\vec{r})} e_1(\vec{r} - \vec{r}') \frac{\delta}{\delta W^\dagger(\vec{r}')} + W^\dagger(\vec{r}) e_2(\vec{r} - \vec{r}') W(\vec{r}') + e_3(\vec{r} - \vec{r}') \right] \qquad (2.16)$$

and then define the effect of $\mathcal{U}_M$ by writing

$$e^{\lambda \mathcal{U}_M} Q[W] = e^{C(\lambda)} Q_\lambda[W] \ . \qquad (2.17)$$

Here $C(\lambda)$ is a constant independent of $W(\vec{r})$ and Q_λ is exactly of the same form as Q, i.e., an exponential of a quadratic form in W, except that Q_λ is defined by a matrix $M_\lambda(\vec{r} - \vec{r}')$ which depends upon the value of the marginal parameter λ.

Equation (2.16) defines (in differential operator form) a rather general marginal operator $\mathcal{U}_M$. Now we wish to specialize this general operator by choosing a very specific set of values of e_1, e_2, and e_3. The goal is to produce a marginal operator $\mathcal{S}$ which leaves the form of $M(q)$ as given in Eqs. (2.10)–(2.13) completely invariant except that in $M_\lambda(q)$ the marginal parameter K in Eq. (2.10) is given some K and q dependence. To see how to do this, imagine for specificity that we choose $e_1(q)$ and $e_2(q)$ to be

$$e_1(q) = \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} \ , \ e_2(q) = \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} [-fB^2(q)] \ , \qquad (2.18)$$

where f is a parameter which we shall adjust to our convenience.

To see the effect of $\mathcal{S}$, take the logarithm of Eq. (2.17) and differentiate with respect to λ to obtain

$$\frac{d}{d\lambda} C(\lambda) + \int d\vec{r}\, d\vec{r}' \tfrac{1}{2} W^\dagger(\vec{r}) \left(\frac{d}{d\lambda} M_\lambda(\vec{r} - \vec{r}') \right) W(\vec{r}') = Q_\lambda^{-1}[W] \mathcal{U}_M Q_\lambda[W] \ ,$$

which then implies that $M_\lambda(q)$ obeys

$$\tfrac{1}{2} \frac{d}{d\lambda} M_\lambda(q) = M_\lambda(q) \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} M_\lambda(q) - \begin{pmatrix} \tau_1 & 0 \\ 0 & 0 \end{pmatrix} fB^2(q) \ . \qquad (2.19)$$

To find this special operator $\mathcal{S}$, then, one substitutes the forms (2.10)–(2.13) into Eq. (2.19) and assumes that the only λ dependence in $M_\lambda(q)$ is via the K's which appear in Eqs. (2.10). One finds that the resulting equation for M_λ will be satisfied if we make the particular choice of f, namely,

$$f = \tfrac{1}{4}\pi^2 \ . \qquad (2.20)$$

Given this choice M_λ does indeed have the form defined by Eqs. (2.10)–(2.13), if K in Eq. (2.10) is replaced by $K(\lambda;q)$ which obeys

$$K(\lambda;q) = K \exp[2\pi\lambda B(q)] \ . \qquad (2.21)$$

Thus $\mathcal{S}$ is indeed a marginal operator which has as its entire effect a modification of the marginal parameter K. The new generator $Q_\lambda[W]$, defined by Eq. (2.17), generates new Gaussian model correlation functions which are exactly of the same form as the old ones in the limit of large separations except that K is replaced by $K_{\text{eff}}(\lambda)$ given by

$$K_{\text{eff}}(\lambda) = K(\lambda; q = 0) = Ke^{2\pi\lambda} \ . \qquad (2.22)$$

With this particular choice of f, e_3 and C are zero. Thence, from Eq. (2.16), (2.18), and (2.20), one can write a differential operator for the density $\mathcal{E}(\vec{r})$, defined as

$$\mathcal{E} = \int d\vec{r}\,\mathcal{E}(\vec{r})$$

with the resulting density being

$$\mathcal{E}(\vec{r}) = 2\frac{\delta}{\delta u_+(\vec{r})}\frac{\delta}{\delta u_-(\vec{r})} + \tfrac{1}{2}\pi^2\int d\vec{r}_1\,d\vec{r}_2 B(\vec{r}-\vec{r}_1)B(\vec{r}-\vec{r}_2)u_+(\vec{r}_1)u_-(\vec{r}_2)\ . \tag{2.23}$$

In Eq. (2.23), $B(\vec{r})$ is the Fourier transform of $B(q)$. It will be a short-ranged function of $\vec{r}$ whenever $B(q)$ is chosen to be sufficiently smooth.

Although the special marginal operator $\mathcal{E}$ is particularly simple, one can consider other marginal operators which equally will generate motion along the critical line. For example, one can define another operator $\mathcal{F}$ by the statement $f=0$. This operator will have a density which is

$$\mathcal{F}(\vec{r}) = \mathcal{F}_{11}(\vec{r}) + \text{const} = 2\frac{\delta}{\delta u_+(\vec{r})}\frac{\delta}{\delta u_-(\vec{r})} + \int d\vec{r}'\,e_3(\vec{r}-\vec{r}')\ . \tag{2.24}$$

Since $f=0$, one can solve Eq. (2.19) in the form

$$M_\lambda^{-1}(q) = M^{-1}(q) - 2\lambda\begin{pmatrix}\tau_1 & 0\\ 0 & 0\end{pmatrix}\ .$$

Once again, one can recast the solution so that $M_\lambda(q)$ takes the form given by Eqs. (2.10)–(2.13). Once again, in this recasting one can structure the result so that a and $\bar{a}$ have exactly the form (2.10) except that K is replaced by the λ-dependent quantity

$$K(\lambda;q) = K\frac{1+\lambda\pi B(q)}{1-\lambda\pi B(q)}\ . \tag{2.25}$$

Now, however, b changes its form to

$$b_\lambda(q) = \begin{pmatrix} \dfrac{1}{1+\pi\lambda\tau_1 B(q)} & \dfrac{1}{\{1-[\pi\lambda B(q)]^2\}^{1/2}} \\ \dfrac{1}{\{1-[\pi\lambda B(q)]^2\}^{1/2}} & \tfrac{1}{2}\end{pmatrix}\ . \tag{2.26}$$

The $\frac{1}{2}$ in the v-v entry of b guarantees that the charged operator correlations remain form invariant for large separations, with the effective marginal parameter being

$$K_{\text{eff}}(\lambda) = K(\lambda;q=0) = K(1+\pi\lambda)/(1-\pi\lambda)\ . \tag{2.27}$$

The other parts of $b_\lambda(q)$ produce a modification in the short-distance form of marginal operator correlations and an additional multiplicative factor which modifies the size of these correlation functions.

However, the only change of real importance to us is the modification in the marginal parameter. Notice that the marginal operator $\mathcal{F}$ gives the λ dependence of Eq. (2.27) while the operator $\mathcal{E}$ gives the different dependences of Eq. (2.22). Thus, the two marginal operators generate different mapping functions of K onto λ. In generating motion along the critical line, we must be very careful in our use of marginal operators, by specifying clearly which operator is to be employed at which time.

The difference between two marginal operators is an irrelevant or even redundant operator. Nonetheless, in the higher-order effects of a given marginal operator, the irrelevant terms have cross terms with the truly marginal ones which renormalize the strength of the marginal operator and thereby generate other marginal terms. Our job is to design a formalism in which these cross effects are fully taken into account.

III. DESCRIPTION OF FORMALISM

A. General structure

We start from some physical problem with a Hamiltonian $\mathcal{H}^*$, and which we expect lies on a line of critical points. The line can be described by a group of Hamiltonians, parametrized by a coupling K, which we write as $\mathcal{H}_0^*(K)$. Our job is to determine which value of K describes $\mathcal{H}^*$, i.e., to find the coupling value $K[\mathcal{H}^*]$ such that at this coupling $\mathcal{H}^*$ and $\mathcal{H}_0^*(K)$ be in the same universality class.

To solve this problem assume that this K has already been found. Then for this special K write

$$\mathcal{H}^* = \mathcal{H}_0^*(K) + \mathcal{U}\ . \tag{3.1}$$

Here, $\mathcal{U}$ must be a totally irrelevant perturbation, in that it cannot modify any of the critical properties of $\mathcal{H}_0^*(K)$.

Wegner[20] has, in large measure, written down conditions which ensure the irrelevance of $\mathcal{U}$. To see these conditions, write the extensive operator $\mathcal{U}$ as an integral over a local density $\mathcal{U}(\vec{r})$ and also write the basic densities of scale invariant operators at the critical point $\mathcal{H}_0^*(K)$, as $\mathcal{O}_\alpha(\vec{r};K)$. The scaling properties of the latter are determined by a critical index $x_\alpha(K)$. They are termed relevant, marginal, and ir-

relevant depending upon whether $x_\alpha(K)$ is less than, equal to, or greater than the dimensionality. In our case, we assume that there is but one marginal operator, with density $\mathcal{U}_M(\vec{r})$, but that $\mathcal{U}_M(\vec{r})$ is nonunique in that we can add any density of redundant operators to it.

If $\mathcal{U}$ is numerically small, the Wegner condition for its irrelevance is easily stated, namely, that $\mathcal{U}(\vec{r})$ contain no components of relevant or marginal operators, i.e., that

$$\langle \mathcal{U}(\vec{r})\mathcal{O}_\alpha(\vec{r}';K)\rangle_{\mathcal{H}_0^*(K)}=0 \tag{3.2}$$

whenever $|\vec{r}-\vec{r}'|$ is very large compared to any short-distance cutoff or lattice constant in the theory. Equation (3.2) must hold for all relevant $\mathcal{O}_\alpha(\vec{r};K)$, the marginal operator $\mathcal{U}_M(\vec{r})$, and it is convenient to demand that it be true for the $\mathcal{O}_\alpha(\vec{r};K)$ being the unit operator as well.

We wish to extend Eq. (3.2) to the determination of the conditions for an irrelevant $\mathcal{U}$ even when $\mathcal{U}$ is not numerically small. We propose a condition which is implicit in Wegner's work. To state this we write

$$\mathcal{U}^\Omega=\int_\Omega d\vec{r}\,\mathcal{U}(\vec{r}) \tag{3.3}$$

for the part of $\mathcal{U}$ which appears in the volume Ω. We choose $\Omega^{1/d}$ to be large compared with short-distance cutoffs and $\vec{r}'$ to obey

$$|\vec{r}-\vec{r}'|\gg\Omega^{1/d},\quad \vec{r}\in\Omega\ . \tag{3.4}$$

An appropriate generalization of Eq. (3.2) to the case of larger $\mathcal{U}$ is

$$\lim_{|\vec{r}-\vec{r}'|\to\infty}|\vec{r}-\vec{r}'|^{2x_\alpha(K)}\langle\mathcal{O}_\alpha(\vec{r}';K)[e^{\mathcal{U}^\Omega}-1]\rangle_{\mathcal{H}_0^*(K)}\to 0 \tag{3.5}$$

for relevant or marginal $\mathcal{O}_\alpha(\vec{r};K)$. We claim that whenever Eq. (3.5) is satisfied $\mathcal{H}_0^*(K)+\mathcal{U}$ and $\mathcal{H}^*$ lie in the same universality class. Justifications for this claim are given in Appendix A.

B. Perturbation theory

To make use of Eq. (3.5), we must imagine starting from a problem with Hamiltonian $\mathcal{H}$ and then constructing a translation of this Hamiltonian into the language of a known problem with a line of critical behavior, i.e., into $\mathcal{H}_0^*(K)$. If, after appropriate identification of variables $\mathcal{H}=\mathcal{H}_0^*(K)$, we have solved our problem by reducing it to one with a known solution. However, in general, one cannot expect to be as lucky as this. The best one can expect is to have $\mathcal{H}$ in a form in which a piece of $\mathcal{H}$ looks like the Hamiltonian of the known problem and a piece looks different. Hence, we might start from a situation in which

$$\mathcal{H}=\mathcal{H}_0^*(K_0)+\mathcal{H}_1\lambda\ . \tag{3.6}$$

Here $\mathcal{H}_1\lambda$ is the error term indicating the difference between the soluble Hamiltonian and the one of interest. We multiply this error term by λ since we shall eventually be interested in calculating the properties of $\mathcal{H}$ via perturbation expansions in the error term and we will use λ to keep track of orders in the perturbation theory.

In general, there is no guarantee that $\mathcal{H}$ will in fact describe any critical point. We may well have to adjust such parameters as temperature and magnetic field to bring $\mathcal{H}$ to criticality. Mathematically this process is represented by the subtraction of terms from $\mathcal{H}$ proportional to the relevant operators $\mathcal{O}_\rho$ [i.e., the operator which has a density $\mathcal{O}_\rho(\vec{r})$], which can force the system away from criticality. We denote the necessary coefficients of these relevant operators by $\Lambda^\rho(\lambda)$. As the notation indicates the size of the terms to be subtracted depends upon the size of the error term. Thus, at this stage we are dealing with a system which we know to be critical and which has a Hamiltonian

$$\mathcal{H}^*=\mathcal{H}_0^*(K_0)+\lambda\mathcal{H}_1-\sum_\rho\mathcal{O}_\rho\Lambda^\rho(\lambda)\ . \tag{3.7}$$

Here the sum extends over all relevant operators.

We still have not recast the Hamiltonian in the form (3.1) in which the correction term is a completely irrelevant perturbation. For $\mathcal{H}_1$ could still include some marginal operator which could shift the value of the marginal parameter K. To take this shift into account, add and subtract a term proportional to the marginal operator for the problem $\mathcal{H}_0(K_0)$ of the form $\Lambda^M(\lambda)\mathcal{U}_M$. Here $\mathcal{U}_M$ is a K-independent marginal operator of the kind discussed in the previous section which has the effect of shifting the effective value of the marginal parameter. Thus, we write

$$\mathcal{H}_0^*(K)=\mathcal{H}_0^*(K_0)+\Lambda^M(\lambda)\mathcal{U}_M\ . \tag{3.8}$$

Thus, K is a known function of K_0 and Λ^M. To find the value of $\Lambda^M(\lambda)$ and hence, of K, we demand that

$$\mathcal{U}=\lambda\mathcal{H}_1-\sum_j\mathcal{O}_j\Lambda^j(\lambda) \tag{3.9}$$

be completely irrelevant in the sense discussed in the previous section. [In Eq. (3.9) and below, sums over j cover all relevant *and marginal* operators.] The appearance of a term proportional to $\mathcal{O}_M\equiv\mathcal{U}_M$ in Eq. (3.9) enables one to make $\mathcal{U}$ completely irrelevant by an appropriate adjustment of the corresponding coefficient $\Lambda^M(\lambda)$.

Once $\Lambda^M(\lambda)$ is known, K is known and the problem has been solved. The values of the coefficients $\Lambda^j(\lambda)$ will be set by demanding that $\mathcal{U}^\Omega$ satisfy Eq.

(3.5). To convert this demand into a practical calculation, we expand everything in a power series in λ.

In particular, expand the unknown coefficients as

$$\Lambda^j(\lambda) = \sum_n \Lambda_n^j \frac{\lambda^n}{n!} \quad . \tag{3.10}$$

Notice here that the subscript on Λ_n^j describes the order of the expansion, $n = 1, 2, \ldots$, while the superscript defines which coefficient is being expanded. For example, $j = M$ describes the coefficient of the marginal operator while $j = 0$ describes the unit operator which has a density $\mathcal{O}_0(\vec{r}) = 1$.

To attack Eq. (3.5), expand $\mathcal{V}^\Omega$ using Eq. (3.9) in the form

$$\mathcal{V}^\Omega = \lambda \mathcal{U}^\Omega - \left(\frac{\lambda^2}{2!}\Lambda_2^j + \frac{\lambda^3}{3!}\Lambda_3^j + \cdots \right) \mathcal{O}_j^\Omega \quad . \tag{3.11}$$

Here the lowest-order term in the expansion involves

$$\mathcal{U}^\Omega = \mathcal{H}_1^\Omega - \Lambda_1^j \mathcal{O}_j^\Omega \quad . \tag{3.12}$$

The index Ω indicates that the extensive operators are defined by an integral over a large but finite volume Ω as that in Eq. (3.3). Furthermore, we use the convention of summation over a repeated index j, i.e., the sum over j in Eqs. (3.11) and (3.12) concerns all relevant and marginal operators.

Next we consider the following Taylor series:

$$e^{\mathcal{V}^\Omega} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n \mathcal{W}_n^\Omega}{n!} \quad . \tag{3.13}$$

Equation (3.5) implies that each $\mathcal{W}_n^\Omega$ is irrelevant. For the first few terms in this expansion we have [using Eq. (3.11)]:

$$\mathcal{W}_1^\Omega = \mathcal{U}^\Omega \quad , \tag{3.14a}$$

$$\mathcal{W}_2^\Omega = \mathcal{U}^\Omega \mathcal{U}^\Omega - \Lambda_2^j \mathcal{O}_j^\Omega \quad , \tag{3.14b}$$

$$\mathcal{W}_3^\Omega = \mathcal{U}^\Omega \mathcal{U}^\Omega \mathcal{U}^\Omega - (3\mathcal{U}^\Omega \Lambda_2^j + \Lambda_3^j)\mathcal{O}_j^\Omega \quad , \tag{3.14c}$$

$$\mathcal{W}_4^\Omega = \mathcal{U}^\Omega \mathcal{U}^\Omega \mathcal{U}^\Omega \mathcal{U}^\Omega - (3!\mathcal{U}^\Omega \mathcal{U}^\Omega \Lambda_2^j + 4\mathcal{U}^\Omega \Lambda_3^j)\mathcal{O}_j^\Omega + 3\Lambda_2^j \Lambda_2^k \mathcal{O}_j^\Omega \mathcal{O}_k^\Omega \quad . \tag{3.14d}$$

In order to find a kind of cumulant expansion for the Λ_n^j, we define operator algebra coefficients $C_{\mathcal{U},\mathcal{V},\mathcal{W},\ldots}^j$ which describe how a product $\mathcal{U}(\vec{r}_1)$, $\mathcal{V}(\vec{r}_2)$, $\mathcal{W}(\vec{r}_3)\cdots$ may be expanded in a set of relevant and marginal operators in the theory. This expansion may be written as

$$\mathcal{U}(\vec{r}_1)\mathcal{V}(\vec{r}_2)\mathcal{W}(\vec{r}_3)\cdots = \mathcal{O}_j(\vec{R})\, C_{\mathcal{U}\mathcal{V}\mathcal{W}}^j(\vec{r}_1\vec{r}_2\cdots)\cdots \quad . \tag{3.15}$$

Here $\vec{R}$ is the average position of the operators; i.e., if there are n operators in the product

$$\vec{R} = (1/n)(\vec{r}_1 + \vec{r}_2 + \cdots + \vec{r}_n) \quad . \tag{3.16}$$

We actually employ Eq. (3.15) in making the statement that as $|\vec{r}' - \vec{R}|$ goes to infinity

$$\langle \mathcal{O}_j(\vec{r}')\mathcal{U}(\vec{r}_1)\mathcal{V}(\vec{r}_2)\mathcal{W}(\vec{r}_3)\cdots\rangle \rightarrow \delta_{jk} \frac{1}{|\vec{r}' - \vec{R}|^{2x_k}} C_{\mathcal{U}\mathcal{V}\mathcal{W}}^k(\vec{r}_1\vec{r}_2\cdots)\cdots \quad . \tag{3.17}$$

Now Eq. (3.5) can be expanded in a power series in λ to read

$$\lim_{r'\to\infty} \langle \mathcal{O}_j(\vec{r}')\mathcal{W}_n\rangle << \frac{1}{|\vec{R} - \vec{r}'|^{2x_j}} \quad .$$

This implies for $n = 1$ that

$$\Lambda_1^j = C_{\mathcal{H}_1}^j \quad . \tag{3.18}$$

This says that the lowest-order terms on the right-hand side of Eq. (3.9) serve to cancel out the relevant or marginal terms in $\lambda\mathcal{H}_1$. To second order, Eqs. (3.5) and (3.14b) give, via the operator product expansion

$$\Lambda_2^j \int_\Omega d\vec{r}\, \mathcal{O}_j(\vec{r}) = \int\int_\Omega d\vec{r}_1\, d\vec{r}_2 C_{\mathcal{U}\mathcal{U}}^j(\vec{r}_1 - \vec{r}_2)\mathcal{O}_j\left(\frac{\vec{r}_1 + \vec{r}_2}{2}\right) \quad . \tag{3.19}$$

Since $\mathcal{U}(\vec{r})$ has no relevant or marginal terms in it, expression (3.19) is bound to converge in Ω and we may write

$$\Lambda_2^j = \Omega^{-1} \int\int_\Omega d\vec{r}_1\, d\vec{r}_2 C_{\mathcal{U}\mathcal{U}}^j(\vec{r}_1 - \vec{r}_2) = \int d\vec{r}\, C_{\mathcal{U}\mathcal{U}}^j(\vec{r}), \quad \Omega \to \infty \quad , \tag{3.20}$$

where the integration extends over whole space. Thus $\Lambda(\lambda)$ is determined through second order. Next, in order to obtain an expression for Λ_3^j, insert Eq. (3.19) in Eq. (3.14c). Equation (3.5) now implies

$$\Lambda_3^j \int_\Omega d\vec{r}\mathcal{O}_j(\vec{r}) = \int\int\int_\Omega d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3 \Lambda_3^j(\vec{r}_1\vec{r}_2\vec{r}_3)\mathcal{O}_j\left(\frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3}{3}\right) \quad , \tag{3.21}$$

with

$$\Lambda_3^j(\vec{r}_1\vec{r}_2\vec{r}_3) = C_{\mathcal{U}\mathcal{U}\mathcal{U}}^j(\vec{r}_1, \vec{r}_2, \vec{r}_3) - C_{\mathcal{U}\mathcal{U}}^k(\vec{r}_1, \vec{r}_2)C_{k\mathcal{U}}^j(\tfrac{1}{2}(\vec{r}_1 + \vec{r}_2), \vec{r}_3) - C_{\mathcal{U}\mathcal{U}}^k(\vec{r}_2, \vec{r}_3)C_{k\mathcal{U}}^j(\tfrac{1}{2}(\vec{r}_2 + \vec{r}_3), \vec{r}_1) - C_{\mathcal{U}\mathcal{U}}^k(\vec{r}_3, \vec{r}_1)C_{k\mathcal{U}}^j(\tfrac{1}{2}(\vec{r}_1 + \vec{r}_3), \vec{r}_2) \quad . \tag{3.22}$$

The terms that are subtracted in Eq. (3.22) cancel out the marginal and relevant terms which arise from the possible short-distance expansions in the product $\mathfrak{U}(\vec{r}_1)\mathfrak{U}(\vec{r}_2)\mathfrak{U}(\vec{r}_3)$. As a consequence, we may take the limit $\Omega^{1/d}\to\infty$ in Eq. (3.21), such that

$$\Lambda_3^J=\Omega^{-1}\int\int\int_\Omega d\vec{r}_1\,d\vec{r}_2\,d\vec{r}_3\Lambda_3^J(\vec{r}_1\vec{r}_2\vec{r}_3)\ , \quad (3.23)$$

where the integrant is given by Eq. (3.22). Again, the expression for Λ_3^J is bound to converge.

One continues along similar lines in order to get the higher-order coefficients Λ_n^J. The result for Λ_4^J is

$$\Lambda_4^J=\Omega^{-1}\int\int\int\int_\Omega d\vec{r}_1\,d\vec{r}_2\,d\vec{r}_3\,d\vec{r}_4 \times\Lambda_4^J(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)\ , \quad (3.24)$$

where the integration is most conveniently written in terms of the following quantities:

$$D_1^J(\vec{r}_a\vec{r}_b;\vec{r}_c\vec{r}_d)=C_{\mathfrak{U}\mathfrak{U}}^k(\vec{r}_a,\vec{r}_b)C_{k\mathfrak{U}\mathfrak{U}}^J(\tfrac{1}{2}(\vec{r}_a+\vec{r}_b),\vec{r}_c,\vec{r}_d)+C_{\mathfrak{U}\mathfrak{U}}^k(\vec{r}_c,\vec{r}_d)C_{k\mathfrak{U}\mathfrak{U}}^J(\tfrac{1}{2}(\vec{r}_c+\vec{r}_d),\vec{r}_a,\vec{r}_b) - C_{\mathfrak{U}\mathfrak{U}}^k(\vec{r}_a,\vec{r}_b)C_{\mathfrak{U}\mathfrak{U}}^l(\vec{r}_c,\vec{r}_d)C_{kl}^J(\tfrac{1}{2}(\vec{r}_a+\vec{r}_b-\vec{r}_c-\vec{r}_d))\ , \quad (3.25a)$$

$$D_2^J(\vec{r}_a;\vec{r}_b\vec{r}_c\vec{r}_d)=C_{\mathfrak{U}k}^J(\vec{r}_a;\tfrac{1}{3}(\vec{r}_b+\vec{r}_c+\vec{r}_d))\Lambda_3^k(\vec{r}_b,\vec{r}_c,\vec{r}_d)\ , \quad (3.25b)$$

such that

$$\Lambda_4^J(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)=C_{\mathfrak{U}\mathfrak{U}\mathfrak{U}\mathfrak{U}}^J(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)-D_1^J(\vec{r}_1\vec{r}_2;\vec{r}_3\vec{r}_4)-D_1^J(\vec{r}_2\vec{r}_3;\vec{r}_4\vec{r}_1)-D_1^J(\vec{r}_1\vec{r}_3;\vec{r}_4\vec{r}_2) - D_2^J(\vec{r}_1;\vec{r}_2\vec{r}_3\vec{r}_4)-D_2^J(\vec{r}_2;\vec{r}_3\vec{r}_4\vec{r}_1)-D_2^J(\vec{r}_3;\vec{r}_4\vec{r}_1\vec{r}_2)-D_2^J(\vec{r}_4;\vec{r}_1\vec{r}_2\vec{r}_3)\ . \quad (3.26)$$

Hence, we have formal expressions for all of the expansion coefficients.

C. Alternative formulation

Expressions (3.18)–(3.26) appear to be—and are—rather complicated. There exists an alternative and simpler approach which gives the same answers as the method outlined above at least to low orders in λ. This formulation arises from replacing Eq. (3.5) by the alternative statement that the two Hamiltonians $\mathcal{H}_0^*(K_0)+\mathfrak{V}_A$ and $\mathcal{H}_0^*(K_0)+\mathfrak{V}_B$ lie in the same universality class if for $r'\to\infty$ and large Ω:

$$\langle\mathfrak{O}_\rho(\vec{r}';K_0)(e^{\mathfrak{V}_A^\Omega}-1)\rangle_{\mathcal{H}_0^*(K_0)}\to 0\ , \qquad \langle\mathfrak{O}_\rho(\vec{r}';K_0)(e^{\mathfrak{V}_B^\Omega}-1)\rangle_{\mathcal{H}_0^*(K_0)}\to 0\ , \quad (3.27)$$

for all relevant operators, $\mathfrak{O}_\rho$, including the unit operator, and, in addition,

$$\langle\mathfrak{V}_M(\vec{r}';K_0)(e^{\mathfrak{V}_A^\Omega}-e^{\mathfrak{V}_B^\Omega})\rangle_{\mathcal{H}_0^*(K_0)}\to 0 \quad (3.28)$$

for $\mathfrak{V}_M$ being the marginal operator.

We are not sure about the range of validity of this formulation. By doing expansions, one can convince oneself that it is derivable from the previous expansion at least through lowest orders. It may be correct to all orders. If it is, there are considerable advantages to the use of Eqs. (3.27) and (3.28) rather than Eq. (3.5). It is both conceptually and computationally simpler to expand about K_0 rather than K.

Further work is needed to establish the range of validity of the alternative formulation.

IV. CALCULATIONS OF MAPPING FUNCTIONS

A. Gaussian example

In Sec. II, we described the effect of two marginal operators $\mathcal{E}$ and $\mathcal{F}$, discovering that the perturbations $\lambda_\mathcal{E}\mathcal{E}$ or $\lambda_\mathcal{F}\mathcal{F}$ added to the fixed-point Hamiltonian $\mathcal{H}_0^*(K_0)$, respectively, changed the system to ones with effective coupling constants

$$K(\lambda_\mathcal{E})=K_0e^{2\pi\lambda_\mathcal{E}} \quad (4.1)$$

and

$$K(\lambda_\mathcal{F})=K_0\frac{1-\pi\lambda_\mathcal{F}}{1+\pi\lambda_\mathcal{F}}\ .$$

Here we wish to see how these results agree with the formalism described in the last section. Consider the effect of $\mathcal{F}$. Write

$$Q_{\lambda_\mathcal{F}}[W]=e^{\lambda_\mathcal{F}\mathcal{F}}Q_0[W]\ , \quad (4.2a)$$

where Q_0 has coupling K_0 and has the form given in Sec. II. Then rewrite this expression as

$$Q_{\lambda_\mathcal{F}}[W]=e^{\lambda_\mathcal{F}\mathcal{F}}e^{-\Lambda^\mathcal{E}(\lambda_\mathcal{F})\mathcal{E}}Q_{\Lambda^\mathcal{E}}[W]\ , \quad (4.2b)$$

where

$$Q_{\Lambda^\mathcal{E}}[W]=e^{\Lambda^\mathcal{E}(\lambda_\mathcal{F})\mathcal{E}}Q_0[W]\ . \quad (4.3)$$

Our job is to calculate $\Lambda^\mathcal{E}(\lambda_\mathcal{F})$ in a form which will ensure that

$$e^{\mathfrak{V}}=e^{\lambda_\mathcal{F}\mathcal{F}}e^{-\Lambda^\mathcal{E}(\lambda_\mathcal{F})\mathcal{E}} \quad (4.4)$$

will produce no motion along the marginal line.

Note the structure of Eq. (4.4). Here, $\mathcal{E}$ and $\mathcal{F}$ are differential operators and are arranged so that $\mathcal{F}$ always appears to the left of and operates on $\mathcal{E}$. In any expansion of Eq. (4.4) we should always be sure that the operators corresponding to $\mathcal{E}$'s and $\mathcal{F}$'s appear in a form in which the $\mathcal{E}$'s and $\mathcal{F}$'s are individually symmetrized in order but all the $\mathcal{F}$'s appear to the left of all the $\mathcal{E}$'s. With this prescription in mind, we can calculate the operator algebra coefficients involving $\mathcal{E}$'s and $\mathcal{F}$'s, put them into Eqs. (3.18)–(3.26) and calculate $\Lambda^{\mathcal{E}}(\lambda_{\mathcal{F}})$.

Products of $\mathcal{E}$'s and $\mathcal{F}$'s will generate via the short-distance expansion more $\mathcal{E}$'s and $\mathcal{F}$'s, but no new relevant operators except for the unit operator. Hence, we can follow the prescription of the last section if we write

$$e^{\mathcal{V}\,\Omega} = e^{\lambda_{\mathcal{F}}\mathcal{F}\,\Omega} \exp[-\Lambda^{\mathcal{E}}(\lambda_{\mathcal{F}})\mathcal{E}\,\Omega - \Lambda^{0}(\lambda_{\mathcal{F}})\Omega] \ , \tag{4.5}$$

where the last term is a reflection of the unit operator and demands that $e^{\mathcal{V}\,\Omega}$ expands to unity plus some irrelevant operators.

Correlation functions of $\mathcal{E}$'s and $\mathcal{F}$'s can be generated according to the prescription defined above. Here $\mathcal{E}(\vec{r})$ and $\mathcal{F}_{11}(\vec{r})$ are simply

$$\mathcal{F}_{11}(\vec{r}) = 2\frac{\delta}{\delta u_+(\vec{r})}\frac{\delta}{\delta u_-(\vec{r})} \ , \tag{4.6}$$

$$\mathcal{E}(\vec{r}) = \mathcal{F}_{11}(\vec{r}) - \tfrac{1}{2}\pi^2 \bar{u}_+(\vec{r})\bar{u}_-(\vec{r}) \ ,$$

where $\bar{u}_{\pm}$ are related to $u_{\pm}$ via

$$\bar{u}_{\pm}(\vec{r}) = \int d\vec{r}\,' B(\vec{r}-\vec{r}\,')u_{\pm}(\vec{r}\,') \ , \tag{4.7}$$

$$B(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^2} e^{i\vec{q}\cdot\vec{r}} B(q) \ .$$

The lowest-order "correlation functions" of $\mathcal{E}$ and $\mathcal{F}$ are simply the statements that the averages of these quantities vanish. Hence, if $\mathcal{O}_0$ represents the unit operator, the first-order coefficient Λ_1^0 vanishes. The other correlation functions are given in terms of the function $G(\vec{r})$, which is defined in terms of the $L(\vec{r})$ of Eq. (2.15) via

$$G(\vec{r}) = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)L(\vec{r}) \tag{4.8}$$

(see Appendix B). Since $L(\vec{r})$ is proportional to $\ln(r)$ for large r:

$$G(\vec{r}) \Rightarrow |x+iy|^{-4} = r^{-4} \tag{4.8a}$$

for large r. Then, the second-order correlation functions take the form

$$\begin{aligned} &\langle \mathcal{F}_{11}(\vec{r}_1)\mathcal{F}_{11}(\vec{r}_2)\rangle = G(\vec{r}_1-\vec{r}_2) \ , \\ &\langle \mathcal{E}(\vec{r}_1)\mathcal{F}_{11}(\vec{r}_2)\rangle = \langle \mathcal{E}(\vec{r}_1)\mathcal{E}(\vec{r}_2)\rangle \\ &\qquad = G(\vec{r}_1-\vec{r}_2) - 4\pi^2 B^2(\vec{r}_1-\vec{r}_2) \ . \end{aligned} \tag{4.9}$$

Equations (4.9) can be converted into two kinds of statements about operator algebra expansion coefficients, as they are defined by Eq. (3.15). The first statement arises from letting $\vec{r}_1$ go to infinity in Eqs. (4.9). Since B is short ranged, we then learn that $C_{\mathcal{F}}^{\mathcal{E}} = 1$ or alternatively that $\mathcal{U}(r)$ in Eqs. (3.11) and (3.12) is simply

$$\mathcal{U}(r) = \mathcal{F}_{11}(\vec{r}) - \mathcal{E}(\vec{r}) \ . \tag{4.10}$$

In addition, we notice that Eq. (3.18) determines the first-order mapping function to be $\Lambda_1^{\mathcal{E}} = 1$.

Alternatively, one can read Eqs. (4.9) as if they had inside the correlation functions a unit operator $1 = \mathcal{O}_0(\vec{r})$. Then Eqs. (4.9) determine second-order operator product coefficients as, for example,

$$C^0_{\mathcal{F}_{11}\mathcal{F}_{11}}(\vec{r}) = G(\vec{r}) \ . \tag{4.11}$$

The important product coefficient for our analysis is

$$\begin{aligned} C^0_{\mathcal{U}\mathcal{U}}(\vec{r}) &= C^0_{\mathcal{F}_{11}\mathcal{F}_{11}}(\vec{r}) - 2C^0_{\mathcal{E}\mathcal{F}_{11}}(\vec{r}) + C^0_{\mathcal{E}\mathcal{E}}(\vec{r}) \\ &= 4\pi^2[B(\vec{r})]^2 \ . \end{aligned} \tag{4.12}$$

Since all odd order correlation functions involving $\mathcal{E}$ and $\mathcal{F}$ vanish we have $C^{\mathcal{E}}_{\mathcal{U}\mathcal{U}}$ also being zero. Hence, from Eq. (3.20) we derive that the appropriate expansion coefficients in $\mathcal{U}$ are

$$\Lambda_2^0 = \int d\vec{r}\, 4\pi^2 B^2(\vec{r}) \ , \quad \Lambda_2^{\mathcal{E}} = \frac{d^2}{d\lambda_{\mathcal{F}}^2}\Lambda^{\mathcal{E}}(\lambda_{\mathcal{F}})|_{\lambda_{\mathcal{F}}=0} = 0 \ . \tag{4.13}$$

The calculation of the third-order term in $\Lambda^{\mathcal{E}}$ proceeds along similar lines. From Eq. (3.22) we notice that the only needed term is the one which is directly an integral of $C^{\mathcal{E}}_{\mathcal{U}\mathcal{U}\mathcal{U}}$ which is

$$C^{\mathcal{E}}_{\mathcal{U}\mathcal{U}\mathcal{U}}(0,\vec{r}_1,\vec{r}_2) = \lim_{r'\to\infty} |\vec{r}\,'|^4 \langle \mathcal{E}(\vec{r}\,')[\mathcal{F}_{11}(0)-\mathcal{E}(0)][\mathcal{F}_{11}(\vec{r}_1)-\mathcal{E}(\vec{r}_1)][\mathcal{F}_{11}(\vec{r}_2)-\mathcal{E}(\vec{r}_2)]\rangle \ . \tag{4.14}$$

This expression is evaluated via a consideration of the fourth-order correlation function. After considerable cancellation, the operator product expansion coefficient takes the form

$$C^{\mathcal{E}}_{\mathcal{U}\mathcal{U}\mathcal{U}}(0,\vec{r}_1,\vec{r}_2) = \tfrac{2}{3}\pi^2[B(\vec{r}_1)B(\vec{r}_2) + B(\vec{r}_1-\vec{r}_2)B(\vec{r}_2) + B(\vec{r}_1-\vec{r}_2)B(\vec{r}_1)] \ .$$

Then the integral of this expression over $\vec{r}_1$ and $\vec{r}_2$ gives

$$\Lambda_3^{\mathcal{S}} = 2\pi^2$$

so that to third order $\Lambda^{\mathcal{S}}$ is finally

$$\Lambda^{\mathcal{S}}(\lambda_{\mathcal{F}}) = \lambda_{\mathcal{F}} + \tfrac{2}{6}\pi^2\lambda_{\mathcal{F}}^3 + \cdots \quad . \tag{4.15}$$

In the notation of Eq. (4.1) this mapping function should obey

$$K_{\mathcal{S}}(\Lambda^{\mathcal{S}}(\lambda_{\mathcal{F}})) \equiv K_{\mathcal{F}}(\lambda_{\mathcal{F}})$$

so that we should have

$$2\pi\Lambda^{\mathcal{S}}(\lambda_{\mathcal{F}}) = \ln\frac{1+\pi\lambda_{\mathcal{F}}}{1-\pi\lambda_{\mathcal{F}}} \quad . \tag{4.16}$$

To the requisite order, the third, the substitution of Eq. (4.15) into Eq. (4.16) gives an identity.

B. Eight-vertex and Ashkin-Teller models

The work of these models will be reported in detail in a later publication by Pruisken and Brown. Here, we only mention that the methodology of Sec. III has been applied to calculating the critical couplings and critical indices to second order in the four-spin interactions in these problems. The specific method used is that of Sec. III C. The results agree with the exactly known answers for these models.

C. Planar model

The Villain version of the planar model can be discussed in exactly the same language as we have used heretofore. This problem is essentially our Gaussian model with thermally excited vortices at a set of lattice sites $\vec{r}_i$ separated by a spacing a on a square lattice.

To get the thermally excited vortices, we make use of the operator (2.4) which introduces a vortex of vorticity m at the lattice site $\vec{r}_i$. If there is a fugacity y_0 for the introduction of these vortices, an appropriate form of the interaction term which brings in these vortices, is $\mathcal{V} = y_0\mathfrak{U}$ with

$$\mathfrak{U} = \sum_{\vec{r}_i}[\mathcal{O}_{0,1}(\vec{r}_i) + \mathcal{O}_{0,-1}(\vec{r}_i)] \quad , \tag{4.17a}$$

which may then be written as

$$\mathfrak{U} = 2\sum_{\vec{r}_i}\cosh\left(\frac{\delta}{\delta v_+(\vec{r}_i)} - \frac{\delta}{\delta v_-(\vec{r}_i)}\right) \quad . \tag{4.17b}$$

In addition, we must set the cutoff to ensure the proper vortex-vortex interaction for a lattice system. To do this choose $\Theta(\vec{r}_i - \vec{r}_i)$ to be exactly the same as the form given in Refs. 7 and 13, and get $L(r)$ to have the same form as in this reference by taking the cutoff function to be (inside and outside the first Brillouin zone, respectively)

$$B(q) = \begin{cases} \dfrac{q_x^2 + q_y^2}{4 - 2\cos q_x - 2\cos q_y}, & \pi < q_x/a_0 < \pi, \ -\pi < q_y/a_0 < \pi \qquad (4.18a) \\ 0. & \qquad (4.18b) \end{cases}$$

The net effect of all of this is to ensure that

$$Q_p[W] = e^{y_0\mathfrak{U}}Q_0[W] \tag{4.19}$$

is a generating function for all the correlations of the Villain planar models. To form correlation functions for the planar model, one uses the differential operators defined in Sec. II, applies them to $Q_p[W]$, then sets $W=0$ and divides by the partition function, $Q_p[0]$. In symbols, if $\mathfrak{X}$ is such a differential operator

$$\langle\mathfrak{X}\rangle_p = (Q_p[0])^{-1}(\mathfrak{X}Q_p[W])_{W=0} \quad . \tag{4.20}$$

Actually, we make a small error in saying that Eq. (4.20) describes exactly the same model as treated by Villain and by Jose *et al.* According to Eq. (4.20), the situation in which two $m=1$ vortices appear at the same lattice site is a given statistical y_0^2. In the earlier treatment it is given a weight of y_0^4. Thus, the multiple vorticity excitations are handled differently in the two formulations. [In Coulomb-gas language, the partition function defined by Eqs. (4.17a) and (4.17b) has the form

$$Q_p = \sum_{\{m\}}\exp\left[\sum_{\vec{r}_i}|m(\vec{r}_i)|\ln y_0 + \pi K\sum_{\vec{r}_i\vec{r}_j} m(\vec{r}_i)L(\vec{r}_i - \vec{r}_j)m(\vec{r}_j)\right]$$

describing a system of interacting electric charges $m(\vec{r})$ located at the dual lattice sites. The sum over $\{m\}$ runs over all configurations with zero total charge. Usually one has $m^2(\vec{r})$ in the exponent instead of $|m(\vec{r})|$. The two descriptions only differ in the definition of the fugacity y_0.] However, this difference will not produce any substantial change in our results.

We have performed quite extensive analyses in which we expanded correlation function expressions of the kind symbolized by Eq. (4.20) in power series in y_0, through order y_0^4. As with Kosterlitz and

Thouless and with Jose *et al.*, the resulting expressions were sums of Gaussian correlation functions of the form

$$y_0^p \sum_{i_1,\ldots,i_p} \sum_{m_1,\ldots,m_p=\pm 1} \langle \mathcal{O}_{0,m_1}(\vec{r}_{i_1}) \cdots \mathcal{O}_{0,m_p}(\vec{r}_{i_p}) \mathfrak{X} \rangle_{K_0} \ . \tag{4.21}$$

Here we want to view $\mathfrak{X}$ as a product of local differential operators like $\mathcal{E}(\vec{r})$ or $\mathcal{O}_{n,m}(\vec{r})$.

Initially the position sums in Eq. (4.21) looked impossibly complex. However, if we stick to the low-temperature phase in which

$$2\pi K_0 > 4 \tag{4.22}$$

detailed analysis can give considerable information about the structure of the correction terms [Eq. (4.21)]. If Eq. (4.22) is satisfied, then the vortex operators are irrelevant, at least in perturbation theory. This implies that the positional sums in Eq. (4.21) get their main contribution when the r_i's are either (a) close to the positions of the operators in $\mathfrak{X}$, or (b) close to each other.

By employing short-distance expansions we can then select out the contributions which decay at least as fast as the zeroth-order term $\langle \mathfrak{X} \rangle_{K_0}$. The task one now faces is to collect all these contributions and see whether and how they reexponentiate to a simple power-law behavior. Such an analysis turns out to be rather complicated, due to interference effects between the various short-distance expansions. We have nevertheless succeeded in performing the expansion in y_0 to order y_0^4. In what follows, we will briefly mention the main effects of the contributions, described under (a) and (b) above.

When the $\vec{r}_i$'s in Eq. (4.21) lie close to the coordinates of the operators in $\mathfrak{X}$ the resulting terms in the expansion can produce a considerable modification of the initial correlation functions. However, this modification can be undone by a simple process of redefinition of the basic operators. That is, there exists a set of new operators $\mathcal{O}_{n,m}(\vec{r},y_0)$ and $\mathcal{F}_{ij}(\vec{r},y_0)$ which are given in terms of linear combinations of the original operators in the theory. These linear combinations are of the form

$$\mathcal{O}_\alpha(\vec{r};y_0) = \sum_\beta f_\alpha^\beta(y_0)\, \mathcal{O}_\beta(\vec{r}) \ ,$$

where $f_\alpha^\beta(y_0)$ is δ_α^β plus a power-series expansion in y_0. This renormalization of the operators is a statement which holds within the expectation brackets and in the limit of infinite separations.

The main effect of the terms in which the $\vec{r}_i$'s are close to one another is to change the value of the critical indices. A product of two or four nearby vortex operators with total vorticity equal to zero, generates terms proportional to the marginal operator. These marginal terms then modify the value of the effective coupling K_0 into $K_{\text{eff}}(y_0)$ in all the correlation functions. Presently, we shall calculate $K_{\text{eff}}(y_0)$ following Kosterlitz and Thouless[4] and Jose *et al.*[7] but going to order y_0^4. But before we embark on this calculation we should notice the primary quantitative result of this qualitative discussion.

By detailed calculations including terms of order up to y_0^4 and by qualitative arguments about all terms in perturbation theory, we have convinced ourselves that one can introduce renormalized operators $\mathcal{O}_\alpha(\vec{r},y_0)$ such that the Villain model correlation functions of these operators have exactly the Gaussian form with a new coupling parameter $K_{\text{eff}}(y_0)$. These arguments apply whenever

$$2\pi K_{\text{eff}}(y_0) > 4 \ . \tag{4.23}$$

Specifically these qualitative arguments support the notion that the usual relationships between the critical indices which hold for the Gaussian model also hold for the planar model, e.g., that the renormalized $\mathcal{E}(\vec{r},y_0)$ remains marginal and that $\mathcal{O}_{n,m}(\vec{r},y_0)$ has critical index

$$x_{n,m} = m^2 \pi K_{\text{eff}} + n^2/4\pi K_{\text{eff}} \ .$$

The index connected with the lowest spin-wave excitation is $x_{1,0} = (4\pi K_{\text{eff}})^{-1}$ so that η ($\eta_{n,m} = 2x_{n,m}$) is given by $\eta_{1,0} = (2\pi K_{\text{eff}})^{-1}$. Similarly the η's associated with the vortex operators $\mathcal{O}_{0,1} \pm \mathcal{O}_{0,-1}$ are equal and given by $\eta_{0,1}^{(\pm)} = 2\pi K_{\text{eff}}$.

D. Calculation of $K_{\text{eff}}(y_0)$

Now choose some marginal operator $\tilde{\mathcal{E}}(\vec{r})$. Rewrite Eq. (4.19) as

$$Q_p[W] \sim e^{\mathcal{U}} e^{\Lambda^{\tilde{\mathcal{E}}}(y_0)\tilde{\mathcal{E}}} Q_0[W] \tag{4.24}$$

and write instead of the general equation (3.9) the specific one which applies to this case:

$$\mathcal{U} = y_0 \mathfrak{U} - \Lambda^{\tilde{\mathcal{E}}}(y_0)\tilde{\mathcal{E}} - \Lambda^0(y_0)\mathcal{O}_0 \ . \tag{4.25}$$

These three operators are sufficient because operator products of $\mathcal{U}(\vec{r})$ lead to only two nonirrelevant operators, the unit operator $\mathcal{O}_0(\vec{r})$ and the marginal one $\tilde{\mathcal{E}}(\vec{r})$.

For this problem we can directly use Eqs. (3.18)–(3.26) in order to obtain an expansion for $\Lambda^{\tilde{\mathcal{E}}}$ and Λ^0 in a power series in y_0. The $\mathfrak{U}$ in Eq. (4.25) has a density

$$\mathfrak{U} = \int d\vec{r}\, \mathfrak{U}(\vec{r})$$

with [see Eq. (4.17a)]

$$\mathfrak{U}(\vec{r}) = \sum_{\vec{r}_i} \delta(\vec{r} - \vec{r}_i)[\mathcal{O}_{0,1}(\vec{r}_i) + \mathcal{O}_{0,-1}(\vec{r}_i)] \ , \tag{4.26}$$

where the sum runs over all (dual) lattice sites $\vec{r}_i$.

The correlation functions containing an odd number of $\mathfrak{u}$'s vanish, so that to order y_0^4 the nonvanishing coefficients are

$$\Lambda_2^0 = \int d\vec{r}\, C^0_{\mathfrak{u}\mathfrak{u}}(\vec{r}) \ , \tag{4.27a}$$

$$\Lambda_2^{\tilde{\mathcal{S}}} = \int d\vec{r}\, C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}) \ , \tag{4.27b}$$

$$\Lambda_4^{\tilde{\mathcal{S}}} = \Omega^{-1} \int\int\int\int_\Omega d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3\, d\vec{r}_4 \Lambda_4^{\tilde{\mathcal{S}}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4) \ , \tag{4.27c}$$

with

$$\Lambda_4^{\tilde{\mathcal{S}}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4) = C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4) - D_1^{\tilde{\mathcal{S}}}(\vec{r}_1\vec{r}_2;\vec{r}_3\vec{r}_4) - D_1^{\tilde{\mathcal{S}}}(\vec{r}_2\vec{r}_3;\vec{r}_4\vec{r}_1) - D_1^{\tilde{\mathcal{S}}}(\vec{r}_1\vec{r}_3;\vec{r}_2\vec{r}_4) \ , \tag{4.27d}$$

and

$$\begin{aligned} D_1^{\tilde{\mathcal{S}}}(\vec{r}_i;\vec{r}_j;\vec{r}_k\vec{r}_l) = {} & C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_i - \vec{r}_j) C^{\tilde{\mathcal{S}}}_{\mathcal{S}\mathfrak{u}\mathfrak{u}}(\tfrac{1}{2}(\vec{r}_i + \vec{r}_j), \vec{r}_k, \vec{r}_l) + C_{\mathfrak{u}\mathfrak{u}}(\vec{r}_k - \vec{r}_l) C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}\tilde{\mathcal{S}}}(\vec{r}_i\vec{r}_j, \tfrac{1}{2}(\vec{r}_k + \vec{r}_l)) \\ & - C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_i - \vec{r}_j) C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_k - \vec{r}_l) C^{\tilde{\mathcal{S}}}_{\mathcal{S}\mathcal{S}}(\tfrac{1}{2}(\vec{r}_i + \vec{r}_j - \vec{r}_k - \vec{r}_l)) \ . \end{aligned} \tag{4.27e}$$

We are free to choose any marginal operator $\tilde{\mathcal{S}}$ for this problem. Let us make the choice

$$\tilde{\mathcal{S}}(\vec{r}) = 2\frac{\delta}{\delta u_+(\vec{r})}\frac{\delta}{\delta u_-(\vec{r})} - \tfrac{1}{2}\pi^2 \bar{u}_+ \bar{u}_-(\vec{r}) \ . \tag{4.28}$$

Thus the $\tilde{\mathcal{S}}(\vec{r})$ is the same as the previously considered operator $\mathcal{S}(\vec{r})$ [see Eqs. (4.6) and (4.7)], with a specific choice for the cutoff function, $B(q)$, namely, Eq. (4.18). According to the analysis in Sec. II, the exponential $\exp[\Lambda^{\tilde{\mathcal{S}}}(y_0)\tilde{\mathcal{S}}]$ effectively replaces the Gaussian parameter K_0 by a K_{eff}, such that [see Eqs. (2.22) and (4.1)]:

$$K_{\text{eff}}(y_0) = K_0 e^{2\pi\Lambda^{\tilde{\mathcal{S}}}(y_0)} \ . \tag{4.29}$$

It is noted that the expansion coefficients of $\Lambda^{\tilde{\mathcal{S}}}(y_0)$ [Eqs. (4.27)] are expressed in terms of operator algebra coefficients which are to be evaluated at the effective parameter $K_{\text{eff}}(y_0)$ in the generating function. This makes the expansion coefficients $\Lambda_n^{\tilde{\mathcal{S}}}$ effectively a function of y_0. However, the expansion of $\Lambda^{\tilde{\mathcal{S}}}(y_0)$ becomes simpler (at least to order y_0^4) if we in addition expand the operator algebra coefficients in a power series in y_0 around K_0. In order to do this, we have to go back to the original Eqs. (3.14). Recall that the coefficient Λ_2^J is determined by the requirement that the operator

$$\mathcal{V}_2^{\Omega} = \mathfrak{u}^{\Omega}\mathfrak{u}^{\Omega} - \Lambda_2^J \mathcal{O}_J^{\Omega} \tag{4.30}$$

is irrelevant, in the sense of Eq. (3.5). If one now wants to evaluate the expansion of the operator products in $\mathcal{V}_2^{\Omega}$ at K_0 rather than at K_{eff}, one then has to demand $\mathcal{V}_2^{\Omega} e^{\Lambda^{\tilde{\mathcal{S}}}(y_0)\tilde{\mathcal{S}}^{\Omega}}$ is irrelevant. In the present case, one has

$$\mathcal{V}_2^{\Omega} e^{\Lambda^{\tilde{\mathcal{S}}}\tilde{\mathcal{S}}^{\Omega}} = (\mathfrak{u}^{\Omega}\mathfrak{u}^{\Omega} - \Lambda_2^{\tilde{\mathcal{S}}}\tilde{\mathcal{S}}^{\Omega} - \Lambda_2^0 \mathcal{O}_0^{\Omega}) e^{\Lambda^{\tilde{\mathcal{S}}}(y_0)\tilde{\mathcal{S}}^{\Omega}} \tag{4.31}$$

In expanding to order y_0^2, this leads to the following result for $\Lambda_2^{\tilde{\mathcal{S}}}$

$$\Lambda_2^{\tilde{\mathcal{S}}} = \int d\vec{r}\, C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r})_0 + \frac{y_0^2}{2!}\Lambda_2^{\tilde{\mathcal{S}}}\left[\Omega^{-1}\int\int\int_\Omega d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3 C^{\tilde{\mathcal{S}}}_{\mathcal{S}\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_2\vec{r}_3)_0 - \Omega\Lambda_2^0\right] + \cdots \ . \tag{4.32}$$

The index "0" on the operator algebra coefficients indicates the point K_0 on the Gaussian fixed line. Thus, collecting the terms in $\Lambda^{\tilde{\mathcal{S}}}(y_0)$ up to order y_0^4, one has

$$\Lambda^{\tilde{\mathcal{S}}}(y_0) = \frac{y_0^2}{2!}\Omega^{-1}\int\int_\Omega d\vec{r}_1\, d\vec{r}_2 \Lambda_2^{\tilde{\mathcal{S}}}(\vec{r}_1 - \vec{r}_2)_0 + \frac{y_0^4}{4!}\Omega^{-1}\int\int\int\int_\Omega d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3\, d\vec{r}_4 \Lambda_4^{\tilde{\mathcal{S}}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)_0 + \cdots \ , \tag{4.33}$$

with

$$\begin{aligned} \Lambda_2^{\tilde{\mathcal{S}}}(\vec{r}_1\vec{r}_2)_0 &= C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_2)_0 = C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_1 - \vec{r}_2)_0 \ , \\ \Lambda_4^{\tilde{\mathcal{S}}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4) &= C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)_0 - C^0_{\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_2)_0 C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_3\vec{r}_4)_0 - C^0_{\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_2)_0 C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_3\vec{r}_4)_0 \\ &\quad - C^0_{\mathfrak{u}\mathfrak{u}}(\vec{r}_2\vec{r}_3)_0 C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_4)_0 - C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_2\vec{r}_3)_0 C^0_{\mathfrak{u}\mathfrak{u}}(\vec{r}_1\vec{r}_4)_0 \\ &\quad - C^0_{\mathfrak{u}\mathfrak{u}}(\vec{r}_3\vec{r}_1)_0 C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_4\vec{r}_2)_0 - C^{\tilde{\mathcal{S}}}_{\mathfrak{u}\mathfrak{u}}(\vec{r}_3\vec{r}_1)_0 C^0_{\mathfrak{u}\mathfrak{u}}(\vec{r}_4\vec{r}_2)_0 \ . \end{aligned} \tag{4.33a}$$

Note that the coefficient $C^{\tilde{\delta}}_{\tilde{\delta}uu}$ has disappeared in Eq. (4.33). It follows that the expansion around K_0, as given above, is independent (to order y_0^4) of the particular choice of the marginal operator $\tilde{\mathcal{O}}(\vec{r})$. (This is a direct consequence of: $C^{\tilde{\delta}}_{\tilde{\delta}\tilde{\delta}}=0$, $\forall\,\tilde{\mathcal{O}}$.) We remark that Eqs. (4.33) exactly correspond to the expansion, derived from the alternative formulation which is described in Sec. III C.

The operator algebra coefficients in Eq. (4.33) can be calculated by straightforward application of the definition (3.15). As a result, Eq. (4.33) can be written as (see Appendix B):

$$\Lambda^{\tilde{\delta}}(y_0)=\pi K_0\left[\frac{y_0^2}{2!}\sum_{\vec{r}_1\vec{r}_2}{}' 2!(\vec{r}_1-\vec{r}_2)^2x_{12}^{-1}+\frac{y_0^4}{4!}\sum_{\vec{r}_1,\ldots,\vec{r}_4}{}' 3!(\vec{r}_1-\vec{r}_2+\vec{r}_3-\vec{r}_4)^2\left[\frac{x_{24}x_{13}-x_{12}x_{34}-x_{23}x_{14}}{x_{12}x_{23}x_{34}x_{41}}\right]\cdots+O(y_0^6)\right], \tag{4.34}$$

with

$$x_{ij}=e^{2\pi K_0 L(\vec{r}_i-\vec{r}_j)}\ . \tag{4.34a}$$

The prime on the summation signs denotes the restriction

$$\sum_n \vec{r}_n=\vec{0} \tag{4.34b}$$

and the sums run over all the sites r_i of the (dual) square lattice. The $L(r)$ represents the lattice Green's function[4–7]

$$L(r)=\int\frac{d\vec{q}}{2\pi}\frac{1-e^{i\vec{q}\cdot\vec{r}}}{4-2\cos q_x-2\cos q_y}\ , \tag{4.34c}$$

which has the asymptotic value

$$L(r)\simeq \text{const}+\ln r\ .$$

Finally we remark that the sums in Eq. (4.34) give finite results for $2\pi K_0>4$. We have convinced ourselves of this fact by a detailed analysis.

E. Concluding remarks

Equation (4.34) forms the central result of this section. We have presented a systematic scheme for expanding in the vortex parameter y_0. We have furthermore proven (at least to order y_0^4) that the low-temperature Villain model has continuously varying critical behavior which is essentially given by the Gaussian model with a renormalized value of the parameter K (i.e., K_{eff}).

Our perturbation expansion holds as long as $2\pi K_{\text{eff}}>4$. This region is characterized by the presence of only one marginal operator in the theory. This marginal operator is generated by sets of nearby vortex operators with zero total vorticity. At exactly $2\pi K_{\text{eff}}=4$, which defines the Kosterlitz-Thouless critical line, the vortex operators $\mathcal{O}_{0,\pm1}$ become marginal themselves. This situation has a more complicated nature and needs separate consideration. For $2\pi K_{\text{eff}}\to 4^+$ the operator algebra coefficients in our original expansion [Eqs. (4.27)] tend to diverge; more study is needed in order to give an adequate description of this limiting situation. Several authors[5,7,21–24] have dealt with this problem, using different methods and different points of view. The commom conclusion is in agreement with Kosterlitz's result. Namely, that the line $2\pi K_{\text{eff}}=4$ exhibits universal values for critical exponents, in particular, $\eta=\frac{1}{4}$. It would be a useful advance to see whether Kosterlitz's conjecture can also be derived in a more systematical way, using the methods developed in this paper. We mention also the recent work of Amit *et al.*[23] which is in several aspects parallel to our work. These authors give a field theoretic treatment of the sine-Gordon theory and derive higher-order corrections to Kosterlitz's renormalization-group equations.

In conclusion we can say that we have set up a theory which provides us with an expansion scheme for marginal operators. The scheme essentially relates different critical lines which are members of the same universality class. We have given a detailed study of marginality for the Gaussian model. The results are put into practice for Villain's version of the planar model: to order y_0^4 we have calculated the mapping of this model onto the Gaussian model. The same techniques can be used in order to expand the Baxter and Ashkin-Teller critical lines around their decoupling point.[25]

The results give stronger evidence for the conjecture that all of these models are members of one and the same universality class.

ACKNOWLEDGMENTS

This research was supported in part by the NSF and by the Material Research Laboratory of Brown University.

APPENDIX A: JUSTIFICATION OF BASIC EXPANSION FORMULA

In order to justify Eq. (3.5), imagine that we can write for the perturbation, instead of $\mathcal{U}$

$$\int_0^{\mu}\mathcal{U}(\xi)\,d\xi,\quad \mathcal{U}(\xi)=\sum_{\vec{r}}\mathcal{U}(\vec{r};\xi)\ . \tag{A1}$$

Here we adjust $\mathcal{U}(\xi)$ so that the perturbation [Eq.

(A1)] obeys Eq. (3.5) for all values of μ. We shall essentially prove Eq. (3.5) by showing that it is true term by term in a power-series expansion in μ.

If Eq. (A1) is to be an irrelevant perturbation to the critical Hamiltonian, $\mathcal{H}^*$, then we must be able to construct for every value of μ a set of critical operators $\mathcal{O}_\alpha(\vec{r};\mu)$ with μ-independent scaling indices

$$\langle \mathcal{O}_\alpha(\vec{r}_\alpha;\mu)\mathcal{O}_\beta(\vec{r}_\beta;\mu)\rangle_\mu = a_{\alpha\beta}|\vec{r}_\alpha-\vec{r}_\beta|^{-x_\alpha-x_\beta}\cdots , \tag{A2}$$

where the expectation, $\langle\cdots\rangle_\mu$, is taken with respect to the Hamiltonian

$$\mathcal{H}(\mu)=\mathcal{H}^*+\int_0^\mu \mathcal{U}(\xi)\,d\xi \ .$$

The dots in Eq. (A2) stand for correction to scaling terms, which show that although $\mathcal{H}^*$ and $\mathcal{H}^*+\mathcal{U}$ are in the same universality class they do have different nonasymptotic behavior.

We assume that $\mathcal{U}(\vec{r};\mu)$ contains no critical operators with scaling indices $x \le d$. This irrelevance condition for $\mathcal{U}(\vec{r};\mu)$ can be formulated as

$$\lim_{R\to\infty} R^{2x_j}\langle \mathcal{U}(\vec{0};\mu)\mathcal{O}_j(\vec{R};\mu)\rangle_\mu = 0 \tag{A3}$$

for every value of μ and for all relevant and marginal operators $\mathcal{O}_j$ in the theory. (For simplicity we will consider only scalar operators. The conclusions of this appendix, however, are unchanged by including nonscalar operators in the analysis.) We will study the consequences of Eq. (A3) by applying the differentiation $d_\mu \equiv d/d_\mu$ to both sides of the equation. First, however, we consider the more general case of Eq. (A3):

$$d_\mu\langle \mathcal{O}_\alpha(\vec{r}_\alpha;\mu)\mathcal{O}_\beta(\vec{r}_\beta;\mu)\rangle_\mu = \left\langle\left[d_\mu+\sum_{\vec{r}}\mathcal{U}(\vec{r};\mu)\right]\mathcal{O}_\alpha(\vec{r}_\alpha;\mu)\mathcal{O}_\beta(\vec{r}_\beta;\mu)\right\rangle_\mu \ . \tag{A4}$$

In order to evaluate Eq. (A4), we assume that the operators $\mathcal{O}_\alpha$, defined for different values of μ, are related to one another via the transformation

$$\mathcal{O}_\alpha(\vec{r};\mu+\Delta_\mu) = \mathcal{O}_\alpha(\vec{r};\mu)+\Delta_\mu f_\alpha^\beta(\mu)\mathcal{O}_\beta(\vec{r};\mu)+\cdots , \tag{A5}$$

where the sum over β runs over a complete set of scaling operators. The $\cdots$ refers to redundant operators which have correlations which fall off with exponential rapidity. To see the meaning of Eq. (A5), consider the sum over lattice sites $\vec{r}$ in the right-hand side correlation of Eq. (A4). Let us, for specificity, define spheres of radii ρ_α and ρ_β around the positions $\vec{r}_\alpha$ and $\vec{r}_\beta$ of the operators $\mathcal{O}_\alpha$, and $\mathcal{O}_\beta$, respectively, such that ρ_α, ρ_β is much larger than a lattice constant, but much less than the separation $|\vec{r}_\alpha-\vec{r}_\beta|$. In this way we can separate out the short-wavelength, model-dependent parts and the long-wavelength parts in the correlation. Notice that the terms in the sum of Eq. (A4) with $\vec{r}$ far from $\vec{r}_\alpha$ and $\vec{r}_\beta$ lead to a power-law behavior of which the leading term is determined by scaling

$$\sum_{\substack{|\vec{r}-\vec{r}_\alpha|>\rho_\alpha\\ |\vec{r}-\vec{r}_\beta|>\rho_\beta}} \langle \mathcal{U}(\vec{r})\mathcal{O}_\alpha(\vec{r}_\alpha;\mu)\mathcal{O}_\beta(\vec{r}_\beta;\mu)\rangle_\mu = r_{\alpha\beta}^{d-x_{\mathcal{U}}-x_\alpha-x_\beta}B(\rho_\alpha/r_{\alpha\beta};\rho_\beta/r_{\alpha\beta})+\cdots , \tag{A6}$$

where we used the abbreviation $r_{\alpha\beta}=|\vec{r}_\alpha-\vec{r}_\beta|$ and where, for the moment, we assumed that $\mathcal{U}(\vec{r};\mu)$ is a single scaling operator with scaling index $x_{\mathcal{U}} > d$. The dots in Eq. (A6) represent all terms, arising from correction to scaling.

On the other hand, when the $\vec{r}$ in Eq. (A4) is close to the positions $\vec{r}_\alpha$ or $\vec{r}_\beta$, we assume that the resulting products of nearby operators $\mathcal{U}$ and $\mathcal{O}_\alpha$, $\mathcal{O}_\beta$ can be expressed in terms of a complete set of critical operators

$$\sum_{|\vec{r}-\vec{r}_\alpha|<\rho_\alpha} \mathcal{U}(\vec{r};\mu)\mathcal{O}_\alpha(\vec{r}_\alpha;\mu) = \sum_{|\vec{r}-\vec{r}_\alpha|<\rho_\alpha} C_{\mathcal{U}\alpha}^\gamma(\vec{r}-\vec{r}_\alpha)\mathcal{O}_\gamma(\vec{r}_\alpha;0) \tag{A7}$$

and a similar expression with $\mathcal{O}_\beta(\vec{r}_\beta;\mu)$. Equations (A6) and (A7) determine the first-order change in the operators as written down in Eq. (A5). We define renormalized operators as follows:

$$d_\mu\mathcal{O}_\alpha(\vec{r}_\alpha;\mu) = -\left[\left(\sum_{\Delta<\rho_\alpha} C_{\mathcal{U}\alpha}^\gamma(\vec{\Delta})\right)-S_{\mathcal{U}\alpha}^\gamma(\rho_\alpha)\right]\mathcal{O}_\gamma(\vec{r}_\alpha;\mu) \tag{A8}$$

and a similar result for $d_\mu\mathcal{O}_\beta$. The term $S_{\mathcal{U}\alpha}^\gamma(\rho_\alpha)$ in Eq. (A8) is such that it subtracts off all ρ_α dependence from the sum over the operator product expansion coefficients. As we have taken ρ_α, ρ_β much larger than a lattice constant, the surface terms $S_{\mathcal{U}\alpha}^\gamma(\rho_\alpha)$ and $S_{\mathcal{U}\beta}^\gamma(\rho_\beta)$ are determined by the long-wavelength behavior of the correlations. In fact, the leading behavior is determined by scaling and we may write

$$\left[d_\mu+\sum_{|\vec{r}-\vec{r}_\alpha|<\rho_\alpha}\mathcal{U}(\vec{r};\mu)\right]\mathcal{O}_\alpha(\vec{r}_\alpha;\mu) = \sum_\gamma\left(a_{\alpha\mathcal{U}\gamma}\rho_\alpha^{d-x_{\mathcal{U}}-x_\alpha+x_\gamma}+\cdots\right)\mathcal{O}_\gamma(\vec{r}_\alpha;\mu) \tag{A9}$$

and a similar expression involving $\mathcal{O}_\beta$.

By inserting Eqs. (A6) and (A9) into Eq. (A4) we can see that the renormalized operators [Eq. (A8)]

indeed have the desired properties

$$d_\mu(\mathcal{O}_\alpha(\vec{r}_\alpha;\mu)\mathcal{O}_\beta(\vec{r}_\beta;\mu))_\mu = b r_{\alpha\beta}^{d-x_\upsilon-x_\alpha-x_\beta} << r_{\alpha\beta}^{-x_\alpha-x_\beta} \quad , \tag{A10}$$

where the amplitude b equals

$$b = B\left(\frac{\rho_\alpha}{r_{\alpha\beta}};\frac{\rho_\beta}{r_{\alpha\beta}}\right) + \sum_\gamma a_{\alpha\upsilon\gamma}a_{\gamma\beta}\left(\frac{\rho_\alpha}{\rho_{\alpha\beta}}\right)^{d-x_\upsilon-x_\alpha+x_\gamma} + \sum_\gamma a_{\beta\upsilon\gamma}a_{\alpha\gamma}\left(\frac{\rho_\beta}{r_{\alpha\beta}}\right)^{d-x_\upsilon-x_\beta+x_\gamma} . \tag{A10a}$$

From Eq. (A10) we see that asymptotic behavior of the correlations is unchanged (remember that $x_\upsilon > d$) and that the only effect of changing μ is contained in corrections to scaling and in a renormalization of the operators. Note that in the amplitude b all ρ_α, ρ_β dependence is collected together such that b itself is a constant, independent of these parameters. On the other hand, all short-wavelength behavior is contained in the renormalization of the operators [Eq. (A8)].

We next apply these results to Eq. (A3) by replacing $\mathcal{O}_\alpha$ by $\mathcal{U}$ and $\mathcal{O}_\beta$ by $\mathcal{O}_j$. Notice that the condition which arises upon $\mathcal{U}(\vec{r};\mu)$, is actually contained in expression (A9), with $\mathcal{O}_\alpha$ replaced by $\mathcal{U}$. Let us first write $\mathcal{U}$ as a linear combination of irrelevant operators $\mathcal{U}_i$ with scaling indices $x_i > d$:

$$\mathcal{U}(\vec{r};\mu) = \sum_i \omega_i(\mu)\mathcal{U}_i(\vec{r};\mu) \tag{A11}$$

then we learn from Eq. (A9) that

$$\left(d_\mu + \sum_{r<\rho_\upsilon}\mathcal{U}(\vec{r};\mu)\right)\mathcal{U}(\vec{0};\mu) = \sum_{i,i',\gamma}[\omega_i(\mu)\omega_{i'}(\mu)a_{\upsilon_i\upsilon_{i'}\gamma}(\mu)\rho_\upsilon^{d-x_i-x_{i'}+x_\gamma}+\cdots]\mathcal{O}_\gamma(\vec{0};\mu) \quad . \tag{A12}$$

Notice that all the coefficients of the relevant operators on the right-hand side of Eq. (A12) vanish in the limit $\rho_\upsilon \to \infty$, for all values of μ. In fact, by repeatedly using Eq. (A9) one is easily convinced that the same conclusion holds for the expression

$$\left(d_\mu + \sum_{r<\rho_\upsilon}\mathcal{U}(\vec{r};\mu)\right)^n \mathcal{U}(\vec{0};\mu) \tag{A13}$$

for arbitrary n. Thus we can form the generating expression

$$\exp\lambda\left(d_\mu + \sum_{r<\rho_\upsilon}\mathcal{U}(\vec{r};\mu)\right)\mathcal{U}(\vec{0};\mu) = \mathcal{U}(\vec{0};\mu+\lambda)\exp\left(\sum_{r<\rho_\upsilon}\int_\mu^{\mu+\lambda}\mathcal{U}(\vec{r};\xi)\,d\xi\right) \quad . \tag{A14}$$

The specific meaning of Eq. (A14) can be formulated by saying that if $\mathcal{U}(\vec{r};\mu)$ is to be an irrelevant perturbation for all μ, then expression (A14), which is defined for a volume of radius ρ_υ around the lattice position $\vec{0}$, will not have any marginal or relevant terms in its expansion, in the limit $\rho_\upsilon \to \infty$. In terms of correlation functions, we can write

$$\left\langle\left\{\left[\exp\left(\int_\mu^{\mu+\lambda}\sum_{r<\rho_\upsilon}\mathcal{U}(\vec{r};\xi)\,d\xi\right)-1\Big/\rho_\upsilon^d\right]\right\}\mathcal{O}_j(R;\mu)\right\rangle_\mu = c_j(\rho_\upsilon;\lambda,\mu)R^{-2x_j}+\cdots, \quad R\to\infty \quad . \tag{A15}$$

The condition for irrelevance of the operator $\mathcal{U}$ can now be formulated as

$$\lim_{\rho_\upsilon\to\infty} c_j(\rho_\upsilon;\lambda,\mu) = 0 \tag{A16}$$

for all values of λ, μ, and all subscripts j, referring to a marginal or relevant operator.

APPENDIX B: CALCULATION OF CORRELATION FUNCTIONS

In this appendix we will work out in some detail some correlation functions for the Gaussian model, using the generating function formalism of Sec. II. Let us denote a product of operators $\mathcal{O}_{n,m}(\vec{r})$ and $\mathcal{T}_{ij}(\vec{r})$ by the symbol $\mathfrak{X}$, then the expectation of $\mathfrak{X}$ is defined by

$$\langle\mathfrak{X}\rangle = \mathfrak{X}Q[W]|_{W=0} \quad , \tag{B1}$$

1. $\mathfrak{X} = \prod_i \mathcal{O}_{0,m_i}(\vec{r}_i)$

In this case, the imaginary part of $\hat{M}^T(\vec{r})$ does not contribute and we only have to consider the part $M_{vv}(\vec{r})$ [see Eqs. (2.5) and (2.6)]. Thus

$$\langle\mathfrak{X}\rangle = \exp\left[\tfrac{1}{2}\int\int d\vec{r}_1\,d\vec{r}'\,\tilde{v}^\dagger(\vec{r})M_{vv}(\vec{r}-\vec{r}')\tilde{v}(\vec{r}')\right] \tag{B2}$$

with

$$\tilde{v}(\vec{r}) = \begin{pmatrix}\tilde{v}_+\\ \tilde{v}_-\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}\sum_i m_i\delta(\vec{r}-\vec{r}_i) \quad . \tag{B3}$$

The matrix elements M_{vv} follow from Eqs. (2.10)–(2.13):

$$M_{vv}(\vec{r}) = \tfrac{1}{2}(2\pi K)^{-\tau_1}\int \frac{d^2q}{(2\pi)^2}\left(-\frac{2\pi}{q^2}B(q)\right)e^{i\vec{q}\cdot\vec{r}} \equiv \tfrac{1}{2}(2\pi K)^{-\tau_1}H(\vec{r}) \ . \tag{B4}$$

Substituting Eqs. (B3) and (B4) in Eq. (B2) leads to:

$$\langle \mathfrak{X} \rangle = \exp\left(\tfrac{1}{2}\sum_{ij} m_i m_j 2\pi K H(\vec{r}_i - \vec{r}_j)\right) \ . \tag{B5}$$

Note that the expression for $H(\vec{r})$ [Eq. (B4)] shows an infrared divergence. We can separate out this divergence by writing

$$\langle \mathfrak{X} \rangle = \exp\left[2\pi K \sum_{i>j} m_i m_j L(\vec{r}_i - \vec{r}_j) + \tfrac{1}{2}2\pi K\left(\sum_i m_i\right)^2 H(0)\right] \ , \tag{B6}$$

where

$$L(\vec{r}) = H(\vec{r}) - H(0) = \int \frac{d^2q}{2\pi}\frac{B(q)}{q^2}(1 - e^{i\vec{q}\cdot\vec{r}}) \tag{B7}$$

and

$$H(0) \to -\infty \ .$$

The potential $L(\vec{r})$ is well behaved; for small r it is zero and it becomes logarithmic for large r. It follows that the expectation (B6) vanishes, unless the total vorticity of $\mathfrak{X}$, i.e., $\sum_i m_i$ equals zero. Therefore

$$\langle \mathfrak{X} \rangle = \delta_{\sum_i m_i, 0}\exp\left[2\pi K \sum_{i>j} m_i m_j L(\vec{r}_i - \vec{r}_j)\right] \ , \tag{B8}$$

which is the desired form.[7,13] Furthermore, by choosing the cutoff function $B(q)$ as in Eq. (4.18), then Eq. (B8) describes precisely the square-lattice Gaussian model with thermally excited vortices at the (dual) lattice sites r_i.[13]

2. $\mathfrak{X} = \mathfrak{F}_{ij}(\vec{r})\mathfrak{F}_{kl}(\vec{r}')$

We start out by giving a precise definition of the operator $\mathfrak{F}_{ij}(\vec{r})$:

$$\mathfrak{F}_{ij}(\vec{r}) = 2^{(i+j)/2}(i!j!)^{1/2}\left(\frac{\delta}{\delta u^+(\vec{r})}\right)^i\left(\frac{\delta}{\delta u^-(\vec{r})}\right)^j \ . \tag{B9}$$

In this case, the important matrix element of $M^T(\vec{r})$ is $M_{uu}(\vec{r})$, which according to Eqs. (1.10)–(2.13) equals

$$M_{uu}(\vec{r}) = \int \frac{d^2q}{(2\pi)^2}e^{i\vec{q}\cdot\vec{r}}\left(-\frac{2\pi}{q^2}B(q)\right)\tfrac{1}{4}\vec{q}^2 = \tfrac{1}{2}\begin{pmatrix}\mathfrak{M}_{++}(\vec{r}) & 0\\ 0 & \mathfrak{M}_{--}(\vec{r})\end{pmatrix} \ , \tag{B10}$$

$$\mathfrak{M}_{\pm\pm}(\vec{r}-\vec{r}') = 2\frac{\partial}{\partial z_\pm}\frac{\partial}{\partial z'_\pm}L(\vec{r}-\vec{r}'), \quad z_\pm = x \pm iy \ , \tag{B11}$$

with $L(\vec{r})$ given by Eq. (B7). Note that

$$\mathfrak{M}_{\pm\pm}(\vec{r}) = \begin{cases}0 \ \text{as} \ r = 0\\ z_\pm^{-2} \ \text{as} \ r \to \infty\end{cases} \ . \tag{B12}$$

Using Eq. (B12), the average of $\mathfrak{X}$ can easily be obtained:

$$\langle \mathfrak{X} \rangle = \delta_{ik}\delta_{jl}\mathfrak{M}_{++}^{-j}(\vec{r}-\vec{r}')\mathfrak{M}_{--}^{-l}(\vec{r}-\vec{r}') \ , \tag{B13}$$

$$\langle \mathfrak{X} \rangle = \delta_{ik}\delta_{jl}|\vec{r}-\vec{r}'|^{-2(i+j)}e^{2(i-j)\Theta(\vec{r}-\vec{r}')}, \quad |\vec{r}-\vec{r}'| \to \infty \ . \tag{B14}$$

Thus the operators $\mathfrak{F}_{ij}(\vec{r})$ exactly have the properties which they are supposed to have[13]; i.e., they form a canonically normalized set with scaling–angular-momentum indices

$$x_{ij} = i + j \ , \quad (l_z)_{ij} = i - j \ , \tag{B15}$$

3. $\mathfrak{X} = \mathfrak{F}_{11}(\vec{R})\prod_i \mathfrak{O}_{0,m_i}(\vec{r}_i)$

The average of this quantity will be calculated in view of Sec. IV B, where we need the operator product coefficients $C^{\mathfrak{F}}_{\mathfrak{O}_{0,m_i}\cdots\mathfrak{O}_{0,m_4}}(\vec{r}_1\vec{r}_2\vec{r}_3\vec{r}_4)$ and $C^{\mathfrak{F}}_{\mathfrak{O}_{0,m_1}\mathfrak{O}_{0,m_2}}(\vec{r}_1\vec{r}_2)$. In keeping only the contributing matrix elements of M^T, we can write

$$\langle \mathfrak{X} \rangle = \mathfrak{F}_{11}(\vec{R})\exp\left[\tfrac{1}{2}\int\int d\vec{r}\,d\vec{r}'(\tilde{v}^\dagger M_{vv}\tilde{v} + \tilde{v}^\dagger M_{vu}u + u^\dagger M_{uv}\tilde{v})\right], \tag{B16}$$

where

$$\tilde{v}(\vec{r}) = \begin{pmatrix}\tilde{v}_+\\ \tilde{v}_-\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}\sum_i m_i\delta(\vec{r}-\vec{r}_i) \ . \tag{B17}$$

The $\tilde{v}$-$\tilde{v}$ part in Eq. (B16) will lead to a factor which is given by Eq. (B8), together with the charge-neutrality condition ($\sum_i m_i = 0$). Furthermore, according to Eqs. (2.10)–(2.13)

$$M_{uv}(\vec{r}) = \int \frac{d^2q}{(2\pi)^2}e^{i\vec{q}\cdot\vec{r}}\left(-\frac{2\pi}{q^2}\right)B(q)\tfrac{1}{2}i\vec{q}(2\pi K)^{-\tau_1/2} \ , \tag{B18}$$

which can be written as

$$M_{uv}(r) = \tfrac{1}{2}\begin{pmatrix}\mathfrak{M}_+ & 0\\ 0 & \mathfrak{M}_-\end{pmatrix}(2\pi K)^{-\tau_1/2} \ ,$$

$$\mathfrak{M}_\pm(\vec{r}) = 2\frac{\partial}{\partial 3_\pm}L(\vec{r}) \to \begin{cases}0 \ \text{as} \ r = 0\\ 3_\pm^{-1} \ \text{as} \ r \to \infty\end{cases} \ , \tag{B19}$$

with $L(\vec{r})$ given by Eq. (B7). Using Eq. (B19) and the equality

$$(2\pi K)^{-\tau_1/2}\tilde{v}(\vec{r}) = (2\pi K)^{1/2}\begin{pmatrix}1\\ -1\end{pmatrix}\sum_i m_i\delta(\vec{r}-\vec{r}_i) \tag{B20}$$

we find

$$\frac{1}{2}\int(\tilde{v}^{\dagger}M_{vu}u+u^{\dagger}M_{uv}\tilde{v})\,d\vec{r}\,d\vec{r}\,'=\frac{1}{2}(2\pi K)^{1/2}\sum_i\int d\vec{r}\,m_i[u_+(\vec{r})\mathfrak{M}_+(\vec{r}-\vec{r}_i)-u_-(\vec{r})\mathfrak{M}_-(\vec{r}-\vec{r}_i)]\ . \quad \text{(B21)}$$

Using Eq. (B8) for the v-v part and using Eq. (B21) we finally get for Eq. (B16):

$$\langle X\rangle=-\pi K\delta_{\sum_i m_i,0}\exp\left(\sum_{i>j}m_im_j2\pi KL(\vec{r}_i-\vec{r}_j)\right)\left(\sum_i m_i\mathfrak{M}_+(\vec{R}-\vec{r}_i)\right)\left(\sum_j m_j\mathfrak{M}_-(\vec{R}-\vec{r}_j)\right)\ . \quad \text{(B22)}$$

Equation (B22) can be used to obtain the operator product coefficient

$$C^{\mathfrak{F}_{11}}_{\mathfrak{O}_{0,m_1}\cdots\mathfrak{O}_{0,m_i}}(\vec{r}_1\vec{r}_2\cdots\vec{r}_i)=\lim_{R\to\infty}R^4\langle\prod_i\mathfrak{O}_{0,m_i}(\vec{r}_i)\mathfrak{F}_{11}(\vec{R})\rangle\ . \quad \text{(B23)}$$

By expanding the $\mathfrak{M}_{\pm}(\vec{R}-\vec{r}_i)$ to first order in r_i in Eq. (B22) one obtains

$$C^{\mathfrak{F}_{11}}_{\mathfrak{O}_{0,m_1}\cdots\mathfrak{O}_{0,m_i}}(\vec{r}_1\vec{r}_2\cdots\vec{r}_i)=-\pi K\delta_{\sum_i m_i,0}\left(\sum_i m_i\vec{r}_i\right)^2\exp\left(\sum_{i>j}m_im_j2\pi KL(\vec{r}_i-\vec{r}_j)\right)\ . \quad \text{(B24)}$$

Equation (B24) is used to obtain the mapping function [Eq. (4.34)] in Sec. IV B.

---

*Present address: Institut für Theoretische Physik der Universität, Philosophenweg 19, 6900 Heidelberg, Federal Republic of Germany.

[1]J. Ashkin and E. Teller, Phys. Rev. 69, 178 (1943).
[2]R. Baxter, Phys. Rev. Lett. 26, 832 (1971).
[3]V. L. Berezinskii, Zh. Eksp. Teor. Fiz. 59, 907 (1970) [Sov. Phys. JETP 32, 493 (1971)]; 61, 1144 (1971) [Sov. Phys. JETP 34, 610 (1971)].
[4]J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).
[5]J. M. Kosterlitz, J. Phys. C 7, 1046 (1974).
[6]J. Villain, J. Phys. (Paris) 36, 581 (1975).
[7]J. V. Jose, L. P. Kadanoff, S. K. Kirkpatrick, and D. R. Nelson, Phys. Rev. B 16, 1217 (1977).
[8]S. T. Chui and J. D. Weeks, Phys. Rev. B 14, 4978 (1976).
[9]H. Van Beyeren, Phys. Rev. Lett. 38, 993 (1977).
[10]H. J. F. Knops, Phys. Rev. Lett. 39, 766 (1977).
[11]J. M. Kosterlitz, J. Phys. C 10, 3753 (1977).
[12]L. P. Kadanoff, J. Phys. A 11, 1399 (1978).
[13]L. P. Kadanoff, Ann. Phys. 120, 39 (1979).
[14]A. Luther and I. Peschel, Phys. Rev. B 12, 3908 (1975).
[15]M. Bander and J. Richardson (unpublished).
[16]L. P. Kadanoff and A. C. Brown, Ann. Phys. 121, 318 (1979).
[17]K. Wilson, Rev. Mod. Phys. 47, 773 (1975).
[18]L. P. Kadanoff and F. Wegner, Phys. Rev. B 4, 3989 (1971).
[19]L. P. Kadanoff, Phys. Rev. Lett. 39, 903 (1977). The following sentence (between brackets) on p. 905 is wrong: "(To get a line of fixed points, one must allow for a second term in $K-4$.)"
[20]F. Wegner, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. 6.
[21]H. J. F. Knops (University of Nymegen, 1978) (unpublished).
[22]P. B. Wiegman, J. Phys. C 11, 1583 (1978).
[23]D. S. Amit, Y. Y. Goldschmidt, and G. Grinstein, J. Phys. A 13, 585 (1980).
[24]A. P. Young, J. Phys. C 11, 453 (1978).
[25]A. M. M. Pruisken and A. C. Brown (unpublished).