Ground-state correlations and universality in two-dimensional fully frustrated systems

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The spin-spin correlation function in the ground state of some two-dimensional fully frustrated spin models are investigated. The correlations are long ranged and decay as $1/r^{\eta}$, where r is the distance between the two spins and $\eta = \frac{1}{2}$. This result is true for both the square and triangular fully frustrated lattices, thus suggesting that these models form a universality class. It is shown that the fully frustrated Ising model on a square lattice can be mapped exactly into a special Baxter model. The spin-spin correlation function of this Baxter model can be calculated exactly. The ground state of the fully frustrated square Ising model corresponds to the free fermion point of the F model and therefore, through duality, to the decoupling point of the critical Baxter line. The relationship between the F model above the critical temperature and the Gaussian model above the multicritical point is discussed.

I. INTRODUCTION

Recent interest in spin-glasses has led to the investigation of the so-called fully frustrated periodic systems. In the case of discrete spins these are models with competing ferromagnetic and antiferromagnetic interactions. The distribution of these interactions however instead of being completely random is periodic. In two dimensions the interactions form a periodic pattern of the kind shown in Fig. 1. Although these models in two dimensions do not show a spin-glass transition at finite temperature, they show frustration effects[1,2] which are believed to be essential for spin-glasses and also their ground state has the character of spin-glasses.

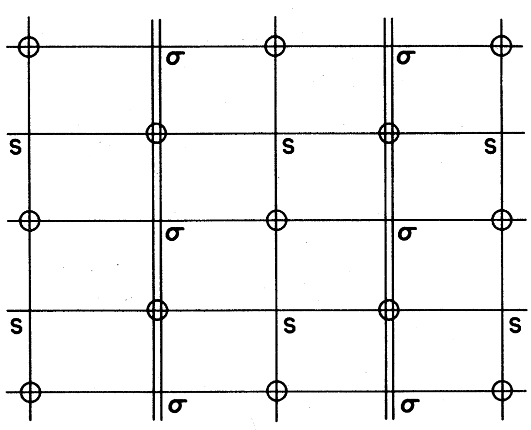

FIG. 1. The "fully frustrated" Ising model (FFSI) on the square lattice. The σ spins are surrounded by two ferromagnetic and two antiferromagnetic bonds, while the s spins are surrounded by four ferromagnetic bonds. The spins denoted by circles are integrated out.

Because of this last property, in this paper we investigate more carefully the correlations in the ground state of several fully frustrated periodic models. It is well known since the work of Stephenson[3] that the spin-spin correlation function of the two-dimensional antiferromagnetic Ising model on a triangular lattice (ATI) at zero temperature (but only there) has a power-law behavior, reminiscent of critical-point correlation functions. The ATI model is a fully frustrated periodic model, that is at $T=0$ (T is the temperature) there are unsatisfied bonds in each unit cell. This leads to an infinitely degenerate ground state in the thermodynamic limit with finite entropy per site.[4]

An interesting question which one would like to answer is whether the power-law decay of the spin-spin correlation function is the property of only the ATI model, or it is also true for other frustrated periodic models. This question was raised by several authors. Andre *et al.*[5] conjectured that this is true also for the model introduced by Villain.[6] Monte Carlo studies[7] also suggest the power-law behavior. However none of the above works gave the concrete value of the η exponent characterizing the power-law decay of the correlation functions. Recently Southern *et al.*[8] showed that Villain's fully frustrated Ising model on a square lattice (FFSI) can be mapped into a special Baxter model. From this, they concluded that the $T=0$ point of the FFSI model corresponds to the F-model limit of the critical Baxter line. On the basis of the results of Barber and Baxter[9] Southern *et al.* conjectured that $\eta=\frac{1}{4}$ for the FFSI model being different from the result for the ATI model where $\eta=\frac{1}{2}$. This would suggest that critical exponents in the case of frustrated periodic systems do depend on the lattice structure thus naive universality is not valid for these systems. There are also other indications that in the case of fully frustrated systems the lattice structure might be important. In three dimensions the antiferromagnetic fcc lattice has a first-order transition,[10] while the fully frustrated simple cubic lattice seems to have a higher-order transition.[11]

In the present paper we analyze the spin-spin correlations in two dimensions more carefully and show that $\eta=\frac{1}{2}$ also for the FFSI model at $T=0$. The result $\eta=\frac{1}{2}$ will be obtained in two different ways. First, the mapping of the FFSI model into a special Baxter model will lead to the identification of the $T=0$ point of the FFSI model with the free-fermion point[12] of the F model. This point is equivalent through duality to the decoupling point of the critical Baxter line.[13] From this duality one obtains that the spin-spin correlation function of the FFSI model at $T=0$ maps into the square of the spin-spin correlation function of the pure Ising model at the critical point. Since the latter correlation function has $\eta=\frac{1}{4}$, we obtain the result $\eta=\frac{1}{2}$ for the FFSI model.

We can also calculate the correlation function of the FFSI model by connecting it with another model originally introduced by Vaks, Larkin, and Ovchinnikov.[14] For this model the correlation function was obtained exactly for all temperatures. This connection will allow us to investigate the vicinity of the $T=0$ point of the FFSI model and will help to understand the special nature of this point. Finally, the relationship between the FFSI model and a special Baxter model gives the exact correlation function of the Baxter model along a particular line in the space of coupling constants.

The value of η at the critical point of the F model is known exactly due to Barber and Baxter.[9] They obtained $\eta=\frac{1}{4}$. At the free fermion point of the F model we get $\eta=\frac{1}{2}$. The F model has long ranged correlations for any temperature above the critical point,[12,15] so η varies continuously with the temperature just like in the Gaussian model.[16] The critical point of the F model can be shown to be equivalent to the multicritical point of the Gaussian model.[17] Using the results of Kadanoff and Brown[18] and those of Knops[19] one can derive a mapping function connecting the coupling constant of the F model with the coupling constant of the Gaussian model.

The paper is organized as follows. In Sec. II we recapitulate the available exact information about the FFSI model and show how this model can be mapped into a particular Baxter model. In Sec. III we analyze the ground state of the FFSI model in terms of this Baxter model and identify this ground state with the free fermion point of the F model. By performing a duality transformation the correlation function of the FFSI model at $T=0$ will be mapped into the square of the pure Ising model correlation function at the critical point. The relationship between the F model and the Gaussian model is discussed also in this section. In Sec. IV we consider another model on a union jack lattice, which contains the FFSI model as a special case and for which the two-spin correlation function is known exactly. In Sec. V we analyze the correlation function of the newly introduced model around $T=0$. In Sec. VI we summarize our results.

II. FFSI MODEL

The FFSI model is shown in Fig. 1. Single lines denote ferromagnetic bonds $J>0$, while double lines denote antiferromagnetic bonds with absolute value equal to J. This model was introduced by Villain[6] in order to investigate frustrated systems without randomness. It was shown that this model does not have a phase transition at any finite temperature. The free energy is an analytic function for any nonzero temperature. The ground state of the FFSI model however is infinitely degenerate in the thermodynamic limit (all the spins denoted by σ in Fig. 1 are "loose" at $T=0$, the energy does not change if one flips any of them). The entropy per site is $S=G/\pi=0.2916$,[5] where G is the catalan constant. The singular part of the free energy, f_s behaves like $e^{-4J/kT}$ at low temperatures[5] (here k is the Boltzmann constant). These results can be compared with those obtained for the ATI model, where $S=0.323\,06$ (Ref. 20) and f_s has the same low-temperature behavior as above.[21] In the case of the ATI model the two-spin correlation function was also calculated exactly for both $T=0$ (Ref. 3) and $T\neq 0$.[22] At $T=0$ the correlation function decays as $r^{-\eta}$, where r is the separation between the two spins and $\eta=\frac{1}{2}$.[3] For T close to zero, the correlation function has the asymptotic form $e^{-r/\xi}$ where $\xi\simeq e^{2J/kT}$ is the correlation length.[22] If one defines a reduced exponent like[23] $f_s\sim\xi^{-(2-\alpha)/\nu}$ then one gets that $2-\alpha=2\nu$ for the ATI model. This implies hyperscaling and it will be shown to be satisfied also for the FFSI model.

Let us now try to determine the behavior of the two-spin correlation function of the FFSI model. In principle this can be expressed as a Toeplitz determinant, but in the present case this is very tedious. Instead, we will map the FFSI model into other models and use the known information about those models. In this section we map the FFSI model into a special eight-vertex (8V) model. This is achieved by a decimation transformation. By summing the spins at lattice sites denoted by circles in Fig. 1 we get two interpenetrating Ising sublattices (formed by the σ and s spins) which are coupled through a four-spin interaction (K_4). The spins on the individual sublattices are coupled through a nearest-neighbor interaction (K_2), but because of the special distribution of the positive and negative bonds in the original lattice there is no two-spin interaction, coupling spins on different sublattices. One then recovers the 8V model as formulated by Kadanoff and Wegner.[24] One obtains

$$Z_F(N;K)=e^{(N/2)A}Z_{8V}(N/2;K_2,K_4) \quad . \tag{1}$$

Here Z_F and Z_{8V} are the partition functions of the FFSI and 8V models, respectively, N is the number of lattice sites in the FFSI model $K=J/kT$ and

$$A=\tfrac{1}{2}\ln\cosh 2K+\tfrac{1}{8}\ln\cosh 4K \quad , \tag{2}$$

$$K_2=\tfrac{1}{8}\ln\cosh 4K \quad , \tag{3}$$

$$K_4=K_2\left[1-4\frac{\ln\cosh 2K}{\ln\cosh 4K}\right]=\lambda K_2 \quad . \tag{4}$$

Our 8V model is special in that K_2 and K_4 are not independent. In the (K_2,λ) plane equations (3) and (4) give a curve which is depicted by a solid line in Fig. 2. The dashed curve of Fig. 2 corresponds to the critical Baxter line and is given by

$$e^{-2K_4}=\sinh 2K_2 \quad . \tag{5}$$

Now, the $T=0$ ($K=\infty$) point of the FFSI model corresponds to the $K_2=\infty$, $\lambda=-1$ of the 8V model, which is the F (Ref. 9) model limit of the 8V model. The ground state of the 8V model becomes infinitely degenerate at its F model limit. In the next section we will show however that along the solid line of Fig. 2, in the limit $T=0$ one reaches not the critical point of the F model, but rather the free-fermion point[12] of the F model. Approaching the $K_2=\infty$, $\lambda=-1$ point along the critical Baxter line one reaches the critical point of the F model.

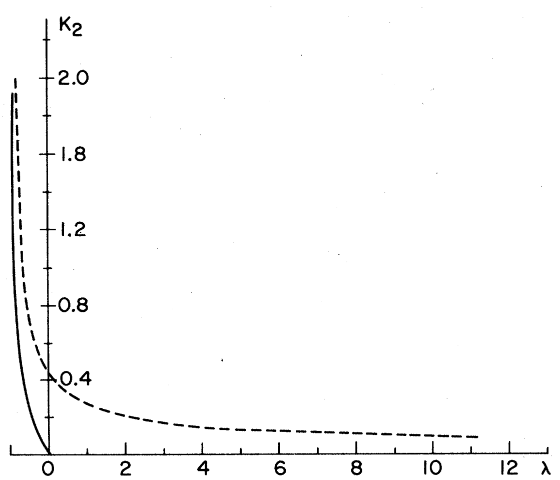

FIG. 2. The dotted line is the Baxter line, or critical line of the 8V model. The full line gives Eq. (4).

It is clear that under the decimation transformation the correlation functions $G_F^{\sigma\sigma}(K), G_F^{ss}(K)$ between two σ or two s spins in the FFSI model map, respectively, into the correlation functions $G_{8V}^{\sigma\sigma}(K_2,K_4)$, $G_{8V}^{ss}(K_2,K_4)$ of the 8V model between spins on the same sublattice.

III. EQUIVALENCE BETWEEN THE GROUND STATE OF THE FFSI MODEL AND THE $\Delta=0$ POINT OF THE F MODEL

The 8 vertices of the 8V model are show in Fig. 3. In the usual parametrization[9,13] the model solved by Baxter corresponds to the following choice of the ω_i weights:

$$\begin{aligned}&\omega_1=\omega_2=a \quad , \quad \omega_3=\omega_4=b \quad ,\\ &\omega_5=\omega_6=c \quad , \quad \omega_7=\omega_8=d \quad .\end{aligned} \tag{6}$$

If $d=0$ and the weights are normalized in such a way that $c=1$, the 8V model reduces to the F model.[19] In order to analyze the $K_2=\infty$, $\lambda=-1$ point of the solid line of Fig. 2, in terms of the weights, we recall the correspondence between a, b, c, d, and K_2,K_4,

$$a=b=Ae^{-K_4} \quad , \tag{7}$$

$$c=Ae^{2K_2+K_4} \quad , \tag{8}$$

$$d=Ae^{-2K_2+K_4} \quad . \tag{9}$$

Here we used the parametrization given in Ref. 9. Since in our case the two-spin interactions on the two sublattices are equal, we have $a=b$. In the above formulas A is constant setting the normalization of the weights. In the limit $K_2\to\infty$, $\lambda\to-1$ the equation of the Baxter line, Eq. (5) reduces to

$$K_2+K_4=\tfrac{1}{2}\ln 2 \quad . \tag{10}$$

Putting this into Eqs. (7)–(9), we get

$$\lim_{K_2\to\infty} a/c=\tfrac{1}{2} \quad , \quad \lim_{K_2\to\infty} d=0 \tag{11}$$

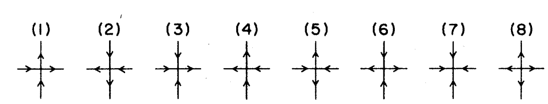

FIG. 3. Vertices in 8V model. In the F-model limit one does not have vertices (7) and (8).

which corresponds to the critical point of the F model.[12] Equations (3) and (4) in the limit $K \to \infty$ lead to

$$K_2 + K_4 = \tfrac{1}{4}\ln 2 \tag{12}$$

which when inserted into Eqs. (7)–(9) gives

$$\lim_{K_2\to\infty} a/c = 1/\sqrt{2}\ , \quad \lim_{K_2\to\infty} d = 0\ . \tag{13}$$

In Fig. 4 the critical Baxter line (dotted line) and the line corresponding to the FFSI model (solid line) are given in terms of the weights. Moving along the $d/c = 0$ line corresponds to varying the temperature in the F model. In terms of the parameter Δ, introduced by Lieb,[12] $\Delta = -1$ is the critical point of the F model, while at the point given by Eq. (13) $\Delta = 0$.[12] We thus conclude, that the ground state of the FFSI model is equivalent to the $\Delta = 0$ point of the F model. At the $\Delta = 0$ the Baxter weights satisfy the

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 \tag{14}$$

free-fermion[12] condition. As was shown by Hurst and Green[25] the most general 8V model can be solved exactly if Eq. (14) is satisfied. Using the exact result of Villain[6] for Z_F in Eq. (1) at $T = 0$ and the result of Lieb for the partition function of the F model at $\Delta = 0$,[12] one can verify that Eq. (1) (where Z_{8V} is replaced by the partition function of the F model) holds.

Now let us calculate the correlation function of the FFSI model at $T = 0$. As has been pointed out at the end of the previous section $G_F^{\sigma\sigma}(K)\,[G_F^{ss}(K)]$ is mapped into $G_{8V}^{\sigma\sigma}(K_2,K_4)$ $[G_{8V}^{ss}(K_2,K_4)]$. By performing a duality transformation[26] on the $\sigma(s)$ sublattice of the 8V model we can map $G_{8V}^{ss}(K_2,K_4)$ $[G_{8V}^{\sigma\sigma}(K_2,K_4)]$ into $G_{AT}^{ss}(L_1,L_2;L_4)$ $[G_{AT}^{\sigma\sigma}(L_1,L_2;L_4)]$. Here G_{AT} is the correlation function of the Ashkin-Teller (AT) model, L_1 and L_2 are the two spin couplings, and L_4 is the four spin coupling of the AT model. It is shown in the Appendix that the spin-spin correlation function of the FFSI model can be transformed into the polarization operator-polarization operator correlation function of an AT model. $T = 0$ of the FFSI model will correspond to the decoupling point of this AT model. At this point the polarization-operator–polarization-operator correlation function is equivalent to the square of the spin-spin correlation function of the Ising model at the critical point. By this we immediately get the result that

$$G_F^{ss}(r,T=0)\ [\text{or } G_F^{\sigma\sigma}(r,T=0)] \underset{r\to\infty}{\longrightarrow} \frac{\text{const}}{r^{1/2}}\ . \tag{15}$$

We now know the value of η at $\Delta = -1$ and $\Delta = 0$ along the F model. At $\Delta = -1$ the exact result of Ref. 9 gives $\eta = \frac{1}{4}$, while at $\Delta = 0$ we have just obtained $\eta = \frac{1}{2}$. η changes continuously along the F model line above the critical point. It is well known that the situation is similar in the Gaussian model[16] and it was shown[18,19] that the AT, 8V, and F models belong indeed to the same universality class as the Gaussian model provided the coupling constants of these models are appropriately mapped into the coupling constant of the Gaussian model. These mapping functions for the AT and 8V model were given by Kadanoff and Brown.[18] Connecting the Gaussian model directly to the F model[19] leads to

$$K_F = \tfrac{1}{2}\ln 2(1-\Delta) = \ln 2\sin(\tfrac{1}{4}\pi^2 K_G)\ . \tag{16}$$

Here K_F (Ref. 12) and K_G are the coupling constants of the F and Gaussian models, respectively. This means that, for example, η of the Gaussian model at K_G is the same as η of the F model at K_F given by Eq. (16). The multicritical point ($K_G = 2/\pi$) of the Gaussian model[17] corresponds to the critical point ($K_F = \ln 2$) of the F model.

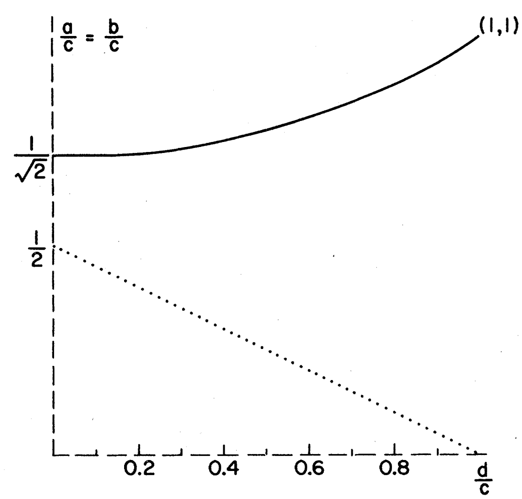

FIG. 4. The critical Baxter line (dotted), the FFSI model (solid line), and the F-model line (the a/c axis).

IV. MODEL OF VAKS, LARKIN, AND OVCHINNIKOV

Consider the model shown in Fig. 5. Spins τ, σ, s are Ising spins on a union-jack lattice. The spins are coupled to each other through two different two-spin couplings. The dashed interaction is denoted by J_1, the solid is denoted by J_2. The τ spins have 4 nearest neighbors, the s and σ spins (which are completely equivalent and are denoted by different symbols because of later convenience) have 8 nearest neighbors. This model was introduced and solved exactly by Vaks, Larkin, and Ovchinnikov for arbitrary J_1 and J_2.[14] We will refer to this model as the

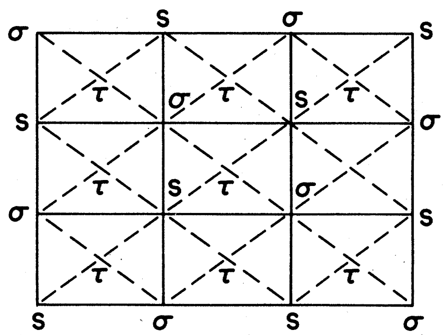

FIG. 5. The VLO model. Dashed lines correspond to J_1, solid lines to J_2. Decimation is performed over the τ spins.

VLO model. They obtained the phase diagram shown in Fig. 6.

In region AF the system is antiferromagnetic, in region FE ferromagnetic, and in region P paramagnetic. The equations of the phase boundaries (solid lines in Fig. 6) are given by[14]

$$(y+1)^2(x^2+1)^2-2(1-x^2)^2=0 \tag{17}$$

and

$$(y+1)^2(x^2+1)^2-2y^2(1-x^2)^2=0 \tag{18}$$

for the ferromagnetic-paramagnetic and antiferromagnetic-paramagnetic transitions, respectively. Here $x=\tanh J_1/kT$, $y=\tanh J_2/kT$. The phase diagram is symmetric under the change of the sign of x. We will consider the case $x \geqslant 0$.

The thermodynamic properties (entropy, specific heat, etc.) of the VLO model can be all calculated exactly using the results of Ref. 14. In Ref. 14 the two-spin correlation function $G(r)$ along the diagonal direction was also calculated exactly with the result

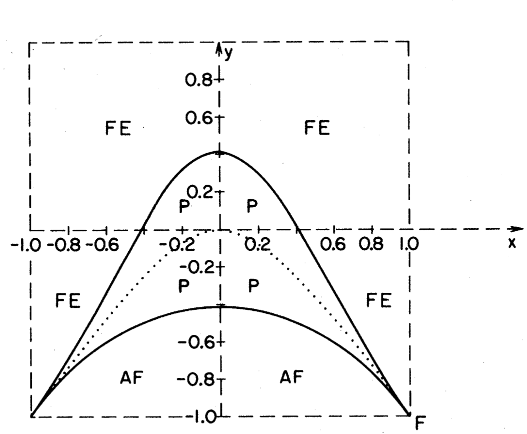

FIG. 6. The phase diagram of the VLO model. The F point is mapped under decimation into the $K_2=\infty$, $\lambda=-1$ point of Fig. 2.

that $G(r) \sim r^{-1/4}$ for large values of r along the whole phase boundary. However as we shall see in the next section this result is not true at the special point F where $-y=+x=1$.

Let us investigate the relationship between the FFSI and VLO models. First of all if $J_2=-J_1<0$ then in each elementary triangle there is one bond which cannot be satisfied, so we get another periodic fully frustrated model. The spins s and σ are "loose" in the ground state and the system has macroscopic entropy in this state. To see the relationship explicitly between the FFSI and VLO models let us perform again a decimation transformation and integrate out the τ spins. The decimation does not involve J_2. After the transformation we get the following interactions:

$$K_{ss}=K_{\sigma\sigma}=\frac{1}{8}\ln\cosh 4\left[\frac{J_1}{kT}\right] , \tag{19}$$

$$K_{s\sigma}=\frac{J_2}{kT}+\frac{1}{4}\ln\cosh 4\left[\frac{J_1}{kT}\right] , \tag{20}$$

$$K_{s\sigma s\sigma}=K_{ss}\left[1-4\frac{\ln\cosh 2J_1/kT}{\ln\cosh 4J_1/kT}\right] . \tag{21}$$

Here $K_{ss}(K_{\sigma\sigma})$, $K_{s\sigma}$, and $K_{s\sigma s\sigma}$ denote the interactions (divided by the Boltzmann constant times the temperature) between two closest $s(\sigma)$ spins, two nearest-neighbor s and σ spins, and four spins belonging to the same square (plaquette), respectively. Now, comparison of Eqs. (19) and (21) with Eqs. (3) and (4) shows that if $K_{s\sigma}=0$ we get the same 8V model which was obtained from the FFSI model in the previous section (with $K=J_1/kT$). Consequently if we choose J_2 in such a way that $K_{s\sigma}=0$ the VLO model becomes equivalent to the FFSI model. $K_{s\sigma}=0$ defines a line in Fig. 6, which is given by the dotted line. Introducing $M_1=J_1/kT$ and $M_2=J_2/kT$, in the limit $M_1 \rightarrow \infty$ the condition $K_{s\sigma}=0$ leads to

$$M_2 \underset{L_1\rightarrow\infty}{=} -M_1+\frac{1}{4}\ln 2 . \tag{22}$$

We see that the ratio $K_{s\sigma s\sigma}/K_{ss}$ goes to -1 as $M_1 \rightarrow \infty$ but to recover the FFSI model we have to take this limit according to Eq. (22). So in order to obtain information about the $T=0$ correlation function of the FFSI model using the exact results of the VLO model we have to approach the point F (where $-M_2=+M_1=\infty$, $-y=+x=1$) in Fig. 6 choosing a path which in the limit $M_1 \rightarrow \infty (x \rightarrow 1)$ reduces to Eq. (22). It is easy to see that the curves given by Eqs. (17) and (18) in the limit $x \rightarrow 1$ give exactly Eq. (22). This means we will have to approach the F point along the phase boundary in Fig. 6. (Since the phase boundaries separating the ferromagnetic, antiferromagnetic, and paramagnetic phases become tangential near the F point we may choose any of them.)

V. CORRELATION FUNCTION OF THE VLO MODEL

As already mentioned the correlation function $G(r)$ of the VLO model between spins along the diagonal in Fig. 5 was calculated exactly in Ref. 14. $G(r)$ is given in terms of a Toeplitz determinant whose elements are

$$T_{kl}=C_{k-l}=\int_0^{2\pi}\frac{d\omega}{2\pi}e^{i(k-l)\omega}f(\omega) \; . \tag{23}$$

Here $1\leq k, l\leq r-1$ and r is the distance between two spins along a diagonal (r is measured in units of the lattice vector in the diagonal direction). The function $f(\omega)$ is given by

$$f(\omega)=\left[\frac{(\alpha_1+\alpha_2)-\alpha_1\alpha_2e^{i\omega}-e^{-i\omega}}{(\alpha_1\alpha_2)-\alpha_1\alpha_2e^{-i\omega}-e^{i\omega}}\right]^{1/2} , \tag{24}$$

where

$$\alpha_1=\frac{1+y}{1-y}\frac{(x^2+1)^2y+4x^2}{(1-x^2)^2} , \tag{25}$$

$$\alpha_2=y\frac{1-y}{1+y}\frac{(1-x^2)^2}{(x^2+1)^2+4xy} . \tag{26}$$

In terms of α_1 and α_2 the regions AF, FE, and P of Fig. 6 correspond to $\{\alpha_1<+1, \alpha_2<-1\}$, $\{\alpha_1>1, \alpha_2>-1\}$, and $\{\alpha_1^2<1, \alpha_2^2<1\}$, respectively. Expressing y through x from Eqs. (17) or (18) and putting it into Eq. (25) we see that along the phase boundaries $\alpha_1=1$ for any value of x.

Now let us approach the F point. If we write

$$x=1-\epsilon_1 , \tag{27}$$

$$y=-1+\alpha\epsilon_1 , \tag{28}$$

where $\epsilon_1=e^{-2M_1}$, then from Eqs. (25) and (26) to lowest order in ϵ_1 we get

$$\alpha_2=\frac{-1}{\alpha_1}=\frac{-2}{\alpha^2} . \tag{29}$$

This means that if we approach the F point, for example, along the

$$y=-\tfrac{1}{2}x(1+x^\gamma) \tag{30}$$

line with $\gamma=2(\sqrt{3}-1)$, then α in Eq. (28) is $\sqrt{3}$ and $\alpha_1=\frac{3}{2}$, $\alpha_2=-\frac{2}{3}$ that is we approach the F point from the ferromagnetic region. Calculating $G(r)=\mathrm{Det}(T_{kl})$ numerically with the above values of α_1 and α_2 we get (see the second column of Table I)

$$G(r)\underset{r\to\infty}{\longrightarrow}0.620 . \tag{31}$$

This is a typical result for a correlation function of a ferromagnetic system below the critical temperature. By approaching the F point from the ferromagnetic side, we get to a ferromagnetic state, which, however, is not a ground state [in which $G(r)=1$ for any r]. If we approach the F point from the antiferromagnetic side, we get $G(r)\rightarrow_{r\to\infty}(-1)^r x$ const, where the value of the constant is again smaller than unity. This analysis shows that reaching the F point from different regions we select from the infinitely many ground states either the ferromagnetic or the antiferromagnetic ones. Since we know that there cannot be a finite magnetization in a fully frustrated ground state, in the above cases the average in $\langle s_is_j\rangle$ is actually taken only over one set of states (either ferromagnetic or antiferromagnetic).

Now let us approach the F point along a phase boundary. In this case we have from Eqs. (17), (25), (26), and (28) $\alpha=\sqrt{2}$ and $\alpha_1=1$, $\alpha_2=-1$. The elements of the Toeplitz determinant can be calculated exactly. We obtain

$$C_0=0 , \tag{32}$$

$$C_n=\frac{1}{\pi n}[(-1)^n-1] , \quad n=1,2,\ldots,(r-1) . \tag{33}$$

This immediately is telling us that there is no correlation between the s, τ or σ, τ spins. The correlation between the s spins and σ spins turns out to follow a power law with an exponent $\eta=\frac{1}{2}$ (see Table I). It is easy to understand that the correlation

TABLE I. Diagonal correlations in the VLO model at the F point (Fig. 5). $G(r)=\langle S_{0,0},S_{r,r}\rangle$. The values in the second column correspond to approaching the F point from the ferromagnetic region. The values in the third column correspond to approaching the F point along the phase boundary.

r	$G(r)$ $\alpha_1=-1/\alpha_2=\frac{3}{2}$	$r^{1/2}G(r)$ $\alpha_1=-\alpha_2=1$
1	0.586 53	0
2	0.638 90	0.573 159
3	0.616 31	0
4	0.622 93	0.584 020
5	0.619 63	0
6	0.620 51	0.586 366
7	0.620 091	0
8	0.620 22	0.587 222
9	0.620 16	0
10	0.620 18	0.587 625
11	0.620 17	0
12	0.620 17	0.587 846
13	0.620 17	0
14		0.587 979
16		0.588 066
18		0.588 126
20		0.588 169

TABLE II. Diagonal correlations at different points on the phase boundary in the VLO model. The values in the second column correspond to the Ising model.

r	$r^{1/4}G(r)$ $x=0.441$	$r^{1/4}G(r)$ $x=0.8$	$r^{1/4}G(r)$ $x=0.98$
1	0.694 24	0.382 11	0.077 30
2	0.700 79	0.514 85	0.482 36
3	0.702 19	0.386 81	0.126 30
4	0.702 70	0.488 28	0.334 87
5	0.702 94	0.396 80	0.137 64
6	0.703 08	0.400 40	0.274 04
7	0.703 16	0.400 24	0.137 05
8	0.703 21	0.401 48	0.238 72
9		0.401 65	0.132 76
10		0.402 17	0.209 45
11		0.402 37	0.126 56
12		0.402 63	0.183 91
13		0.402 80	0.119 12
14		0.402 96	0.161 95
34			0.050 60
35			0.044 17

function between two nearest s (or σ) and τ spins is zero. As pointed out before, at $T=0$ the VLO model is fully frustrated and the relative direction of the s (or σ) and τ spins is arbitrary. Our result, however, shows that the correlation between s and σ is very strong. Notice that the $r^{-1/2}$ behavior of $G(r)$ is reached already for rather small r values, just as in the case of the ATI model.[3] It is interesting to investigate numerically how the crossover from $\eta=\frac{1}{4}$ to $\frac{1}{2}$ takes place. In Table II we gave the values of $G(r)$ for different finite temperatures (x) along the phase boundary. At the pure Ising model critical point where $x=\sqrt{2}-1$ ($y=0$) the $r^{1/4}$ behavior is reached very quickly. The larger the value of x is, the larger the r values are at which the power-law decay sets in. Very close to $x=1$, the correlation between s (or σ) and τ spins becomes very small and at $T=0$ it disappears. Using the exact expression [Eq. (23)] for $G(r)$ it can also be shown that if J_2 is chosen such that $K_{s\sigma}=0$ in Eq. (21) (in which case we recover the FFSI model) in the limit $T\to 0$, $G(r)\sim e^{-r/\xi}$ and $\xi\sim e^{2J_1/T}$. This, together with the expression for the singular part of the free energy, given in Sec. II, shows that $2-\alpha=2\nu$ also for the FFSI model.

VI. DISCUSSION AND CONCLUSIONS

Using an exact relationship between the FFSI and VLO models we were able to determine the spin-spin correlation function of the FFSI model for arbitrary temperature. At $T=0$ we obtained that this correlation function decays as $r^{-\eta}$ at large distances where $\eta=\frac{1}{2}$. This is in agreement with the results obtained for the ATI model. We also found that hyperscaling holds in both models. These facts suggest that two-dimensional fully frustrated models with finite ground state entropy per particle form a universality class.

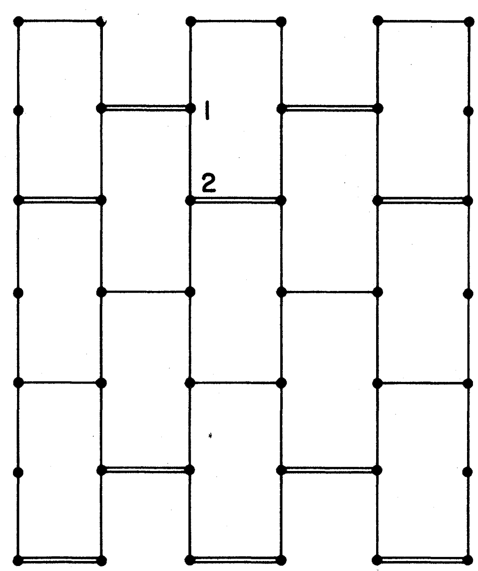

FIG. 7. The hexagonal fully frustrated lattice. Single lines denote ferromagnetic and double lines denote antiferromagnetic bonds. Dots denote lattice points. Flipping spins 1 and 2 simultaneously in the ground state does not change the energy.

In two dimensions all fully frustrated lattices seem to have finite entropy per particle, so $\eta=\frac{1}{2}$ probably for all two-dimensional fully frustrated lattices. The neutral boundaries (regions of spins, which can be flipped without cost in energy of the ground state) are not, however, necessarily single spins like in the cases considered in this work. On the hexagonal fully frustrated lattice shown in Fig. 7 there are no "loose" spins because the coordination number is odd, but one can flip clusters of two spins without the change of energy.

Through another exact transformation we showed that the FFSI model corresponds to a line in the coupling constant space of the Baxter model.

The correlation function of the Baxter model along the solid line of Fig. 2 or Fig. 4 is therefore known exactly. We also showed that the $T=0$ point of the FFSI model maps into the $\Delta=0$ point of the F model. The $\eta=\frac{1}{2}$ result for the correlation function of the FFSI model therefore also follows from exact duality relations between the F model and the AT model (see Appendix).

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APPENDIX

Consider the two-spin correlation function $G^{ss}_{8V}(r)$ of Baxter's 8V model between spins on the s sublattice. Performing duality on the other sublattice of the 8V model, we get an Ashkin-Teller model,[26] so $G^{ss}_{8V}(r)$ goes over to $G^{ss}_{AT}(r)$. If K_2 and K_4 are the two-spin and four-spin couplings of the 8V model, then

$$L_1=K_2+\frac{1}{4}\ln\frac{\cosh(K_2+K_4)\sinh(K_2+K_4)}{\cosh(K_2-K_4)\sinh(K_2-K_4)}\ , \tag{A1}$$

$$L_2=\frac{1}{4}\ln\coth(K_2+K_4)\coth(K_2-K_4)\ , \tag{A2}$$

$$L_4=\frac{1}{4}\ln\frac{\coth(K_2+K_4)}{\coth(K_2-K_4)}\ . \tag{A3}$$

Here L_1 is the two-spin coupling between the s spins (which are not affected by the duality transformation), L_2 is the two-spin coupling between the dual spins μ, and L_4 is the four-spin interaction. We have

$$G^{ss}_{8V}(r;K_2,K_4)=G^{ss}_{AT}(r;L_1,L_2,L_4)\ . \tag{A4}$$

Let us define a new spin variable T in the AT model by $T=s\mu$. With this change of the spins G^{ss}_{AT} in Eq. (A4) transforms into $\langle T_0\mu_0T_r\mu_r\rangle_{L_4,L_2,L_1}$, which is the polarization operator-polarization operator correlation function of an AT model. The four-spin interaction of this AT model is L_1 given by Eq. (A1). At the special point $K_2+K_4=\frac{1}{4}\ln 2$ in the limit $K_2\to\infty$ (which corresponds to $T=0$ in the FFSI model) one can easily verify that $L_1=0$, $L_2=L_4=\frac{1}{2}\ln(\sqrt{2}+1)$. This is exactly the decoupling point of the AT model with the T,μ spins, so $\langle T_0\mu_0T_r\mu_r\rangle=\langle T_0T_r\rangle\langle\mu_0\mu_r\rangle$ where $\langle T_0T_r\rangle$ and $\langle\mu_0\mu_r\rangle$ are two-spin correlation functions of the ordinary Ising model at the critical point. Since $\langle T_0T_4\rangle(\langle\mu_0\mu_r\rangle)=r^{-1/4}$, for $r\gg 1$ we get the result quoted in the text.

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