One-dimensional resistance reduces to one-dimensional Ising partition function

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A resistance ρ of an arbitrary one-dimensional system of elastic scatterers exactly reduces to a one-dimensional partition function of an inhomogeneous Ising Hamiltonian. Except for special cases, $\ln\rho$ is proportional to the length of the system, in accordance with previous work. The dependence of ρ on other parameters of the system is discussed. For certain random systems an explicit formula for ρ is presented.

The theory of a 1D (one-dimensional) resistance of random systems is important for the very understanding of a metal conductivity.[1-5] An efficient approach to the problem is related[4,5] to the Landauer formula,[6] which reduces the resistance ρ to the transmission coefficient of the system. However, this formula is complemented by a somewhat less efficient composition law of scatters. In this paper, I introduce a complex "resistance amplitude" R^- [such that $\rho = (\pi\hbar/e^2)|R^-|^2$], whose composition law for a sequence of scatterers is linear with respect to each of them. The law provides an exact relation between the resistance of an arbitrary 1D system of elastic scatterers and the free energy of a certain 1D (nearest-neighbor interaction) Ising Hamiltonian with inhomogeneous (e.g., random for a random system) "magnetic field" and "exchange integrals." Such a Hamiltonian has already been considered[7-13] (e.g., with relation to a helix-coil transition in polymers). So, apart from general statements on the resistivity dependence on the system length L ($\ln\rho \propto L$, in accordance with Refs. 1–5), one learns the ρ dependence (which may be oscillatory) on the parameters of the scatterers and their position. Also, an exact formula for ρ for a certain class of systems (including certain random ones) is presented. For any given system, the obtained equations allow for simple and convenient computer calculations.

We start with the Landauer formula[6] in the form

$$\rho = \frac{\pi\hbar}{e^2}\left|\frac{S}{T}\right|^2 , \qquad (1)$$

where S and T are correspondingly the amplitudes of the reflected and transmitted wave functions ($|S|^2 + |T|^2 = 1$). I introduce a complex "resistance amplitude" R^- and "inverse transmission amplitude" R^+:

$$R^- = -(S/T)^* , \quad R^+ = 1/T , \quad |R^+|^2 - |R^-|^2 = 1 ; \quad (2)$$

a star denotes a complex conjugation.

Suppose successive elastic scatterers are located at the points $x = x_1, x_2, \ldots, x_n, \ldots$, and R_n^-, R_n^+ refer

to the composition of the first n scatterers, while r_n^-, r_n^+ describe the scattering by a single nth scatterer. Then one easily obtains[14] the (linear in R) relation which is of the same nature as the usual law[6]

$$\vec{R}_{n+1} = \begin{bmatrix} r_{n+1}^+ , & (r_{n+1}^-)^* \\ r_{n+1}^- , & (r_{n+1}^+)^* \end{bmatrix}\vec{R}_n , \quad \vec{R}_n = \begin{pmatrix} R_n^+ \\ R_n^- \end{pmatrix} . \quad (3)$$

To exclude in this relation the dependence on the absolute position of the scatterers, we perform the transformation

$$r_n^+ = \tilde{r}_n^+ \exp[ik(x_{n+1} - x_n)] ,$$

$$r_n^- = \tilde{r}_n^- \exp[-ik(x_{n+1} + x_n)] , \qquad (4)$$

$$R_n^s = \tilde{R}_n^s \exp(iskx_{n+1}) , \quad s = \pm 1 ;$$

k being a wave vector. Now Eq. (3) can be written as

$$G_{n+1,\sigma} = \tilde{r}_{n+1}^+ G_{n\sigma} + i\sigma(\tilde{r}_{n+1}^-)^* G_{n\sigma}^* , \qquad (5)$$

$$G_{n\sigma} = \tilde{R}_n^+ + i\sigma(\tilde{R}_n^-)^* , \quad \sigma = \pm 1 . \qquad (6)$$

Thus, the equation for the real $A_{n\sigma}^+$ and imaginary $A_{n\sigma}^-$ parts of $G_{n\sigma}$ reads

$$A_{n+1,\sigma}^s = a_{n+1,\sigma}^{ss'} A_{n\sigma}^{s'} , \quad \sigma, s, s' = \pm 1 , \qquad (7)$$

$$A_{n\sigma}^+ = \text{Re}G_{n\sigma} , \quad A_{n\sigma}^- = \text{Im}G_{n\sigma} , \qquad (7a)$$

where $a_{n\sigma}^{ss'}$ is real; its relation to $\tilde{r}_n^\pm$ is obvious. Later, until Eq. (12), we omit the subscript σ for brevity.

Equation (7) is very convenient for computer calculations for any given (random, periodic with incommensurate periods, etc.) system.

Quantities $a_n^{ss'}$ can be presented in the form

$$a_n^{ss'} = (-1)^{\theta_n^{ss'}} \exp(\ln|a_n^{ss'}|) , \quad 1 - 2\theta_n^{ss'} = a_n^{ss'}/|a_n^{ss'}| , \qquad (8)$$

$$-\ln|a_n^{ss'}| = \gamma_n + \delta_n s + \kappa_n s' + \chi_n ss' , \qquad (9)$$

$$\theta_n^{ss'} = \zeta_n + \eta_n s + \xi_n s' + \nu_n ss' . \qquad (10)$$

The coefficients in Eqs. (9) and (10) are easily related to a, θ; the coefficients in Eq. (10) change only when $a_n^{ss'}$ changes its sign.

Equations (7)–(10) provide

$$A_n^s = \sum_{\{s_k\}} (-1)^{H_\theta} \exp(-H_a) = \sum_{H_\theta} (-1)^{H_\theta} Z_a(H_\theta) \quad , \quad (11)$$

where H_θ and H_a are Ising nearest-neighbor interaction Hamiltonians [with H_θ always integer and, e.g., $H_a = -\sum_{k=1}^n \ln|a_k^{s_k s_k - 1}|$, $s_n = s$, with $\ln|a|$ from Eq. (9)] and $Z_a(H_\theta)$ is the partition function for H_a and the fixed value of H_θ; "temperature" equals one. The calculation of $Z_a(H_\theta)$ reduces to the calculation of the partition function for the 1D Ising Hamiltonian $H_\mu = H_a - \mu H_\theta$ with μ being a chemical potential.[15] This completes the exact evaluation of the resistivity of an arbitrary 1D system, as, by Eqs. (1), (2), (4), (6), and (7a):

$$\rho = (4e^2/\pi\hbar)[(A_{N+}^+ - A_{N-}^+)^2 + (A_{N+}^- - A_{N-}^-)^2] \quad , \quad (12)$$

N being the total number of scatterers.

When absolute values of exchange integrals are large, the free energy for the Hamiltonian of H_μ type was determined[12,13] as an explicit function of its parameters (see also Refs. 11 and 10). The main qualitative results (unrelated to the values of exchange integrals) are as follows.

(a) All sequences of scatterers (except for those of measure zero), and in particular random ones, have a negative extensive free energy.[16] Thus, $\ln\rho \propto N$ (in accordance with Refs. 1–5). This is vividly demonstrated by the case of real $\tilde{r}_{n+1}^+ = \alpha_{n+1}$ and $i\tilde{r}_{n+1}^- = \beta_{n+1}$ in Eq. (5), when $A_{N\sigma}^s$ in Eq. (12) exactly equals

$$A_{N\sigma}^s = \prod_{k=1}^N (\alpha_k - \sigma s \beta_k) \quad . \quad (13)$$

In particular, for a random sequence of two different

scatterers (denoted by superscripts) with concentrations $C^{(1)}$ and $C^{(2)}$

$$\ln|A_{N\sigma}^s| = N(C^{(1)} \ln|\alpha^{(1)} - \sigma s \beta^{(1)}| + C^{(2)} \ln|\alpha^{(2)} - \sigma s \beta^{(2)}|) \quad . \quad (13a)$$

(b) When a "magnetic field" in H_μ changes its sign, an oscillatory dependence of ρ on the characteristics of scatterers and their position is possible.

(c) A "special-type" situation is illustrated by a periodic sequence of identical scatterers; $\tilde{r}_n^\pm = \tilde{r}^\pm$. Then in Eq. (12)

$$A_{N\sigma}^s = U_\sigma^s \cos(N\omega) + V_\sigma^s \sin(N\omega) \quad , \quad \cos\omega = \text{Re}\,\tilde{r}^+ \quad , \quad (14)$$

the constants U_σ^s, and V_σ^s are easily obtained from the boundary conditions (for $N = 1, 2$). When $\text{Re}\,\tilde{r}^+ < 1$, i.e., the Fermi energy is inside the band, created by scatterers (this must always be the case in metals), the resistance is finite (and may be zero) and oscillates with N.

The reduction of the resistance problem to the Ising free energy allows for the application to the problem of powerful thermodynamic methods (e.g., scaling) and for the investigation of various kinds of 1D systems and of transitions between different ρ dependences on the parameters.

For an arbitrary dimensionality, Eq. (3) remains valid, once its parameters are replaced by the appropriate angle dependent ones, and the presented approach is applicable.

Note added in proof. After this paper was submitted, I learned about a similar approach by E. Abrahams and M. Stephen, J. Phys. C **13**, L377 (1980).

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[14]For the waves $\exp(\pm ikx)$, incident from opposite directions, R^+'s coinside, while (iR^-)'s are complex conjugated. Thus, if in the region x, where $x_n < x < x_{n+1}$ and the potential energy is zero, the wave function is $p\exp(ikx) + g\exp(-ikx)$, and $r_n^+ \equiv 1/\tau_n$; $r_n^- \equiv -(\lambda_n/\tau_n)^*$, then [accounting also for Eq. (2)] $S_{n+1} = S_n + gT_n$; $T_{n+1} = p\tau_{n+1}$; $g = p\lambda_{n+1} = -p\tau_{n+1}(r_{n+1}^-)^*$; $p = T_n + gT_n R_n^-$. These relations provide Eq. (3).

[15]$Z_a(H_\theta)$ exactly equals the factor, multiplying $\exp(\mu H_\theta)$ in the partition function for H_μ; the calculation of A_n^s in some cases may imply more accurate than usual calculation of $Z_a(H_\theta)$. Alternatively, Eq. (11), due to $(-1)^\theta = \exp(i\pi\theta)$, reduces to the analytical continuation of the thermodynamic potential Ω to $\mu = i\pi$.

[16]This immediately follows from $\min_\mu E_\mu < 0$, where E_μ is the ground-state energy of H_μ.