

Renormalization-group calculation of first- and second-order phase transitions in the Potts model

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A renormalization-group calculation has been carried out in $d = 6 - \epsilon$ dimensions to lowest order in ϵ , using the continuum generalization of the Potts model. The equation of state and the various thermodynamic functions have been calculated explicitly by integrating up the recursion relations in both the ordered and disordered phases. The transition is found to be first order for $p > 2$. For $2 < p < \frac{10}{3}$ the transition is first order in spite of the fact that there is an accessible stable fixed point for all $p < \frac{10}{3}$. At the first-order transition, instead of the usual expansion in temperature, the various thermodynamic functions are studied as functions of $(p - 2)$. Finally it is shown that the free energy of the Potts model has an essential singularity at the first-order transition temperature T_c , due to instantons, which is of the same form as that of the Ising model at $H = 0$.

I. INTRODUCTION

In renormalization-group calculations based on Landau-Ginzburg-Wilson free-energy functionals, second-order transitions are characterized by accessible stable fixed points corresponding to infinite correlation lengths $\xi = \infty$. First-order transitions are (usually) characterized by the absence of accessible stable fixed points. Under the iterations of the renormalization group the parameters flow off to infinity, out of the region of parameter space where the theory is applicable. Using a real-space renormalization approach Nienhuis and Nauenberg¹ have shown that first-order transitions are characterized by discontinuity fixed points for which $\xi = 0$.

The present calculation is based on the continuum generalization of the Potts model² due to Priest and Lubensky.³ The critical properties are studied in $d = 6 - \epsilon$ dimensions, to lowest order in ϵ . An accessible stable fixed point is obtained³ for all values $p < \frac{10}{3}$. For $p > \frac{10}{3}$ there is no stable fixed point, and the resulting runaway is believed associated with a first-order transition in the usual way. However, as will be shown in detail below, for $2 < p < \frac{10}{3}$ a first-order transition is obtained in spite of the presence of an accessible stable fixed point.

These results are consistent with recent real-space Monte Carlo renormalization-group calculations by Blöte and Swendsen.⁴ They find that the transition is first order for $p = 3$ in three and four dimensions, but that the transition is *not* associated with a discontinuity fixed point. Instead (in three dimensions) evidence was found for the existence of a second-order fixed point associated with singularities in the metastable region just beyond the first-order transition. The present calculations suggest that if their

calculations were repeated for larger p values, a conventional first-order transition with a discontinuity fixed point might be expected. For $d = 6$ the boundary value separating the two types of first-order transitions is $p_c = \frac{10}{3}$. The value of p_c in lower dimensions is not known. When corrections of order ϵ are included the value of p_c increases.³

There appear to be no other examples, as yet discussed, where there exists an accessible stable fixed point but where the transition is nonetheless first order. Further, within the same model, there appears to be two distinct types of first-order transitions depending on the value of p .

Instantons, which are solutions of nonlinear equations associated with tunneling phenomena, are known to play an important role at first-order phase transitions.⁵⁻⁷ This suggests that the kind of low-order perturbation expansion which characterize momentum space renormalization-group calculations, may not be a good description for $2 < p < \frac{10}{3}$, even though a fixed point with values of the parameters $\sim \epsilon$ is obtained. Clearly for $p > \frac{10}{3}$ no such expansion is possible.

Much of the work on the Potts model has been concerned with determining the order of the transition. The three-state Potts model is known to be of second order in two dimensions.^{8,9} After some early conflicting results, it now seems well established that the transition of the three-state Potts model is first order in three and four dimensions.^{4,10-21} Thus finding the transition to be first order for $p > 2$, makes the results of the $\epsilon = 6 - d$ expansion consistent with those obtained by other methods.

The results of the renormalization-group calculations in the disordered phase is presented in Sec. II. In Sec. III, the ordered phase is discussed. The

mean-field theory is reviewed briefly followed by the results of the second-order transition for $p < 2$. The properties of the first-order transition at $T = T_c$ are then described as functions of $(p - 2)$. Finally there is a brief discussion of the role of instantons in the neighborhood of the transition temperature and the two stability limits.

II. DISORDERED PHASE

Following Priest and Lubensky³ the effective reduced Hamiltonian is written

$$\mathfrak{H} = -\frac{1}{4} \int (r + k^2) \sum Q_{ij}(k) Q_{ij}(-k) + \omega \int \sum Q_{ij}(k) Q_{jk}(k') Q_{kl}(-k - k') - u \sum Q_{ij}(k) Q_{ij}(k') Q_{kl}(k'') Q_{kl}(-k - k' - k'') - v \sum Q_{ij}(k) Q_{jk}(k') Q_{kl}(k'') Q_{li}(-k - k' - k'') , \quad (1)$$

where Q_{ij} are symmetric traceless tensors and in the case of the Potts model also diagonal. The corresponding propagator has the form,

$$\langle Q_{ii}(k) Q_{jj}(-k) \rangle = \frac{2}{r + k^2} \left[\delta_{ij} - \frac{1}{p} \right] . \quad (2)$$

The tensor components Q_{ii} are related to the components A_α of the p -state Potts model,

$$\mathfrak{H} = -J \sum_{\langle xx' \rangle} \bar{A}(x) \cdot \bar{A}(x') \quad (3)$$

by

$$Q_{ii} = \sum_{\alpha=1}^p A_\alpha a_{ii}^\alpha , \quad (4)$$

where

$$a_{ii}^\alpha = \left(\frac{p - \alpha}{p - \alpha + 1} \right)^{1/2} \times \begin{cases} 0 & \text{if } i < \alpha \\ 1 & \text{if } i = \alpha \\ -1/(p - \alpha) & \text{if } i > \alpha \end{cases} . \quad (5)$$

When written in differential form the recursion relations are given by,

$$\begin{aligned} \frac{dr}{dl}(l) &= [2 - \eta(l)]r(l) - 144K_6\omega^2(l) \left[1 - \frac{2}{p} \right] \frac{1}{[1+r(l)]^2} + 2^4K_6 \left[(p+1)u(l) + 3 \left[1 - \frac{1}{p} \right] v(l) \right] \frac{1}{1+r(l)} , \\ \frac{d\omega}{dl}(l) &= \left[\frac{1}{2}\epsilon - \frac{3}{2}\eta(l) \right] \omega(l) + 288K_6 \left[1 - \frac{3}{p} \right] \frac{\omega^3(l)}{[1+r(l)]^3} - 2^4K_6 \left[2u(l) + 3 \left[1 - \frac{2}{p} \right] v(l) \right] \frac{\omega(l)}{[1+r(l)]^2} , \\ \frac{du}{dl} &= [-2 + \epsilon - 2\eta(l)]u(l) + 3^22^6K_6u(l)\omega^2(l) \left[1 - \frac{4}{p} \right] \frac{1}{[1+r(l)]^3} + 3^32^6K_6v(l)\omega(l)^2 \frac{1}{p^2} \frac{1}{[1+r(l)]^3} , \\ \frac{dv}{dl} &= [-2 + \epsilon - 2\eta(l)]v(l) + 3^22^7K_6u(l)\omega^2(l) \frac{1}{[1+r(l)]^3} + 3^32^6K_6v(l)\omega^2(l) \left[1 - \frac{3}{p} \right] \frac{1}{[1+r(l)]^3} , \end{aligned} \quad (6)$$

where $\epsilon = 6 - d$ and,

$$\eta(l) = 48K_6(1 - 2/p)\omega^2(l) . \quad (7)$$

Only linear terms in $u(l)$ and $v(l)$ have been included. This will be justified later.

For $d > 4$ the quartic coefficients are irrelevant with an l dependence $\sim e^{-2l}$ for $d \sim 6$. In the disordered phase the $u(l)$ and $v(l)$ terms can then be neglected, and we obtain results derived previously,²²

$$\begin{aligned} r(l) &= t(l) + 72K_6\omega^2(l) \left[1 - \frac{2}{p} \right] \left[1 - 2t(l) \ln[1+t(l)] - \frac{t^2(l)}{1+t(l)} \right] , \\ \omega^2(l) &= \omega^2 e^{\epsilon l} / W(l) , \\ t(l) &= t e^{2l} W(l)^{-5/3(2-p)/(10-3p)} , \\ W(l) &= 1 + 144K_6 \left[\frac{10}{p} - 3 \right] \frac{\omega^2}{\epsilon} (e^{\epsilon l} - 1) , \end{aligned} \quad (8)$$

and where for later reference

$$\exp\left\{\int_0^l \eta(l') dl'\right\} = W(l)^{-1(2-p)/(10-3p)/3} \quad (9)$$

In these equations l and ω are initial values,

$$l = l(l=0), \quad \omega = \omega(l=0) .$$

The critical behavior is determined by relating properties in the critical region to those far from criticality by means of the recursion relations. The free energy is given by²²

$$F = \frac{1}{2} K_6 (p-1) \int_0^{l^*} \ln[1+r(l')] e^{-dl'} dl' \\ \sim \frac{l^3}{12^3 \omega^2} (p-1) [W(l^*)^{2p/(10-3p)} - 1] , \quad (10)$$

where l^* is determined by setting $l(l^*) = 1$. Similarly the susceptibility is given by

$$\chi^{-1} = \exp\left[-2l^* + \int_0^{l^*} \eta(l') dl'\right] \chi^{-1}(l^*) \\ \sim l W(l^*)^{-2(2-p)/(10-3p)} \quad (11)$$

since to leading order, $2\chi^{-1}(l^*) = l(l^*) = 1$. Equations (7), (10), and (11) yield the exponents^{3,22}

$$\alpha = -1 + [p/(10-3p)]\epsilon , \\ \gamma = 1 + [(2-p)/(10-3p)]\epsilon , \quad (12) \\ \eta = -\frac{1}{3}\epsilon(2-p)/(10-3p) .$$

These are the appropriate exponents at the second-

$$\mathfrak{K} = -\frac{1}{4} \int (r_L + k^2) \mathfrak{L}(k) \mathfrak{L}(-k) - \frac{1}{4} \int (r_T + k^2) \sum_{i \neq 1} q_{ii} q_{ii} - \int \tilde{h}(k) \mathfrak{L}(-k) \\ + [(p-2)c\omega - 4Q(u+bv)] \int \mathfrak{L}(k) \mathfrak{L}(k') \mathfrak{L}(-k-k') - [3\omega c + 4Q(u+3vc^2)] \int \mathfrak{L}(k) \sum_{i \neq 1} q_{ii}(k') q_{ii}(-k-k') \\ + (\omega + 4cvQ) \int \sum_{i \neq 1} q_{ii}(k) q_{ii}(k') q_{ii}(-k-k') - (u+bv) \int \mathfrak{L}(k) \mathfrak{L}(k') \mathfrak{L}(k'') \mathfrak{L}(-k-k'-k'') \\ - 2(u+3vc^2) \int \mathfrak{L}(k) \mathfrak{L}(k') \sum_{i \neq 1} q_{ii}(k'') q_{ii}(-k-k'-k'') + 4vc \int \mathfrak{L}(k) \sum_{i \neq 1} q_{ii}(k') q_{ii}(k'') q_{ii}(-k-k'-k'') \\ - u \sum_{\substack{i \neq 1 \\ j \neq 1}} q_{ii}(k) q_{ii}(k') q_{jj}(k'') q_{jj}(-k-k'-k'') - v \sum_{i \neq 1} q_{ii}(k) q_{ii}(k') q_{ii}(k'') q_{ii}(-k-k'-k'') \quad (17)$$

and its fluctuation-independent (mean-field) part,

$$\mathfrak{K}_{MF} = -\frac{1}{4} r Q^2 + (p-2)c\omega Q^3 - hQ - (u+bv)Q^4 . \quad (18)$$

In these equations

$$r_L = r - 12(p-2)\omega c Q + 24(u+bv)Q^2 , \\ r_T = r + 12\omega c Q + 8(u+3vc^2)Q^2 , \\ \tilde{h} = h - \frac{1}{2} r Q + 3\omega c (p-2)Q^2 - 4(u+bv)Q^3 , \\ c = [p(p-1)]^{-1/2}, \quad b = (p^2-3p+3)c^2 . \quad (19)$$

order transition for $p \leq 2$ and at the spinoidal point for $2 < p < \frac{10}{3}$. For $p=2$, which corresponds to the Ising model, $\text{Tr}Q^3=0$, and we obtain exponents appropriate to the Gaussian model.

III. ORDERED PHASE

To describe the ordered phase we set

$$A_\alpha(x) = \langle A_\alpha \rangle + \mathfrak{L}_\alpha(x) , \quad (13)$$

where the brackets denote the thermal average and \mathfrak{L}_α is the fluctuating part. Following Priest and Lubensky³ only uniaxial order will be considered. Thus,

$$\langle A_\alpha \rangle = \begin{cases} Q, & \alpha = 1 \\ 0, & \text{otherwise} . \end{cases} \quad (14)$$

The corresponding expression for Q_{ii} is given by Eq. (4),

$$Q_{11} = \left[\frac{p-1}{p} \right]^{1/2} (Q + \mathfrak{L}) , \quad (15)$$

$$Q_{ii} = \frac{-1}{p-1} \left[p - \frac{1}{p} \right]^{1/2} (Q + \mathfrak{L}) + q_{ii} \quad (16)$$

for $i \neq 1$,

where q_{ii} is a traceless diagonal tensor of dimension $(p-1)$ and where the subscript on \mathfrak{L}_1 has been dropped.

We add a fictitious field $-A_1(x)h(x)$ to the Hamiltonian and separate it into its fluctuating part,

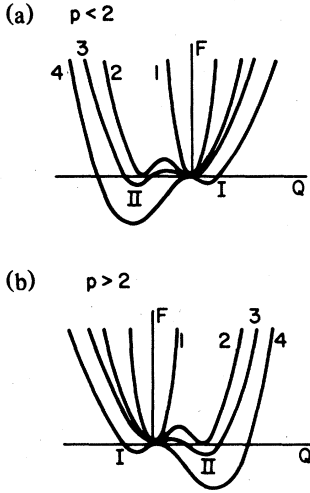


FIG. 1. (a), (b) Mean-field free energy as a function of order parameter Q with decreasing temperature from 1 to 4.

A. Mean-field results

In Fig. 1 the mean-field free energy has been plotted as a function of Q for $p > 2$ and $p < 2$ for decreasing temperatures from curves 1 to 4. The stability of the mean-field solutions is determined by the requirement that the coefficients of the quadratic terms r_L and r_T , describing fluctuations about the mean-field order parameter, be positive. Or, equivalently that the longitudinal and transverse susceptibilities $\chi_{L,T}^{-1} = \frac{1}{2} r_{L,T}$ be positive. By Eq. (18) the longitudinal susceptibility is also given by

$$\chi_L^{-1} = \frac{\partial^2 \mathcal{F}_{\text{MF}}}{\partial Q^2} . \quad (20)$$

Thus it follows that $\chi_L^{-1} > 0$ for all the local minima shown in Fig. 1. We shall find, however, that $\chi_T^{-1} < 0$ for several of the local minima showing that these are, in fact, unstable saddle points. The extrema of the mean-field free energy are determined by $(\partial/\partial Q)\mathcal{F}_{\text{MF}} = 0$, or

$$Q \left[\frac{1}{2} r - 3\omega c (p-2)Q + 4(u+bv)Q^2 \right] = 0 . \quad (21)$$

For $Q = 0$, $r_L = r_T = r$ and the $Q = 0$ extremum changes from a local minimum to a local maximum at $r = 0$ in the usual way. For the $Q \neq 0$ extrema we obtain

$$r_L = -r + 8(u+bv)Q^2 , \quad (22)$$

$$r_T = \frac{p}{(p-2)}r + 8uQ^2 \frac{p}{(p-2)} + \frac{8vQ^2(2p-3)}{(p-1)(p-2)} .$$

For the extrema denoted by I in Fig. 1 which develop

continuously with $Q \propto |r|$, the Q^2 terms can be neglected in the neighborhood of $r \leq 0$, and we obtain simply

$$r_L = |r| ; \quad r_T = \frac{-p}{(p-2)}|r| . \quad (23)$$

We note that $r_T > 0$ for $p < 2$ while $r_T < 0$ for $p > 2$. Thus the apparently locally stable minimum I Fig. 1(b) is, in fact, unstable.²²

In order to test the stability of the large Q minimum denoted by II in Fig. 1 consider case 2. Equating the free energies for $Q = 0$ and $Q \neq 0$ gives

$$\frac{1}{4}rQ^2 + \omega c(p-2)Q^3 - (u+bv)Q^4 = 0 , \quad (24)$$

where Q is determined by Eq. (21) as before. Solving these two equations, we obtain

$$r_c = \frac{(p-2)^2 c^2 \omega^2}{(u+bv)} , \quad Q_c = \frac{1}{2} \frac{(p-2)c\omega}{(u+bv)} . \quad (25)$$

The corresponding values of $r_{T,L}$ are

$$r_L = \frac{(p-2)^2 c^2 \omega^2}{(u+bv)} , \quad (26)$$

$$r_T = \frac{6(p-2)c^2\omega^2}{(u+bv)} + \frac{(p-2)^2 c^2 \omega^2}{(u+bv)^2} [3u + (b+6c^2)v] .$$

However, for $p > 2$ all terms in r_T are positive while $r_T < 0$ for $p \leq 2$. Thus the (apparent) minimum II is stable for $p > 2$ but is unstable for $p \leq 2$.

From the above discussion it follows that the mean-field approximation provides a consistent description of the model for $p > 2$. A first-order transition is obtained at $r = r_c$ when $Q = 0$ and $Q \neq 0$ minima are equal as in Fig. 1 curve 2. For $p > 2$ both of these minima are locally stable with $r_T > 0$. The $Q = 0$ minimum is an absolute minimum for $r > r_c$ and remains metastable for $0 < r < r_c$. At the stability limit $r = 0$, the order parameter changes discontinuously to the positive value

$$Q_{c1} = \frac{3}{2} Q_c ,$$

where Q_c is the discontinuity at the first-order transition. The large Q minimum, which is an absolute minimum for $r < r_c$, remains metastable for $r > r_c$ until the stability limit,

$$r_{c2} = \frac{9}{8} r_c ; \quad Q_{c2} = \frac{1}{2} Q_c ,$$

is reached. At the stability limit r_{c2} , $\chi_L^{-1} = 0$ while $\chi_T^{-1} > 0$. We note that Q is nonzero at the superheating stability limit while the slope, $dQ/dr \sim 1/\sqrt{r_{c2}-r}$ has a square-root singularity.

For $p > 2$ the minimum that develops continuously from $Q = 0$ is not stable because $r_T < 0$. Thus a continuous transition is not possible. Also, the stable large Q minimum is always lower.

In Secs. III B–III G the results of the renormalization-group calculations will be presented. The most interesting result is that the presence of metastable state persists although only for a limited range of p values. Whereas in mean field a metastable state with $Q = 0$ is obtained in the temperature range $0 < r < r_c$ for all $p > 2$, in the renormalization-group calculation a metastable state is obtained only for $2 < p < \frac{10}{3}$. For $p > \frac{10}{3}$ a first-order transition is obtained without metastable states. The properties of the first-order transition are calculated explicitly for $p \geq 2$. It should be noted that the value $p = \frac{10}{3}$ enters only through the renormalization-group calculations. In mean field there is no indication that $p = \frac{10}{3}$, or any other value apart from $p = 2$, should play a special role.

For $p < 2$ the picture is much less clear. In this case the large Q minimum (with $Q < 0$) is unstable because $\chi_T < 0$. However, for $r < r_c$ this is the deepest minimum. Thus in order to have a well-defined model for $p < 2$ higher-order terms in Q need be included. It is possible that if terms to all orders in Q are included there will be no large Q minimum with $Q < 0$. This would be analogous to the random ferromagnet where for certain choices of parameters a first-order transition is obtained in

mean field when only quartic terms are included,²³ whereas it is believed that when terms to all orders are kept in the free-energy functional, the additional minimum disappears and the transition is second order.²⁴ Priest and Lubensky have argued on physical grounds that, at least for $p = 1$, a solution with $Q < 0$ is not permissible.³ Thus in this paper it will be assumed that the transition is second order for $p < 2$. With this assumption the renormalization-group calculation presented below, keeping only the low-order terms and thus ignoring any large Q minimum, will give a correct description of the transition. All the higher-order terms are irrelevant at a second-order transition. Because of the instability of the large Q minimum, the model given by Eq. (1) cannot describe a first-order transition for $p < 2$.

B. Solutions of the recursion relations

From the effective Hamiltonian for the ordered phase Eq. (17) we derive the following recursion relations for $r_L(l)$, $r_T(l)$, and $\tilde{h}(l)$, the solutions of which are required in order to calculate the longitudinal and transverse susceptibilities, the equation of state, and the free energy.

$$\begin{aligned} \frac{dr_L}{dl}(l) = & [2 - \eta(l)]r_L(l) - 2^4 3^2 K_6 (p-2) c^2 \omega^2(l) [(p-2)g_L^2(l) + g_T^2(l)] + 2^4 3 K_6 [u(l) + b v(l)] g_L(l) \\ & + 2^4 K_6 (p-2) [u(l) + 3c^2 v(l)] g_T(l) + 2^7 3^2 K_6 (p-2) c \omega(l) Q(l) [u(l) + b v(l)] g_L^2(l) \\ & - 2^7 3 K_6 (p-2) c \omega(l) Q(l) [u(l) + 3c^2 v(l)] g_T^2(l) , \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{dr_T}{dl}(l) = & [2 - \eta(l)]r_T(l) - 2^5 3 K_6 c^2 \omega^2(l) g_T(l) g_L(l) - 2^4 3 K_6 \frac{(p-3)}{(p-1)} \omega^2(l) g_T^2(l) \\ & + 2^4 K_6 p [u(l) + 3c^2 (p-2) v(l)] g_T(l) + 2^4 K_6 [u(l) + 3c^2 v(l)] g_L(l) \\ & - 2^8 3 K_6 c \omega(l) Q(l) [u(l) + 3c^2 v(l)] g_T(l) g_L(l) - 2^7 3^2 K_6 c \omega(l) Q(l) \frac{(p-3)}{(p-1)} g_T^2(l) , \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{d\tilde{h}}{dl}(l) = & [4 - \frac{1}{2} \epsilon - \frac{1}{2} \eta(l)] \tilde{h}(l) + 6 K_6 (p-2) c \omega(l) [g_L(l) - g_T(l)] \\ & - 2^3 3 K_6 Q(l) [u(l) + b v(l)] g_L(l) - 2^3 K_6 (p-2) Q(l) [u(l) + 3c^2 v(l)] g_T(l) , \end{aligned} \quad (29)$$

where

$$g_L(l) = \frac{1}{1+r_L(l)}, \quad g_T = \frac{1}{1+r_T(l)} . \quad (30)$$

For $\omega(l)$, $u(l)$, and $v(l)$ the recursion relations for the disordered phase will be sufficient.

The solutions of these equations are given by,

$$\begin{aligned}
 r_L(l) = & T_L(l) + 2^3 3^2 K_6 \left[1 - \frac{2}{p} \right] \omega^2(l) + 2^6 3 K_6 (p-2) \left[2u(l) + 3 \left[1 - \frac{2}{p} \right] v(l) \right] c \omega(l) Q(l) \\
 & - 2^4 3^2 K_6 (p-2) \{ (p-2) c^2 \omega^2(l) - 8[u(l) + b v(l)] c \omega(l) Q(l) \} \left[T_L(l) \ln[1 + T_L(l)] + \frac{1}{2} \frac{T_L^2(l)}{1 + T_L(l)} \right] \\
 & - 2^4 3^2 K_6 (p-2) \{ c^2 \omega^2(l) - 8[u(l) + 3c^2 v(l)] c \omega(l) Q(l) \} \left[T_T(l) \ln[1 + T_T(l)] + \frac{1}{2} \frac{T_T^2(l)}{1 + T_T(l)} \right] \\
 & + 2^3 3 K_6 [u(l) + b v(l)] \{ [T_L(l) - \frac{1}{2}] + T_L^2 \ln[1 + T_L(l)] \} \\
 & + 2^3 K_6 (p-2) [u(l) + 3c^2 v(l)] \{ [T_T(l) - \frac{1}{2}] + T_T^2 \ln[1 + T_T(l)] \} , \tag{31}
 \end{aligned}$$

$$\begin{aligned}
 r_T(l) = & T_T(l) + 2^3 3^2 K_6 \left[1 - \frac{2}{p} \right] \omega^2(l) + 2^6 3 K_6 \left[2u(l) + 3 \left[1 - \frac{2}{p} \right] v(l) \right] c \omega(l) Q(l) \\
 & - 2^4 3 K_6 \{ 3c^2 \omega^2(l) + 8[u(l) + 3c^2 v(l)] c \omega(l) Q(l) \} \\
 & \times \left[\frac{[T_T^2(l) + T_L^2(l) + T_L(l) T_T(l)]}{[T_T(l) - T_L(l)]} \ln \left[\frac{1 + T_T(l)}{1 + T_L(l)} \right] \right. \\
 & \left. + \frac{[T_T(l) + T_L(l)]}{[T_T(l) - T_L(l)]} \{ T_T(l) \ln[1 + T_L(l)] - T_L(l) \ln[1 + T_T(l)] \} \right] \\
 & - 2^4 3^2 K_6 \frac{(p-3)}{(p-1)} [\omega^2(l) + 8v(l) c \omega(l) Q(l)] \left[T_T(l) \ln[1 + T_T(l)] + \frac{1}{2} \frac{T_T^2(l)}{1 + T_T(l)} \right] \\
 & + 2^3 K_6 p [u(l) + 3(p-2) c^2 v(l)] \{ T_T(l) - \frac{1}{2} + T_T^2(l) \ln[1 + T_T(l)] \} \\
 & + 2^3 K_6 [u(l) + 3c^2 v(l)] \{ T_L(l) - \frac{1}{2} + T_L^2 \ln[1 + T_L(l)] \} , \tag{32}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{h}(l) = & h(l) - \frac{1}{2} t(l) Q(l) + 3(p-2) c \omega(l) Q^2(l) - 4[u(l) + b v(l)] Q^3(l) \\
 & + 3K_6 (p-2) c \omega(l) \{ T_L(l) - T_T(l) - T_L(l) \ln[1 + T_L(l)] + T_T(l) \ln[1 + T_T(l)] \} \\
 & - 2^2 3 K_6 [u(l) + b v(l)] Q(l) \{ -\frac{1}{2} + T_L(l) - T_L^2(l) \ln[1 + T_L(l)] \} \\
 & - 2^2 K_6 (p-2) [u(l) + 3c^2 v(l)] Q(l) \{ -\frac{1}{2} + T_T(l) - T_T^2(l) \ln[1 + T_T(l)] \} , \tag{33}
 \end{aligned}$$

where

$$\begin{aligned}
 T_L(l) = & t(l) - 12(p-2) c \omega(l) Q(l) + 24[u(l) + b v(l)] Q^2(l) , \\
 T_T(l) = & t(l) + 12c \omega(l) Q(l) + 8[u(l) + 3c^2 v(l)] Q^2(l) , \tag{34}
 \end{aligned}$$

are the leading order solutions of the dr_L/dl and dr_T/dl recursion relations,

$$h(l) = h \exp \left[\left(4 - \frac{1}{2} \epsilon \right) l - \frac{1}{2} \int_0^l \eta(l') dl' \right] , \quad Q(l) = Q \exp \left[\left(2 - \frac{1}{2} \epsilon \right) l + \frac{1}{2} \int_0^l \eta(l') dl' \right] , \tag{35}$$

and where $t(l)$, $\omega(l)$, $u(l)$, and $v(l)$ are determined by the set of equations,

$$\begin{aligned} \frac{dt}{dl}(l) &= [2 - \eta(l)]t(l) + 2^5 3 K_6 \left[1 - \frac{2}{p}\right] \omega^2(l) t(l) + 2^4 K_6 \left[(p+1)u(l) + 3\left[1 - \frac{1}{p}\right]v(l)\right] t^2(l) , \\ \frac{d\omega}{dl}(l) &= \left[\frac{\epsilon}{2} - \frac{3}{2}\eta(l)\right] \omega(l) + 2^5 3^2 K_6 \left[1 - \frac{3}{p}\right] \omega^3(l) + 2^5 3 K_6 \left[2u(l) + 3\left[1 - \frac{2}{p}\right]v(l)\right] \omega(l) t(l) , \\ \frac{du}{dl}(l) &= [-2 + \epsilon - 2\eta(l)]u(l) + 2^6 3^2 K_6 \left[1 - \frac{4}{p}\right] u(l) \omega^2(l) + 2^6 3^3 K_6 \frac{1}{p^2} v(l) \omega^2(l) , \\ \frac{dv}{dl}(l) &= [-2 + \epsilon - 2\eta(l)]v(l) + 2^7 3^2 K_6 u(l) \omega^2(l) + 2^6 3^3 K_6 \left[1 - \frac{3}{p}\right] v(l) \omega^2(l) . \end{aligned} \quad (36)$$

The solutions of this set will be needed only for various special cases.

C. Susceptibilities

The susceptibilities in the critical region $\chi_{T,L}$ are related to those far from criticality $\chi_{T,L}$ by

$$\chi_{T,L}^{-1} = \exp\left[-2l^* + \int_0^{l^*} \eta(l) dl\right] \chi_{T,L}^{-1}(l^*) , \quad (37)$$

where $\chi_{T,L}(l^*)$ is calculated by fluctuation corrected Landau theory. The diagrams needed are the same as for the $r_{T,L}$ recursion relations. We obtain,

$$\begin{aligned} 2\chi_L^{-1}(l^*) &= r_L(l^*) - 2^4 3^2 (p-2) c^2 \omega^2(l^*) \left[(p-2) K_6 \int_0^1 \frac{k^5 dk}{[r_L(l^*) + k^2]^2} + K_6 \int_0^1 \frac{k^5 dk}{[r_T(l^*) + k^2]^2} \right] \\ &+ 2^4 3 [u(l^*) + b v(l^*)] K_6 \int_0^1 \frac{k^5 dk}{r_L(l^*) + k^2} + 2^4 (p-2) [u(l^*) + 3c^2 v(l^*)] K_6 \int_0^1 \frac{k^5 dk}{r_T(l^*) + k^2} \\ &+ 2^7 3^2 (p-2) c \omega(l^*) Q(l^*) [u(l^*) + b v(l^*)] K_6 \int_0^1 \frac{k^5 dk}{[r_L(l^*) + k^2]^2} \\ &- 2^7 3 (p-2) c \omega(l^*) Q(l^*) [u(l^*) + 3c^2 v(l^*)] K_6 \int_0^1 \frac{k^5 dk}{[r_T(l^*) + k^2]^2} , \end{aligned} \quad (38)$$

$$\begin{aligned} 2\chi_T^{-1}(l^*) &= r_T(l^*) - 2^5 3^2 c^2 \omega^2(l^*) K_6 \int_0^1 \frac{k^5 dk}{[r_T(l^*) + k^2][r_L(l^*) + k^2]} - 2^4 3^2 \frac{(p-3)}{(p-1)} \omega^2(l^*) K_6 \int_0^1 \frac{k^5 dk}{[r_T(l^*) + k^2]^2} \\ &+ 2^4 p [u(l^*) + 3c^2 (p-2) v(l^*)] K_6 \int_0^1 \frac{k^5 dk}{r_T(l^*) + k^2} + 2^4 [u(l^*) + 3c^2 v(l^*)] K_6 \int_0^1 \frac{k^5 dk}{r_L(l^*) + k^2} \\ &- 2^8 3 c \omega(l^*) Q(l^*) [u(l^*) + 3c^2 v(l^*)] K_6 \int_0^1 \frac{k^5 dk}{[r_L(l^*) + k^2][r_T(l^*) + k^2]} \\ &- 2^7 3^2 \frac{(p-3)}{(p-1)} c \omega(l^*) Q(l^*) v(l^*) K_6 \int_0^1 \frac{k^5 dk}{[r_T(l^*) + k^2]^2} . \end{aligned} \quad (39)$$

Performing the integrals and substituting Eqs. (31) and (32) for r_L and r_T we obtain

$$\begin{aligned} 2\chi_L^{-1}(l^*) &= T_L(l^*) - 2^4 3^2 K_6 (p-2) c^2 \omega^2(l^*) \\ &\times \left\{ (p-2) \left[\frac{1}{2} T_L(l^*) + T_L(l^*) \ln T_L(l^*) \right] + \frac{1}{2} T_T(l^*) + T_T(l^*) \ln T_T(l^*) \right\} \\ &+ 2^7 3^2 K_6 (p-2) c \omega(l^*) Q(l^*) [u(l^*) + b v(l^*)] \left[\frac{1}{2} T_L(l^*) + T_L(l^*) \ln T_L(l^*) \right] \\ &- 2^7 3 K_6 (p-2) c \omega(l^*) Q(l^*) [u(l^*) + 3c^2 v(l^*)] \left[\frac{1}{2} T_T(l^*) + T_T(l^*) \ln T_T(l^*) \right] \\ &- 2^3 3 K_6 [u(l^*) + b v(l^*)] T_L^2(l^*) \ln T_L(l^*) - 2^3 K_6 (p-2) [u(l^*) + 3c^2 v(l^*)] T_T^2(l^*) \ln T_T(l^*) , \end{aligned} \quad (40)$$

$$\begin{aligned}
2\chi_T^{-1}(l^*) &= T_T(l^*) - 2^4 3^2 K_6 c^2 \omega^2(l^*) \frac{[T_T^2(l^*) \ln T_T(l^*) - T_L^2(l^*) \ln T_L(l^*)]}{[T_T(l^*) - T_L(l^*)]} \\
&\quad - 2^4 3^2 K_6 \frac{(p-3)}{(p-1)} \omega^2(l^*) \left[\frac{1}{2} T_T(l^*) + T_T(l^*) \ln T_T(l^*) \right] \\
&\quad - 2^7 3 K_6 c \omega(l^*) Q(l^*) [u(l^*) + 3c^2 v(l^*)] \frac{[T_T^2(l^*) \ln T_T(l^*) - T_L^2(l^*) \ln T_L(l^*)]}{[T_T(l^*) - T_L(l^*)]} \\
&\quad - 2^7 3 K_6 \frac{(p-3)}{(p-1)} c \omega(l^*) Q(l^*) v(l^*) \left[\frac{1}{2} T_T(l^*) + T_T(l^*) \ln T_T(l^*) \right] \\
&\quad - 2^3 K_6 p [u(l^*) + 3(p-2)c^2 v(l^*)] T_T^2(l^*) \ln T_T(l^*) - 2^3 K_6 [u(l^*) + 3c^2 v(l^*)] T_L^2(l^*) \ln T_L(l^*) . \quad (41)
\end{aligned}$$

D. Equation of state

The magnetization is determined by the self-consistency condition $\langle \mathbf{L} \rangle = 0$. When evaluated at $l = l^*$ the diagrams that need be included are the same as for the $\tilde{h}(l)$ recursion relation. We obtain,

$$\begin{aligned}
0 &= \tilde{h}(l^*) + 6(p-2)c\omega(l^*)K_6 \int_0^1 k^5 dk \left[\frac{1}{r_L(l^*) + k^2} - \frac{1}{r_T(l^*) + k^2} \right] \\
&\quad - 2^3 3 K_6 [u(l^*) + b v(l^*)] Q(l^*) K_6 \int_0^1 \frac{k^5 dk}{r_L(l^*) + k^2} \\
&\quad - 2^3 K_6 (p-2) [u(l^*) + 3c^2 v(l^*)] Q(l^*) \int_0^1 \frac{k^5 dk}{r_T(l^*) + k^2} . \quad (42)
\end{aligned}$$

Performing the integrals and substituting for $\tilde{h}(l^*)$ the solution obtained by integrating up the recursion relations, gives,

$$\begin{aligned}
0 &= h(l^*) - \frac{1}{2} t(l^*) Q(l^*) + 3(p-2)c\omega(l^*) Q^2(l^*) - 4[u(l^*) + b v(l^*)] Q^3(l^*) \\
&\quad - 3K_6(p-2)c\omega(l^*) [T_L^2(l^*) \ln T_L(l^*) - T_T^2(l^*) \ln T_T(l^*)] \\
&\quad + 2^2 3 K_6 [u(l^*) + b v(l^*)] Q(l^*) T_L^2(l^*) \ln T_L(l^*) \\
&\quad + 2^2 K_6 (p-2) [u(l^*) + 3c^2 v(l^*)] Q(l^*) T_T^2(l^*) \ln T_T(l^*) . \quad (43)
\end{aligned}$$

E. Free energy

The free energy is given by,

$$\begin{aligned}
F &= \frac{1}{4} r Q^2 - (p-2)c\omega Q^3 + (u + b v) Q^4 \\
&\quad + \frac{1}{2} K_6 \int_0^{l^*} dl \{ \ln[1 + r_L(l)] \\
&\quad \quad + (p-2) \ln[1 + r_T(l)] \} e^{-ld} . \quad (44)
\end{aligned}$$

This consists of a mean-field part and a trajectory integral describing the effect of fluctuations. To leading order the trajectory integral is given by,

$$\frac{1}{6} K_6 \int_0^{l^*} dl [T_L^3(l) + (p-2)T_T^3(l)] e^{-ld} .$$

Part of the contribution from the lower limit cancels

the mean-field contribution to Eq. (44). We obtain

$$\begin{aligned}
F &= \frac{1}{6} K_6 (p-1) \int_0^{l^*} dl t^3(l) e^{-ld} \\
&\quad + \left\{ \frac{1}{4} t(l^*) Q^2(l^*) - (p-2)c\omega(l^*) Q^3(l^*) \right. \\
&\quad \quad \left. + [u(l^*) + b v(l^*)] Q^4(l^*) \right\} e^{-l^* d} , \quad (45)
\end{aligned}$$

where $t(l)$, $\omega(l)$, $u(l)$, and $v(l)$ are solutions of Eqs. (36).

F. Second-order transition $p < 2$

In the case of a second-order transition the irrelevant variables $u(l)$ and $v(l)$ do not contribute in the neighborhood of the critical point. The results

then reduce to those derived previously with only cubic terms included.²²

From the equation of state we obtain, to leading order, in the limit $h(l^*)/Q(l^*) \rightarrow 0$

$$Q(l^*) = \frac{t(l^*)}{6(p-2)c\omega(l^*)}, \quad (46)$$

where $t(l)$ and $\omega(l)$ are given by Eqs. (8). The order parameter in the critical region is therefore determined by

$$Q = \frac{t}{6(p-2)c\omega} W(l^*)^{2/(10-3p)}. \quad (47)$$

At $l = l^*$ the susceptibilities are, again to leading order, given by

$$2\chi_{TL}^{-1}(l^*) = T_{TL}(l^*), \quad (48)$$

where now

$$T_L(l) = t(l) - 12(p-2)c\omega(l)Q(l), \quad (49)$$

$$T_T(l) = t(l) + 12c\omega(l)Q(l).$$

The free energy takes the form

$$F = -\frac{p-1}{(p-2)^2} \frac{|t|^3}{12^3\omega^2} \times [(p^2+4)W(l^*)^{2p/(10-3p)} - (p-2)^2]. \quad (50)$$

From Eq. (49) it follows that $T_T(l) \geq T_L(l)$ for $p \geq 1$. The solutions of the r_L and r_T recursion relations are only valid for $r_L, r_T \leq 1$. In the ordered phase we therefore choose l^* such that²⁵ $T_T(l^*) = 1$. By Eqs. (46) and (49) $T_T(l^*) = 1$ is equivalent to

$$-t(l^*) = \left[\frac{p-2}{-p} \right] \quad (51)$$

such that to leading order

$$e^{2tl^*} = |t|^{-1} \left[\frac{p-2}{-p} \right]. \quad (52)$$

For $T_T(l^*) = 1$, it further follows from Eqs. (46) and (49) that

$$T_L(l^*) = \left[\frac{p-2}{-p} \right] \quad (53)$$

and the susceptibilities take the form,

$$2\chi_{L,T}^{-1} = \left\{ -\frac{1}{p/(p-2)} \right\} |t| W(l^*)^{-2(2-p)/(10-3p)}, \quad (54)$$

where we have made use of Eq. (9). From Eqs.

(47), (50), and (54) we obtain for $p < 2$

$$\begin{aligned} Q &\propto |t|^\beta, \\ c_v &\propto |t|^{-\alpha}, \\ \chi_{L,T} &\propto |t|^{-\gamma}, \end{aligned} \quad (55)$$

with α and γ given by Eqs. (12) and β given by

$$\beta = 1 - \epsilon/(10-3p), \quad (56)$$

in agreement with scaling.

From Eq. (54) it follows that $\chi_T > 0$ for $p < 2$, and the assumed second-order transition is stable. For $p > 2$ $W(l^*)$ becomes complex such that all the thermodynamic functions develop an imaginary part due to the fluctuations. Because there is no real solution for Q a second-order transition is not possible in this case. Instead the stable transition is found to be first order.

According to Eq. (53) $T_L(l^*)$ vanishes as $p \rightarrow 2$. We note from Eqs. (40), (41), and (43) that this does not give rise to any singularities in higher order in $\chi_{L,T}$ or the equation of state as $p \rightarrow 2^-$.

G. First-order transition $p \geq 2$

In the case of a second-order transition, as just discussed, $u(l)$ and $v(l)$ can be neglected because they are irrelevant variables with an l dependence (neglecting ϵ corrections) $\sim e^{-2l}$. In the disordered phase $e^{-2l^*} \sim t$ and these terms vanish as $t \rightarrow 0$. In the ordered phase products of $u(l)$ and $v(l)$ with $Q(l)$ can be constructed with an l dependence of the same order as the leading terms. However, because $Q \rightarrow 0$ as $t \rightarrow 0$ the u and v terms again do not contribute in the neighborhood of the critical point. In the case of a first-order transition, on the other hand, neither t nor Q vanish at the transition point and the quartic terms, as well as other higher-order terms may all be equally important, unless another small parameter can be found in which to expand. The transition is second order for $p \leq 2$, and $(p-2)$ will turn out to be a convenient parameter in which to expand.

Equating the free energies of the disordered and ordered phases determines the first-order transition temperature. We note that the first term of the ordered phase free energy Eq. (45) is equal to the free energy of the disordered phase. Thus we obtain,

$$\begin{aligned} \frac{1}{4} t(l^*) Q^2(l^*) - c\omega(l^*) Q^3(l^*) \\ + [u(l^*) + b v(l^*)] Q^4(l^*) = 0. \end{aligned} \quad (57)$$

Taking the limit $h(l^*)/Q(l^*) \rightarrow 0$, we obtain from

the equation of state Eq. (43) a second equation,

$$\frac{1}{2}t(l^*) - 3c\omega(l^*)Q(l^*) + 4[u(l^*) + bv(l^*)]Q^2(l^*) = 0 \quad (58)$$

Solving these two equations, we obtain,

$$t_c(l^*) = \frac{(p-2)^2 c^2 \omega^2(l^*)}{[u(l^*) + bv(l^*)]} \quad (59)$$

$$Q_c(l^*) = \frac{1}{2} \frac{(p-2)c\omega(l^*)}{[u(l^*) + bv(l^*)]} \quad (60)$$

Making use of Eqs. (59) and (60) the expressions for $T_L(l)$ and $T_T(l)$ given by Eqs. (34) may be written

$$T_L(l) = t_c(l), \quad T_T(l) = \frac{6}{p-2} t_c(l) \quad ,$$

with the latter expression valid only for $p \sim 2$. As before l^* is determined by the condition $T_T(l^*) = 1$, or equivalently,

$$t_c(l^*) = \frac{1}{6}(p-2) \quad (61)$$

The latter condition will be used to determine l^* .

It follows from Eqs. (59) and (61) that irrelevant variables will be of higher order in $(p-2)$. Thus $u(l)$ and $v(l)$ can again be neglected in the dt/dl and $d\omega/dl$ equations. Thus the solutions for $t(l)$ and $\omega(l)$ will as before be given by Eqs. (8). Furthermore it will be sufficient to solve the du/dl and dv/dl equations only for $p = 2$. In this case Eqs. (36) reduce to

$$\frac{du}{dl}(l) = (-2 + \epsilon)u(l) - 2^4 3^2 K_6 [4u(l)\omega^2(l) - 3v(l)\omega^2(l)] \quad , \quad (62)$$

$$\frac{dv}{dl}(l) = (-2 + \epsilon)v(l) + 2^5 3^2 K_6 [4u(l)\omega^2(l) - 3v(l)\omega^2(l)] \quad , \quad (63)$$

with solutions

$$\begin{aligned} v(l) &= -\frac{4}{5} u e^{(-2+\epsilon)l} [W(l)^{-5} - 1] \\ &\quad + \frac{1}{5} v e^{(-2+\epsilon)l} [3W(l)^{-5} - 2] \quad , \\ u(l) &= \frac{2}{5} u e^{(-2+\epsilon)l} [W(l)^{-5} + \frac{3}{2}] \\ &\quad - \frac{3}{10} v e^{(-2+\epsilon)l} [W(l)^{-5} - 1] \quad , \end{aligned} \quad (64)$$

where $W(l)$ is given by Eq. (8). From Eqs. (8), (59)–(61), and (64) l^* is determined by

$$e^{2l^*} = \left[\frac{u + bv}{\omega^2 c^2} \right] \frac{W(l^*)}{6(p-2)} \quad (65)$$

From Eq. (59) it follows that:

$$t_c = \frac{\omega^2 c^2}{(u + bv)} (p-2)^2 W(l^*)^{-1} \quad (66)$$

or

$$t_c \sim (p-2)^{2+\epsilon/2} \quad , \quad (67)$$

where t_c is the shift in the transition temperature from the spinodal point. From Eq. (60)

$$Q_c = \frac{1}{2}(p-2) \frac{c\omega}{(u + bv)} W(l^*)^{-1/2} \quad (68)$$

or

$$Q_c \sim (p-2)^{1+\epsilon/4} \quad , \quad (69)$$

where Q_c is the discontinuity of the order parameter at the first-order transition.

For $p \sim 2$ the susceptibilities take the form,

$$2\chi_L^{-1} = e^{-2l^*} = \frac{\omega^2 c^2}{u + bv} (p-2)^2 W(l^*)^{-1} \quad , \quad (70)$$

$$2\chi_T^{-1} = \frac{6\omega^2 c^2}{u + bv} (p-2) W(l^*)^{-1} \quad ,$$

or

$$\begin{aligned} \chi_L^{-1} &\sim (p-2)^{2+\epsilon/2} \quad , \\ \chi_T^{-1} &\sim (p-2)^{1+\epsilon/2} \quad . \end{aligned} \quad (71)$$

These expressions show that the assumed first-order transition is stable for $p > 2$ but unstable for $p \leq 2$ when $W(l^*)$ becomes complex. For $p \leq 2$ there is no real solution for Q_c , and a first-order transition is not possible.

In the above discussion only the leading-order solution for χ_T , χ_L , and \tilde{h} have been made use of. It is, however, easy to check from the next-order corrections that no divergences develop in higher order, as $p \rightarrow 2^+$.

H. Essential singularity at T_c . The role of instantons

Above it has been shown explicitly that the transition is first order for $p > 2$ by calculating the discontinuity in the order parameter as a function of $(p-2)$ and by testing the stability of the transition. It is generally believed that the free energy has an essential singularity at a first-order transition.^{5-7, 26} The most detailed investigation has been carried out for the ϕ^4 continuum Ising model in a magnetic field h . In the ordered phase this model undergoes a first-order transition at $h = 0$. For the Potts model the free-energy functional for the ordered phase is given by Eq. (17). It is reasonable to assume that

any instanton, which is associated with tunneling phenomena, results only from the longitudinal degrees of freedom (see also Fig. 1). Neglecting the transverse degrees of freedom Eq. (17) reduces to

$$H = - \int d^d x \left[\frac{1}{4} (\nabla \mathcal{L})^2 + \frac{1}{4} r_L \mathcal{L}^2 - \tilde{\omega} \mathcal{L}^3 + \lambda \mathcal{L}^4 + \tilde{h} \mathcal{L} \right], \quad (72)$$

where

$$\begin{aligned} \tilde{\omega} &= (p-2)c\omega - 4\lambda Q, \\ \lambda &= u + bv \end{aligned} \quad (73)$$

and where r_L and \tilde{h} are given by Eq. (19). To make contact with an Ising model in an external field we translate $\mathcal{L}(x)$

$$\mathcal{L}(x) = \mathcal{L}_0 + \phi(x).$$

By choosing $\mathcal{L}_0 = \tilde{\omega}/4\lambda$, the cubic term in $\phi(x)$ is eliminated, and we obtain a ϕ^4 continuum Ising model,

$$\mathcal{K} = - \int d^d x \left[\frac{1}{4} (\nabla \phi)^2 + \frac{1}{4} \tilde{r} \phi^2 + \lambda \phi^4 - h \phi \right], \quad (74)$$

where

$$\begin{aligned} \tilde{r} &= -\frac{1}{2}(p-2)^2 \frac{c^2 \omega^2}{\lambda}, \\ h &= \frac{(p-2)c\omega}{8\lambda} (r - r_c), \end{aligned} \quad (75)$$

and where we have made use of Eqs. (25). We note that $\tilde{r} < 0$, and that the effective field h is linear in $(r - r_c)$. For the ϕ^4 model described by Eq. (74) there is an instanton,^{6,7} which is usually referred to as a critical droplet. The imaginary part of the free energy as $h \rightarrow 0^-$ has been calculated by Langer⁶ for three dimensions and has been generalized to d dimensions by Günther, Nicole, and Wallace.⁷ When expressed in terms of the Potts model the result is

$$\text{Im}F \sim \exp(-B|p-2|^{d+2}/|r-r_c|^{d-1}) \quad (76)$$

for $r \rightarrow r_c +$, where only the leading exponential dependence has been kept. This expression has an essential singularity at T_c . When the coexistence curve is approached from the stable phase an essential singularity is expected in the real part of the free energy. A calculation for this case has been carried out by Klein, Wallace, and Zia²⁶ using a real-space renormalization-group approach.

As discussed in Sec. III A the upper (superheating) stability limit is determined by $r_L = 0$, while $r_T > 0$. The transverse degrees of freedom are noncritical, and will therefore vanish under the iterations of the renormalization group. Thus again only the longitudinal part of Eq. (17) need be considered. In this case a shift in \mathcal{L} to eliminate the cubic term is not useful, as this gives rise to a constant field indepen-

dent of $(r - r_{c2})$ for $r \approx r_{c2}$, $Q \approx Q_{c2}$. Thus at the upper stability limit, $r \rightarrow r_{c2}^-$, the behavior of the Potts model is the same as for an Ising model with a cubic term

$$\mathcal{K} = - \int d^d x \left[\frac{1}{4} (\nabla \phi)^2 + \frac{1}{4} \tilde{r} \phi^2 - \tilde{\omega} \phi^3 + \lambda \phi^4 \right], \quad (77)$$

where

$$\begin{aligned} \tilde{r} &= (-r + r_{c2}), \\ \tilde{\omega} &= -\frac{1}{2}\omega c(p-2), \end{aligned} \quad (78)$$

for $r \approx r_{c2}$, at its lower stability limit $\tilde{r} \rightarrow 0^+$. It is well known that the ϕ^3 model does not have any stable fixed points. It does however, have instanton solutions.²⁷ Thus in the same way as the properties near T_c may be described in terms of instantons of the ϕ^4 model, it is possible that the properties near these stability limits may be described in terms of instanton solutions of the ϕ^3 model.

Finally we discuss briefly the lower stability limit (spinodal point). Although a first-order transition occurs at $t = t_c$ for $p > 2$, within the renormalization-group calculation the $Q = 0$ minimum remains locally stable for all $p < \frac{10}{3}$ until the stability limit $t = 0$ is reached. It follows from Eqs. (8) and (11) that $\chi^{-1} > 0$ for $t > 0$ and $p < \frac{10}{3}$. It is not clear whether the exponents given by Eqs. (12), which describe the second order transition for $p < 2$, in any sense describe the spinodal point for $2 < p < \frac{10}{3}$, or whether instead the usual small-amplitude and low-order perturbation approximations of the renormalization group breaks down in this case. The latter seems more likely. The possible role of instantons has not been investigated.

For $p > \frac{10}{3}$ there is no stable fixed point, χ^{-1} is no longer real, and no description of the spinodal points is possible by renormalization-group calculations based on a perturbation expansion in ω^2 . As discussed above, the behavior of the Potts model at the upper stability limit is for all $p > 2$ identical to that at the lower stability limit of a ϕ^3 Ising model. This model has no stable fixed points and the behavior is presumably similar to that obtained for the Potts model at the lower stability limit for $p > \frac{10}{3}$.

The results for $2 < p < \frac{10}{3}$ at the lower stability limit remain anomalous. However, the present calculations demonstrate that the presence of a stable accessible fixed point is no guarantee that the transition is second order as has generally been assumed.

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