PHYSICAL REVIEW B VOLUME 22, NUMBER 9 1 NOVEMBER 1980

Critical behavior of the four-dimensional Ising model

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(Received 15 February 1980)

The phase transition of the spin-one-half, four-dimensional hypercubic Ising model has been investigated by means of the Monte Carlo renormalization-group technique. The results agree with mean-field critical exponents.

I. INTRODUCTION

While the four-dimensional (4D) Ising model is not directly applicable to real magnetic systems, it is useful to investigate the influence of dimensionality on phase transitions. A comparison of results for the discrete hypercubic model to those from continuous models may shed some light on problems concerning universality and ϵ expansions. Further, the four-dimensional Ising model is of interest for the Z_2 lattice gauge theory[1] in elementary-particle physics.

The four-dimensional simple hypercubic Ising lattice (the 4D analog of the simple cubic lattice, with one particle per unit cell) has already been investigated by high-temperature series expansions. Fisher and Gaunt[2] derived terms up to 11th order in the nearest-neighbor coupling, from which the transition was estimated to be at $J/kT_c = 0.29976$. The critical exponent for the divergence of the susceptibility ($\chi \sim |T - T_c|^{-\gamma}$) was estimated as $\gamma = 1.094 \pm 0.0025$. Another estimate of the critical coupling from series expansions was given by Moore.[3] He obtained $J/kT_c = 0.29962 \pm 0.00013$, close to the value of Fisher and Gaunt.[2] He also provided estimates for the critical exponents α, γ, and ν for the temperature dependence of the heat capacity, susceptibility, and correlation length, respectively ($\alpha = -0.12 \pm 0.03$, $\gamma = 1.065 \pm 0.003$, $\nu = 0.536 \pm 0.003$). However, the series of Moore were also consistent with a set of mean-field type exponents ($\alpha = 0$, $\gamma = 1$, $\nu = 0.5$) if logarithmic correction factors were allowed. Unfortunately, the increase in the number of parameters that had to be fitted to the series expansions, was found to inhibit an accurate determination of the exponents in that case. An approximation using an expansion of γ in the inverse number of dimensions by Abe[4] suggested that γ approached 1 continually with increasing dimensionality, which was in qualitative agreement with results of Fisher and Gaunt.[2] Also Baker[5] found, using series expansions, that the critical behavior in four dimensions was nonclassical. Further, he concluded that his results did not support hyperscaling.

However, most recent results are in favor of mean-field exponents. It is interesting to note that for $d < 4$ dimensions, molecular-field theory gives internally inconsistent results[6] for the influence of fluctuations at the transition. For $d \geqslant 4$, these inconsistencies disappear, allowing for the possibility that mean-field theory gives an accurate description of the critical behavior. Kadanoff *et al.*[7] have used an analytic renormalization-group (RG) method (one-hypercube approximation) to derive the critical exponents of the four-dimensional hypercubic Ising model. Their results are close to the mean-field predictions.

Further results have been derived from related Hamiltonians. Starting from a Landau-Ginzburg Hamiltonian (continuous in space as well as in the length of the spin) Wilson[8] found, using renormalization-group transformations, that the critical behavior was determined by the Gaussian fixed point; i.e., the critical behavior is essentially molecular-field-like. This result was confirmed and supplemented by calculations of Larkin and Khmelnitski[9] and of Wegner and Riedel[10] who showed that the existence of a marginal eigenvalue at the critical dimensionality $d = 4$ leads to a modification of the classical critical behavior of the specific heat and the susceptibility by correction factors $|\ln|T - T_c||^{1/3}$. As already noted by Moore,[3] these correction factors could account for the difference between series results and mean-field theory. Recent series expansion results of McKenzie *et al.*[11] and Gaunt *et al.*[12] are in agreement with classical critical behavior modified by such correction factors. Further, these authors did not observe (using longer series than were available to Baker[5]) any apparent violations of hyperscaling. Monte Carlo (MC) simulations have also been found to be consistent with the presence of logarithmic corrections.[13]

A further confirmation that the four-dimensional Ising model plays the role of a borderline case between molecular field and nonclassical critical behavior is given by Knops, van Leeuwen and Hemmer.[14] Their position-space RG treatment included weak long-range interactions and showed that the infinite-range fixed point is attractive for $d > 4$.

II. RESULTS

We have applied the Monte Carlo—renormalization-group (MCRG) technique to the spin-one-half four-dimensional simple hypercubic Ising model. MCRG calculations were first described by Ma.[15] The MCRG technique used here derives critical exponents, which are related to the eigenvalues of the linearized RG transformation, from correlations which are obtained by MC simulations. The principle of this MCRG method is described elsewhere.[16] The method has previously been applied to the two- and three-dimensional Ising models[16–18] with results that are in good agreement with values for the critical exponents obtained by other methods. It was shown in Ref. 17 that a RG transformation with a linear scale factor 2 and a modified majority rule (using a "tie breaker") works well in two dimensions. Therefore, a similar RG transformation was applied to the four dimensional model. The critical parameter $J/kT_c = 0.29962$ was taken from Moore.[3] Although the MCRG method itself can be used to estimate the transition temperature, the error margins of Moore's result are smaller than the present MCRG calculations on a 12^4 lattice can produce. The interactions used in analysis were divided into an odd and an even group as explained in Ref. 19. Even interactions used in the present MCRG analysis are between nearest neighbors, 2nd neighbor, 3rd neighbors, and 4th neighbors. Odd interactions are: one spin, and three spin (corresponding to a 90° angle) interaction. The lattice size permitted two RG transformations, yielding a 3^4 lattice, still allowing for a large number of interactions.

Results for the eigenvalue exponent $y_{o,e} = \ln\lambda_{o,e}/\ln l$ are shown in Table I, together with results for mean-field theory. The subscripts o,e stand for the odd and even interaction subspaces, λ stands for a relevant eigenvalue of the linearized RG transformation matrix, and l is the linear scale factor of the RG transformation. The MCRG results in Table I show a clear trend towards mean-field exponents.

The accuracy of our data is determined by statistical errors, finite size effects, uncertainty in the transition temperature, convergence with increasing number of interactions included in the analysis, and convergence with increasing number of RG iterations.

(i) The statistical accuracy was estimated from a comparison with other MCRG runs using different initial configurations. These comparisons suggested that the statistical error in the eigenvalue exponents is about 0.1 for the second RG step, which is more than the difference with the mean-field eigenvalue exponents. Time dependent correlations further showed that the relaxation time of the system was very short (about 15 MC steps per site for the nearest-neighbor correlation function of a 12^4 lattice) in comparison with the length of the calculation.

(ii) Finite size effects were estimated by comparing the results in Table I with those from a calculation on a 6^4 lattice. The first RG step yielded, within statistical error, the same results so that finite size effects probably are unimportant.

(iii) The inaccuracy as a consequence of the uncertainty in the transition temperature was inferred from a MCRG calculation at $J/kT = 0.294$, 2% above the transition temperature. The eigenvalue ex-

TABLE I. Eigenvalue and critical exponents for the four-dimensional, spin-one-half Ising model. Results for a MCRG calculation of 6000 MC steps per site (after 600 steps per site to allow the system to reach equilibrium) on a 12^4 lattice are presented as even (temperaturelike) and odd (magnetic) eigenvalue exponents y_e and y_o. The calculation was performed at the critical coupling as given by Moore (Ref. 3). The number of digits in this table is not meant as an indication of the accuracy.

RG step	Number of interactions	y_e	α	ν	y_o	δ	η
1	1	1.57	−0.55	0.64	2.97	2.87	0.06
	2	1.64	−0.44	0.61	2.96	2.84	0.08
	3	1.63	−0.45	0.61			
	4	1.63	−0.45	0.61			
2	1	2.00	0.00	0.50	3.06	3.28	−0.13
	2	2.04	0.04	0.49	2.96	2.83	0.09
	3	2.04	0.04	0.49			
	4	2.04	0.04	0.49			
Mean field		2	0	0.5	3	3	0

ponents for the second RG step decreased by about 0.4 (even) and 0.2 (odd). This indicates that the error as a consequence of the uncertainty in and between the values of the critical parameter as given by Fisher and Gaunt,[2] Moore,[3] and Gaunt *et al.*[12] is probably unimportant.

(iv) The effect of the number of interactions included in the analysis can be seen from Table I. In accordance with MCRG results[17,18] on analogous, lower dimensional Ising systems using the same type of RG transformation, we observe that only a very small number of interactions are important in this case. The accuracy of the entries in Table I is therefore mainly determined by statistics, and the error in the eigenvalue exponents is about 0.1.

Actually, the rapid apparent convergence of the MCRG data to the classical values is somewhat surprising. A marginal operator should be present at the critical dimensionality giving slow convergence toward the mean-field exponents and logarithmic corrections to scaling.[9,10] We do not see this effect, which suggests that the component of the 4D nearest-neighbor Ising Hamiltonian in the direction of the marginal field is relatively small. This is in basic agreement with calculations of Kadanoff *et al.*[7] for the 4D hypercubic Ising model, which yielded a fixed point very close to the nearest-neighbor Hamiltonian. In such a case, i.e., when the marginal scaling field is small, one may still expect to find "effectively" rapid convergence (the effects of slow convergence are smaller than the statistical error).

Thus the accuracy in our results is mainly determined by statistics and by the fact that only two RG transformations could be performed together with the possibility of slow convergence. The inaccuracy is tentatively estimated as 0.1 in the eigenvalue exponents. We conclude that our results agree with a small marginal field and the classical set of critical exponents.

ACKNOWLEDGMENTS

The authors are indebted to Dr. V. J. Emery, Dr. M. Creutz, and Professor J. M. J. van Leeuwen for some valuable and interesting discussions. This research was supported by the Division of Basic Energy Sciences, U.S. DOE, under Contract No. EY-76-C-02-0016.

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