Critical dynamics of the Potts model

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The dynamical critical index z is calculated for the two-dimensional Potts model on square lattice using a Migdal-type recursion method generalized to dynamics.

I. INTRODUCTION

Physisorbed films have gained much interest in the last few years because they provide a possibility to study experimentally[1] higher-order phase transitions in two dimensions. In order to study these systems theoretically Potts lattice gas models on a triangular lattice were constructed.[2–4] Applying Migdal's recursion method to these Potts models good agreement with experiments was obtained.[3,5] The experiments until now studied only the time independent properties of the physisorbed systems. By studying time dependent processes in these systems, we may hope to learn about dynamical critical phenomena in two dimensions. To encourage such experiments we investigated the dynamics of the q-state Potts model using Migdal's recursion method[6] which we have recently generalized to study the two-dimensional (2D) kinetic Ising model.[7] The method of Ref. 7 is based on the linear-response theory worked out by Achiam and Kosterlitz,[8] using a finite number of perturbations. Recently Mazenko *et al.*[9] developed a method for the kinetic Ising model in which they were able to follow the full dynamical operator under renormalization-group transformation (RGT). Memory effects within their approach are eliminated by a clever choice of the RGT. Unfortunately, because of the complexity of this method it is very difficult to apply it to the study of the q-state Potts model. The question of memory effects is not fully solved within the method of Ref. 8, where instead of the full master equation only the linear part is treated. Certainly this remains an open question in our Migdal-type approach, too, in which the invariance of only this linear part is required under the RGT. On the other hand, the numerical values of the dynamical critical indices obtained by the methods of Refs. 7 and 8 and by the method of Mazenko *et al.* are very close to each other. This may indicate that memory effects are perhaps irrelevant in the sense of RG theory. For more details on the Migdal-type RG method we refer to Ref. 7, in which we obtained results for the 2D kinetic Ising model, which were in excellent agreement with the available high-temperature series-expansion results. Because of the similarity between that and the present problem, we expect our results for the Potts model to be just as reasonable (at least for small q; see below). Applying this method to the Potts model on a square lattice we have obtained, for the first time (whereas high-temperature series calculations have been performed for the Ising system, none is available for the Potts model), numerical results for the dynamical critical index of the Potts model in two dimensions.

II. THE MODEL AND RESULTS

To study the dynamics of the q-state Potts model we start with the following master equation:

$$\tau\frac{dP}{dt}=\sum_i\sum_{\{\hat{s}_i\}}[-w(s_i\to\hat{s}_i)P(\{s\})+w_i(\hat{s}_i\to s_i)P(\{s\},\hat{s}_i)]\ . \tag{1}$$

Here P is the time dependent normalized probability distribution of the spin configuration, $\sum_{\{\hat{s}_i\}}$ means summation over all values of $\hat{s}_i$, and $w_i(s_i\to\hat{s}_i)$ is a transition probability from a spin configuration with $\{s_i\}$ into one, in which the ith spin has a value given by $\hat{s}_i$. τ sets the time scale. The $q=2$ case in Eq. (1) corresponds to the model studied by Glauber.[10] The transition probabilities satisfy the detailed balance condition and are chosen as

$$w_i(s_i\to\hat{s}_i)=\left(\frac{P_e(\hat{s}_i)}{P_e(s_i)}\right)^{1/2}, \tag{2}$$

where

$$P_e=\frac{1}{Z}\exp\left(\frac{J}{KT}\sum_{i,\omega}\delta_{s_i,s_{i+\omega}}\right),\quad K\to k \tag{3}$$

is the equilibrium Boltzmann distribution, and Z is the partition function. $\sum_\omega$ means summation over nearest neighbors. Following Achiam and Kosterlitz[8] we introduce a time dependent perturbation and use linear response. In this paper we use energylike per-

turbation. The case of the magneticlike perturbation is technically more complicated and will not be discussed here. In any case it is believed that the critical exponent z should be the same for the energylike and magneticlike perturbations.[11] The details of the present calculation are similar to our earlier calculations[7] and will not be presented. Here we recapitulate our key approximations. We assume for $P(t)$ the following form:

$$P(t)=\frac{1}{Z}\exp\left[\frac{1}{2}K\sum_{i,\omega}\delta_{s_i,s_{i+\omega}}\right]\left[1+\frac{1}{2}h(t)\sum_{i,\omega}\delta_{s_i,s_{i+\omega}}\right] , \quad (4)$$

where $K=J/kT$.

Putting (4) into Eq. (1) and keeping only terms linear in h we obtain

$$\tau\frac{dP(t)}{dt}=\frac{h(t)}{2}\sum_i\sum_{\{\hat{s}_i\}}\left[w_i(s_i\to\hat{s}_i)\sum_\omega(\delta_{s_i s_{i+\omega}}-\delta_{\hat{s}_i s_{i+\omega}})\right]P_e(s_i) . \quad (5)$$

In Eq. (5) we have used the detailed balance condition. We assume Eq. (5) to be valid for a finite system (in $d=2$ a square) with linear size L and average both sides of Eq. (5) over the internal spins of this finite system. Furthermore we assume that all the spins along the edges of the square domain (in $d=2$) are aligned and therefore can be replaced by one average spin. We then find the most general functional form of the right-hand side of the averaged equation (5), which satisfies the factorization hypothesis[6,7] and which also is invariant under the appropriate Migdal's recursion (for details see Ref. 7). The recursion equation for the time scale factor τ is obtained by requiring that the master equation in the Migdal variables had the same form when written in terms of quantities corresponding to linear scales L and λL. The actual Migdal transformation for the master equation follows closely that given in Ref. 7. We finally get

$$\tau_{\lambda L}=(h_{\lambda L}/h_L)C(K^*,q)\tau_L , \quad (6)$$

where $C(K^*,q)$ depends only on the fixed-point value of the K and the value of q. Since $h_{\lambda L}$ and h_L are taken at the same time, we can use the static result for their ratio, namely,

$$h_{\lambda L}/h_L=\lambda^{1/\nu} . \quad (7)$$

It is well known that the best numerical results obtained by the Migdal's method correspond to $\lambda=1$ (in a limiting procedure); therefore, we also take the limit $\lambda\to 1$. The value of K^* and ν for this case had also been calculated by Stephen[12] and are given by

$$K^*=\ln(1+\sqrt{q}) , \quad (8)$$

$$\nu^{-1}=2[1-\ln(1+\sqrt{q})/\sqrt{q}] . \quad (9)$$

Equation (8) is the exact result obtained by Potts using duality arguments.[13] According to dynamical scaling,[14] $\tau_{\lambda L}=\lambda^z\tau_L$.

Using Eq. (8) in Eq. (6), in the limit $\lambda\to 1$, we obtain

$$z=\frac{1}{\nu}+\frac{2+\sqrt{q}}{2\sqrt{q}}\ln(1+\sqrt{q})=\frac{1}{\nu}+\pi_E . \quad (10)$$

For $q=2$ Eq. (10) gives the result obtained in Ref. 7 for the Glauber-Ising model. Equation (10) involves the value of ν, which is known to be badly determined by the Migdal approximation. As it was shown in Ref. 7 if we use the exact result for ν in Eq. (10) for $q=2$ and calculate only π_E using the above method, the agreement with the results of high-temperature series expansions is very good, moreover magnetic and energylike perturbations give essentially the same z. For $q>2$ neither exact result for ν nor high-temperature expansion for z exist. What we can do is to use the value of ν for the $q=3$ and 4 Potts models obtained from high-temperature series expansions and scaling. If we had estimated z by using magnetic perturbation, we would get (see Ref. 7) $z=y_H+\pi_M$, where y_H is the usual magnetic eigenvalue of the linearized static recursion matrix and π_M is an exponent determined purely by the dynamics. y_H is believed to be given by the Migdal's procedure rather accurately. Unfortunately the magnetic (or odd) perturbation is more difficult to deal with in the case of the Potts model then in the Ising model, therefore it will be discussed elsewhere.

In Table I we listed the values of z for $q=3$ and 4 obtained with ν determined by Eq. (9) and by high-temperature series expansions. There exists strong numerical evidence that reduced static exponents like $\gamma/\nu, \beta/\nu$ are the same for many two-dimensional discrete models (including the Baxter model). Suzuki[20] argues that this perhaps could be expected on the basis of rather general theoretical arguments. Now, $z=\Delta/\nu$ where Δ is defined by $\tau=(T-T_c)^{-\Delta}$. It means that z is a reduced exponent of the same type as those mentioned above. This, together with values of z_1, in Table I could suggest that perhaps z should be the same for those two-dimensional discrete models for which the reduced static exponents are the same. However, we would like to stress that this is at best only a conjecture. There are several reasons why only the $q=3$ and 4 cases are listed. According to Baxter, and more recently Nienhuis *et al.*[21] in the two-dimensional Potts model with $q>4$ a first-order phase transition takes place, while for $q\leq 4$ the transition is of higher order. The results of the Migdal type calculation predict first-order transition in the limit $q\to\infty$. Comparing the result [Eq. (9)] with the conjecture of den Nijs[22] one can see that the agreement is excellent for small q. The physisorbed systems of experimental interest are modeled by two-dimensional Potts models with $q=3$ and 4 so if these theoretical models are correct we may expect the experiments to give some information on z for $q=3$ and 4.

Our Migdal-type method for critical dynamics can

TABLE I. z_1 is obtained using Eqs. (9) and (10), and z_2 is obtained using Eq. (10) and the results of series expansions and scaling for ν^{-1}, that is $\nu^{-1}=(\beta+\gamma/2)^{-1}$.

	$q=2$			$q=3$		$q=4$	
Exponents	This work	Exact	Series	This work	Series and scaling	This work	Series and scaling
ν^{-1}	0.75	1		0.840	1.17[a]	0.900	1.45[b,c]
π_E	1.065			1.082		1.099	
z_1	1.819			1.922		2	
z_2	2.065		2.135[d]			2.55	
			2.00[e]	2.25			

[a]Reference 17.
[b]Reference 18 for the value of β.
[c]Reference 19 for the value of γ.
[d]Reference 15.
[e]Reference 16.

be used in any dimension. It is easy to see that the method is exact in one dimension where we obtain $z=2$ independently of q. In principle we could predict z for the three-dimensional $q=3$ Potts model too, which according to Bak and Domany[23] could describe the order-disorder transition in the stage-1 graphite intercalation compound C_6Li. However, it is still not clear what is the nature of the phase transition in this model, also, the results of a Migdal-type calculation are not as reliable in three dimensions as in two dimensions.

III. CONCLUSIONS

In conclusion we may say that the present calculation is the first one which gives results for the dynamics of the two-dimensional Potts model. Since this model describes rather well the static critical properties of krypton and nitrogen submonolayers adsorbed on graphite, we hope that the above results will help us learn about the dynamical critical properties of these systems. Experimentally, probably the exponent $\Delta=\nu z$ is more accessible than z. Its value can be obtained from Table I, using either Eq. (9) or results for ν from series expansions and scaling. By comparing with other numerical results, obtained on the basis of Migdal-type calculations involving the exponent ν, we think that the above results should be at worst within 20–25% of the real values. Either high-temperature series expansions for Δ or experiments on the dynamics of physisorbed systems would be welcome to support these results.

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