Universal critical amplitude ratios for percolation

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The hypothesis of universality implies that for every scaling relation among critical exponents there exists a universal ratio among the corresponding critical amplitudes. If one writes $B|t|^{\beta}$, $A_F^{\pm}|t|^{2-\alpha}$, $C^{\pm}|t|^{-\gamma}$, and $\xi_0|t|^{-\nu}$ [where $t=(p_c-p)/p_c$, p being the concentration of nonzero bonds, and $+(-)$ stands for $p<p_c$ ($p>p_c$)] for the leading singular terms in the probability to belong to the infinite cluster, the mean number of clusters, the clusters' mean-square size, and the pair connectedness correlation length, then it is shown that the ratios A_F^+/A_F^-, C^+/C^-, $A_F^+B^{-2}C^+$, ξ_0^+/ξ_0^-, and $A_F^+(\xi_0^+)^d$ (d is the dimensionality) are universal. Similar quantities are found for the behavior at $p=p_c$ (as a function of a "ghost" field). All of these universal ratios are derived from a universal scaled equation of state, which is calculated to second order in $\epsilon=6-d$. The (extrapolated) results are compared with available information in dimensionalities $d=2,3,4,5$, with reasonable agreements. The amplitude relations become exact at $d=6$, when logarithmic corrections appear. Additional universal ratios are obtained for the confluent correction to scaling terms.

I. INTRODUCTION AND SUMMARY OF RESULTS

The behavior of percolation models near the percolation threshold involves singularities which are very similar to those of systems undergoing thermodynamic phase transitions.[1] As the concentration p of nonzero bonds approaches the percolation concentration p_c from above, the probability of a bond to belong to the infinite cluster decreases as

$$P_{\infty}(p)=1-\sum_{s=1}^{\infty} s n_s(p) \simeq B|t|^{\beta}, \quad |t| << 1 \quad , \tag{1.1}$$

where $t=(p_c-p)/p_c$. The function $n_s(p)$ denotes the average number (per bond) of clusters containing s bonds. Equation (1.1) is very similar to that describing the order parameter in a thermodynamic phase transition. Similarly, other moments of $n_s(p)$ also exhibit singularities. In particular, one can define the singular parts of the mean number of clusters F and their mean-square size S,[1]

$$F(p)=\left[\sum_s n_s(p)\right]_{\text{sing}} = A_F^{\pm}|t|^{2-\alpha} \quad , \tag{1.2}$$

$$S(p)=\left[\sum_s s^2 n_s(p)\right]_{\text{sing}} = C^{\pm}|t|^{-\gamma} \quad , \tag{1.3}$$

where the amplitudes A_F^+(and C^+) and A_F^-(C^-) refer to $t>0$($p<p_c$) or $t<0$($p>p_c$), in analogy to the free energy and the order-parameter susceptibility. One can also define correlation lengths for the pair connectedness

$$\xi(p)=\xi_0^{\pm}|t|^{-\nu} \quad . \tag{1.4}$$

Introduction of a "ghost field"[1] H yields the critical "isotherm"

$$P(p_c,H)=\left[1-\sum_{s=1}^{\infty} s n_s(p_c)e^{-sH}\right]_{\text{sing}} = EH^{1/\delta} \quad . \tag{1.5}$$

One should note that throughout this paper we always refer to *bond* percolation and not to *site* percolation. Similar relations (1.1)–(1.5) may be written for the latter problem.

Much progress has been achieved in recent years in the determination of the various exponents appearing in Eqs. (1.1)–(1.5), using series expansions, Monte Carlo simulations, renormalization-group calculations, and experiments.[1] The *exponents* turn out to be *universal;* i.e., they depend only on the *dimensionality* of the systems considered and not on any other specifics (such as the lattice structure). In the theory of phase transitions it has become clear that in addition to the exponents, the *amplitudes* which appear in Eqs. (1.1)–(1.5) must also obey *universal relations.*[2,3] This led various authors to search for analogous relations in the percolation problem.[4] The present paper is devoted to this question.

Much of the current understanding of percolation processes is due to the realization[5] that these processes can in fact be described by the limit $q\to 1$ of the lattice-gas phase transitions associated with the q-state Potts model.[6] In particular, this correspondence has been used to identify $d=6$ dimensions as the upper critical dimensionality, above which mean-field theory (i.e., percolation theory on Cayley trees[1]) becomes correct, and to expand critical exponents in powers of $\epsilon=6-d$.[7–10] This ϵ expansion has also been used by Stephen[11] to derive the distribution function $n_s(p)$ and the equation of state $P_\infty(p,H)$ [Eq. (1.5)] to first order in ϵ. However, we are not aware of any earlier discussion of *universal amplitude ratios* in this context. In the present paper the universality of such ratios is discussed, and various ratios are calculated as expansions in $\epsilon=6-d$. Some of the present results can also be derived, after some algebra, to a lower order in ϵ, from Ref. 11.

For convenience, the final results of this paper are listed here. The amplitudes in Eqs. (1.1)–(1.5) are found to obey the following universal relations:

$$A_F^+/A_F^- = -\tfrac{1}{5}(1+\tfrac{26}{35}\epsilon)+O(\epsilon^2)\ , \tag{1.6}$$

$$C^+/C^- = \gamma/\beta + O(\epsilon^3) = \delta - 1 + O(\epsilon^3)\ , \tag{1.7}$$

$$R_\chi = C^+E^{-\delta}B^{\delta-1} = 2^{\delta-2}+O(\epsilon^3)\ , \tag{1.8}$$

$$\begin{aligned} R_C &= \alpha(2-\alpha)(1-\alpha)A_F^+B^{-2}C^+ \\ &= \tfrac{1}{4}(1+\tfrac{5}{7}\epsilon)+O(\epsilon^2)\ , \end{aligned} \tag{1.9}$$

$$\xi_0^+/\xi_0^- = 1+\tfrac{5}{42}\epsilon+O(\epsilon^2) = 2\nu+O(\epsilon^2)\ , \tag{1.10}$$

and

$$\begin{aligned} (R_\xi^+)^d &= \alpha(1-\alpha)(2-\alpha)A_F^+(\xi_0^+)^d \\ &= \frac{7K_d}{2\epsilon}\left[1-\frac{397}{2^2 3^2 7^2}\epsilon\right]+O(\epsilon)\ , \end{aligned} \tag{1.11}$$

where $K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)$.

In these expressions, the exponents should be interpreted as their corresponding ϵ expansions[9,10]

$$\alpha = -1+\tfrac{1}{7}\epsilon-\frac{443}{2^2 3^2 7^3}\epsilon^2+O(\epsilon^3)\ , \tag{1.12}$$

$$\beta = 1-\tfrac{1}{7}\epsilon-\frac{61}{2^2 3^2 7^3}\epsilon^2+O(\epsilon^3)\ , \tag{1.13}$$

$$\gamma = 1+\tfrac{1}{7}\epsilon+\frac{565}{2^2 3^2 7^3}\epsilon^2+O(\epsilon^3)\ , \tag{1.14}$$

$$\delta = 2+\tfrac{2}{7}\epsilon+\frac{565}{(2)3^2 7^3}\epsilon^2+O(\epsilon^2)\ , \tag{1.15}$$

and

$$2\nu = 1+\tfrac{5}{42}\epsilon+\frac{589}{(2)3^3 7^3}\epsilon^2+O(\epsilon^3)\ . \tag{1.16}$$

At $d=6$, the power laws (1.1)–(1.5) have *logarithmic corrections*[12,13]

$$P_\infty(p) = \hat{B}|t|\,|\ln|t/t_0||^{2/7}\ , \tag{1.17}$$

$$F(p) = \hat{A}_F^\pm|t|^3|\ln|t/t_0||^{2/7}\ , \tag{1.18}$$

$$S(p) = \hat{C}^\pm|t|^{-1}|\ln|t/t_0||^{2/7}\ , \tag{1.19}$$

$$\xi(p) = \hat{\xi}_0^\pm|t|^{-1/2}|\ln|t/t_0||^{5/42}\ , \tag{1.20}$$

$$P_\infty(p_c,H) = \hat{E}|H|^{1/2}|\tfrac{1}{2}\ln|H||^{2/7}\ , \tag{1.21}$$

where t_0 is some nonuniversal constant. The amplitudes in Eqs. (1.17)–(1.21) obey the same universal relations (1.6)–(1.10) in the limit $\epsilon\to 0$. The relation (1.11) must however be replaced by[14]

$$F(p)\xi(p)^6 = \tilde{D}|\ln|t/t_0||,\quad d=6\ , \tag{1.22}$$

with the *universal* amplitude

$$\tilde{D} = 7K_6/24 = 7/2^9 3\pi^3\ . \tag{1.23}$$

In addition to these results, one can discuss confluent *corrections to the leading singular terms.*[15] Equations (1.1)–(1.4) should in principle be multiplied by the factors $(1+a_p|t|^{\omega\nu})$, $(1+a_F^\pm|t|^{\omega\nu})$, $(1+a_S^\pm|t|^{\omega\nu})$, and $(1+a_\xi^\pm|t|^{\omega\nu})$, with[10,16]

$$\omega = \epsilon-\frac{671}{(2)3^2 7^2}\epsilon^2+O(\epsilon^3)\ . \tag{1.24}$$

As in the thermodynamic case, the ratios

$$\begin{aligned} &a_p/a_F^\pm = 1+O(\epsilon),\quad a_p/a_S^\pm = 1+O(\epsilon)\ , \\ &a_p/a_\xi^\pm = 12/5+O(\epsilon)\ , \end{aligned} \tag{1.25}$$

are also universal.[17]

The outline of the paper is as follows: In Sec. II we recapitulate the *hypotheses of scaling and universality*, and show how all the amplitudes can in fact be determined once two of them are known. Section III contains a detailed calculation, to order $-\epsilon^2$, of the *scaled equation of state*. The calculation is performed using diagrammatic expansions for the q-state Potts model in the limit $q\to 1$. Details of some calculations are given in Appendixes A and B. The correlation length amplitudes are derived, to order $-\epsilon$, in Sec. IV. Appendix C contains an alternative calculation, based on the renormalization-group recursion relations. These results are used in Sec. V to obtain the amplitudes of the confluent corrections to scaling and the logarithmic corrections at $d=6$. The results are evaluated numerically, and compared to existing data (from series and Monte Carlo) in Secs. VI and VII. Section VIII contains final comments and conclusions. The structure of the paper allows anyone not interested in the technical derivation of the results to skip Secs. III–V, and go over to the numerical results.

II. SCALING AND UNIVERSALITY

Using the concept of the "ghost field," one can generalize Eq. (1.5) to the form[1]

$$P_\infty(p,H) = \left[1-\sum_{s=1}^{\infty} s n_s(p)e^{-sH}\right]_{\text{sing}} \tag{2.1}$$

Similarly, Eqs. (1.2) and (1.3) may be generalized

$$F(p,H)=\left[\sum_s n_s(p)e^{-sH}\right]_{\text{sing}} , \tag{2.2}$$

$$S(p,H)=\left[\sum_s s^2 n_s(p)e^{-sH}\right]_{\text{sing}} . \tag{2.3}$$

Note that P_∞ and S are related to the first and second derivatives of F with respect to H, in full analogy with the phase transition case. This analogy leads one to the *scaling hypothesis*[18]

$$F(p,H)=|t|^{2-\alpha}\mathfrak{F}_\pm(H/|t|^{\beta\delta}) . \tag{2.4}$$

Appropriate derivatives then yield the usual scaling relations among the exponents, e.g., $\beta=2-\alpha-\beta\delta$, $\gamma=\beta(\delta-1)$. In fact, the hypothesis (2.4) is a consequence of the assumption that the function $n_s(p)$ has the scaling form[1]

$$n_s(p)=q_0 s^{-\tau}f(q_1|t|s^\sigma),\quad s>>1 , \tag{2.5}$$

with q_0 and q_1 being nonuniversal amplitudes, and

$$\tau=2+1/\delta,\quad \sigma=1/\beta\delta=1/(\beta+\gamma) . \tag{2.6}$$

This is due to the fact that $n_s(p)$ is simply the inverse Laplace transform of $F(p,H)$, Eq. (2.2).[1]

Instead of using Eqs. (2.4) or (2.5), we find it more convenient to write $P_\infty(p,H)$ in a scaling form, and then invert it to find[19]

$$H/P_\infty^\delta=h(t/P_\infty^{1/\beta}) . \tag{2.7}$$

For completeness, we now summarize the discussion of Refs. 2 and 3. We define two nonuniversal constants x_0 and h_0

$$h_0=h(0),\quad h(-x_0)=0 . \tag{2.8}$$

Rescaling $h(x)$ by h_0 and x by x_0 we arrive at a *universal equation of state scaling functions*[2,20]

$$\tilde{h}(\tilde{x})=\tilde{h}(x/x_0)=h_0^{-1}h(x) . \tag{2.9}$$

All the amplitudes in Eqs. (1.1)–(1.3), and (1.5) are then directly related to x_0 and h_0[2,20]

$$B=x_0^{-\beta} , \tag{2.10}$$

$$C^+=\lim_{x\to\infty}[x^\gamma/h(x)]=x_0^\gamma h_0^{-1}\tilde{C}^+ , \tag{2.11}$$

$$C^-=\beta x_0^{\gamma-1}/h'(-x_0)=x_0^\gamma h_0^{-1}\tilde{C}^- , \tag{2.12}$$

$$\alpha(1-\alpha)(2-\alpha)A_F^+=-\beta\int_0^\infty dy y^\alpha h'''(y)=h_0x_0^{\alpha-2}\tilde{A}^+ , \tag{2.13}$$

$$\alpha(1-\alpha)(2-\alpha)A_F^-=\beta\left[\alpha x_0^{\alpha-1}h'(-x_0)+x_0^\alpha h''(-x_0)+\int_{-x_0}^0 dy|y|^\alpha h'''(y)\right]=h_0x_0^{\alpha-2}\tilde{A}^- , \tag{2.14}$$

and

$$E=h_0^{-1/\delta} . \tag{2.15}$$

The universal constants $\tilde{C}^\pm$, $\tilde{A}^\pm$ are given by

$$\tilde{C}^+=\lim_{\tilde{x}\to\infty}[\tilde{x}^\gamma/\tilde{h}(\tilde{x})] , \tag{2.16}$$

$$\tilde{A}=-\beta\int_0^\infty dy y^\alpha\tilde{h}'''(y) , \tag{2.17}$$

etc. These results immediately yield the universal ratios (1.6)–(1.9),

$$A_F^+/A_F^-=\tilde{A}^+/\tilde{A}^- , \tag{2.18}$$

$$C^+/C^-=\tilde{C}^+/\tilde{C}^- , \tag{2.19}$$

$$R_x=\tilde{C}^+ , \tag{2.20}$$

and

$$R_c=\tilde{A}^+\tilde{C}^+ . \tag{2.21}$$

We next turn to the correlation functions. Denoting the Fourier transform of the pair connectedness function[1] by $\hat{\chi}(k,t,P_\infty)$, the scaling assumption implies the form[3]

$$\hat{\chi}(k,t,P_\infty)=|t|^{-\gamma}Z(tP_\infty^{-1/\beta},k|t|^{-\nu}) . \tag{2.22}$$

The function $Z(x,0)$ is directly related to the function $h(x)$, since $\hat{\chi}(0,t,P_\infty)=S(p,H)$. Therefore, one needs only one additional nonuniversal parameter to obtain a fully universal function $\tilde{Z}(\tilde{x},\tilde{y})$. For $t>0$, this is chosen so that[3,21] for $y<<1$ one has

$$Z(0,y)=C^+[1+(\xi_0^+y)^2+O((\xi_0^+y)^4)]^{-1} , \tag{2.23}$$

with ξ_0^+ having been defined in Eq. (1.4). For $t<0$, a similar expression is applied to $Z(-x_0,y)$. The *hypothesis of two-scale-factor universality*[22] states that in fact h_0, x_0, and ξ_0^+ are not independent. They are related via the universality of the combination (1.11). In Ref. 3 it was shown (in the context of the usual critical phenomena in $d=4-\epsilon$ dimensions) that the universality of R_ξ^+ indeed follows from the basic assumptions of the renormalization-group approach.[15] Exactly the same analysis can be carried over to the present case. We shall comment further on the renormalization-group approach to the percolation problem below, in Appendix C.

We have thus expressed all the universal ratios in terms of the universal functions $\tilde{h}(\tilde{x})$, or $\tilde{z}(\tilde{x},\tilde{y})$. We next turn to the calculations of these functions. There are various ways to carry out this calculation. To order $-\epsilon$, most of the necessary results are hidden in Ref. 11, which used a direct diagrammatic expansion. Alternatively, one can follow Pytte's extension of the recursion relations approach[9] to the "ordered" phase.[23] In what follows, both approaches will be used.

III. EQUATION OF STATE

In what follows we adopt the formulation of Refs. 7 and 9 of the q-state Potts model. Since we aim here at order $-\epsilon^2$ results, we shall describe in detail a diagrammatic calculation, similar to that of Brézin *et al.*[24] for the usual phase transition problem. However, various aspects of the problem are easier to see from a renormalization-group calculation like that done by Pytte.[23] We shall describe some to these in Appendix C.

The Potts-model Hamiltonian can be written in the form[9]

$$\mathcal{H} = -\tfrac{1}{4}\int (r_0+k^2)\sum_{i=1}^{q} Q_{ii}(\vec{k})Q_{ii}(-\vec{k}) + w\int \sum_{i=1}^{q} Q_{ii}(\vec{k})Q_{ii}(\vec{k}^1)Q_{ii}(-\vec{k}-\vec{k}^1) \ , \qquad (3.1)$$

where Q_{ii} are the diagonal elements of a $q \times q$ dimensional traceless tensor. Integrals are performed over a spherical Brillouin zone $|\vec{k}|<1$. Expectation values of correlation functions in the disorderd phase have the form

$$\langle Q_{ii}(\vec{k})Q_{jj}(-\vec{k})\rangle = (\delta_{ij}-1/q)G(k) \ , \qquad (3.2)$$

with $G(0)=2/r=\chi$ being the susceptibility (proportional to S). In terms of the components A_α of the q-state Potts vector model,

$$H=-J\sum_{\langle \vec{x}\vec{x}^1\rangle} \vec{A}(\vec{x})\cdot\vec{A}(\vec{x}^1) \ , \qquad (3.3)$$

one has

$$Q_{ii}=\sum_{\alpha=1}^{q-1} A_\alpha a_{ii}^\alpha \ , \qquad (3.4)$$

with

$$a_{ii}^\alpha = \left[\frac{q-\alpha}{q-\alpha+1}\right]^{1/2} \times \begin{cases} 0 & \text{if } i<\alpha, \\ 1 & \text{if } i=\alpha, \\ -1/(q-\alpha) & \text{if } i>\alpha. \end{cases} \qquad (3.5)$$

In the "ordered" phase we now set[23]

$$A_\alpha(\vec{x}) = \langle A_\alpha\rangle + \mathcal{L}(\vec{x}) = Q\delta_{\alpha 1} + \mathcal{L}(\vec{x}) \ , \qquad (3.6)$$

(Q is proportional, when $q\to 1$, to P_∞) and therefore

$$Q_{11}=c(q-1)(Q+\mathcal{L}), \quad c=[q(q-1)]^{-1/2} \ ,$$
$$Q_{ii}=-c(Q+\mathcal{L})+q_{ii}, \quad i\neq 1 \ . \qquad (3.7)$$

Adding a fictitious field $HA_1(\vec{x})$, the Hamiltonian can be written in the form

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1 \ , \qquad (3.8)$$

with

$$\mathcal{H}_0 = -\tfrac{1}{4}\int (r_L+k^2)\mathcal{L}(\vec{k})\mathcal{L}(-\vec{k}) - \tfrac{1}{4}\int (r_T+k^2)\sum_{i=2}^{q} q_{ii}(\vec{k})q_{ii}(-\vec{k}) \ , \qquad (3.9)$$

$$\mathcal{H}_1 = [H-\tfrac{1}{2}r_0Q+3w(q-2)Q^2]\mathcal{L}(\vec{k}=0) - \tfrac{1}{4}[r_0-r_L-12(q-2)wcQ]\int \mathcal{L}\mathcal{L}$$
$$-\tfrac{1}{4}(r_0-r_T+12wcQ)\int \sum q_{ii}q_{ii} + (p-2)cw\int \mathcal{L}\mathcal{L}\mathcal{L} - 3wc\int \mathcal{L}\sum q_{ii}q_{ii} + w\int \sum q_{ii}q_{ii}q_{ii} \ . \qquad (3.10)$$

We have introduced r_L and r_T as the actual renormalized "masses,"[24] defined via

$$\langle \mathcal{L}(0)\mathcal{L}(0)\rangle = G_L(0)=2/r_L \ ,$$

$$\langle q_{ii}(0)q_{jj}(0)\rangle = \left[\delta_{ij}-\frac{1}{q-1}\right]G_T(0) = \left[\delta_{ij}-\frac{1}{q-1}\right]2/r_T \ . \qquad (3.11)$$

The angular bracket now means a thermodynamic average with the full Hamiltonian $\mathcal{H}$.

We next expand all averages in a perturbation expansion in $\mathcal{H}_1$. In addition to the conditions (3.11), the equation of state will be given by $\langle A_\alpha\rangle = Q\delta_{\alpha 1}$; i.e.,

$$\langle \mathcal{L}(\vec{x})\rangle = 0 \ . \qquad (3.12)$$

Figure 1 shows the diagrams which contribute to this equation. Explicity, this yields

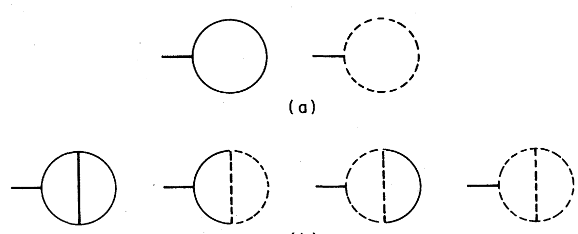

FIG. 1. Diagrams contributing to the equation of state [Eq. (3.13)]. (a) Order $-\epsilon$, (b) order $-\epsilon_2$. Full (broken) lines denote $G_L(G_T)$.

$$H=\tfrac{1}{2}r_0Q-3wc(q-2)Q^2+3wc(q-2)\int(G_L-G_T)$$
$$-54w^3c^3(q-2)\int\int\{(q-2)^2G_L^2(\vec{k})G_L(\vec{k}^1)[G_L(\vec{k}+\vec{k}^1)-G_L(\vec{k})]$$
$$+(q-2)G_L^2(\vec{k})G_T(\vec{k}^1)[G_T(\vec{k}+\vec{k}^1)-G_T(\vec{k})]-2G_T^2(\vec{k})G_L(\vec{k}^1)[G_T(\vec{k}+\vec{k}^1)-G_T(\vec{k})]$$
$$-q(q-3)G_T^2(\vec{k})G_T(\vec{k}^1)[G_T(\vec{k}+\vec{k}^1)-G_T(\vec{k})]\}+\cdots\ , \tag{3.13}$$

with the propagators $G_L=2/(r_L+k^2)$, $G_T=2/(r_T+k^2)$. The subtractions in the last term arise from "mass" renormalization terms which were already included in the term preceding it.

Similary, Eq. (3.11) yields (Fig. 2)

$$r_L=t-12wc(q-2)Q-36w^2c^2(q-2)\left[(q-2)\int\left(G_L^2-\frac{4}{k^4}\right)+\int\left(G_T^2-\frac{4}{k^4}\right)\right]+\cdots\ ,$$
$$r_T=t+12wcQ-36w^2c^2\left[2\int\left(G_TG_L-\frac{4}{k^4}\right)+q(q-3)\int\left(G_T^2-\frac{4}{k^4}\right)\right]+\cdots\ , \tag{3.14}$$

where we define $t=r_0-r_{0c}$, with

$$r_{0c}=36w^2c^2(q-2)(q-1)\int(4/k^4)+\cdots \tag{3.15}$$

denoting the critical value of r_0, at which $r_L=r_T=Q=0$ when $H=0$; i.e., $p=p_c$.

Note that r_L satisfies the relation $r_L=\partial(2H)/\partial Q$, whereas in the general case there exists no simple relation between r_T and $2h/Q$. (A Ward identity can be constructed, relating $r_T-2H/Q-6wcqQ$ to higher-order correlations.) Substitutions of the results [Eqs. (3.14)] into Eq. (3.13), together with replacement of r_0 by t, in the limit $q\rightarrow1$, finally gives our equation of state. The parameter w must be chosen at its appropriate fixed point value. Noting the equivalence between the recursion relations with large rescale factor b and the ϵ expansion,[25] we use the value found by Priest and Lubensky[9]; i.e.,

$$144K_dw^2=\tfrac{1}{7}\epsilon+\frac{629}{(2)3^27^3}\epsilon^2+\cdots\ . \tag{3.16}$$

Details of the calculations of the various integrals are given in Appendix A. It should be noted that the percolation limit $q\rightarrow1$ is rather delicate. Clearly, in that limit one does not expect to have two different susceptibilites. Indeed, Eqs. (3.14) yield

$$r_T-r_L=12wc(q-1)Q-36w^2c^2\left(2\int G_T(G_L-G_T)-(q-2)^2\int(G_L^2-G_T^2)\right)+\cdots$$
$$=12wc(q-1)Q+36w^2c^2(r_T-r_L)\left((q-1)(q-3)\int G_T^2G_L+\tfrac{1}{4}(r_T-r_L)(q-2)^2\int G_L^2G_T^2\right)+\cdots\ . \tag{3.17}$$

Thus, we see that in the limit $q\rightarrow1$ one has $r_T=r_L=r$, the limit being approached as

$$(r_T-r_L)/(q-1)\rightarrow12wcQ\left(1-72w^2\int G^3+108w^3cQ\int G^4+\cdots\right)\ . \tag{3.18}$$

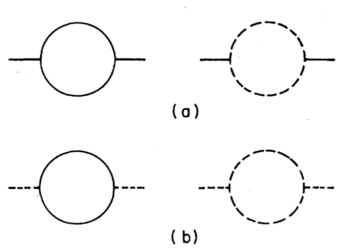

FIG. 2. Diagrams contributing to the inverse susceptibilities [Eq. (3.14)] and correlation functions [Eq. (4.2)]. (a) Contribution to r_L, (b) contribution to r_T.

One must be careful with this limit, since all the terms in Eq. (3.13) involve combinations of integrals which tend to zero as $r_L\rightarrow r_T$. These are multiplied by powers of $c^2=[q(q-1)]^{-1}$, finally yielding finite contributions as $q\rightarrow1$. Note also that Q is always multiplied by c, so that the actual order parameter to be used is $P_\infty=Q/c$.

Leaving all further technicalities to the Appendices, the final equation of state in the limit $p\rightarrow1$ has the form

$$2H/Q = t + 6wcQ + 72K_d w^2[1 + 144K_d w^2(2\ln r + 3 + 2wcQ/r)][r(2\ln r + 1) - \tfrac{1}{2}\epsilon r(\ln r + \ln^2 r)]$$
$$- 1296(K_d w^2)^2[r(12\ln^2 r + 104\ln r) + 6wcQ(16\ln^2 r + \tfrac{256}{3}\ln r)] , \quad (3.19)$$

with r given by Eq. (A5). We note here that the coefficients in the last term were deduced assuming that the critical exponents obey scaling (Appendix B).

We can now follow Eqs. (2.8) and (2.9), and eliminate the nonuniversal amplitudes h_0 and x_0. In our case, these are given by Eqs. (B4) and (B6). A tedious but straightforward calculation then yields

$$\tilde{h}(\tilde{x}) = 2^{2-\delta}\left[\tilde{x} + 1 + \frac{\epsilon}{7}\left(1 + \frac{565\epsilon}{2^2 3^2 7^2}\right)(\tilde{x}+2)\ln(\tilde{x}+2) + \frac{\epsilon^2}{(2)7^2}(\tilde{x}+4)\ln^2(\tilde{x}+2)\right] . \quad (3.20)$$

The fact that x_0 and h_0 indeed cancel out from this result proves the universality hypothesis to order ϵ^2. Note that the coefficients c_1 and c_2 [and in fact all the order ϵ^2 constants appearing in Eq. (3.19) or in Eq. (B1)] drop out of the final result. Note also that the powers of $c = [q(q-1)]^{-1/2}$ also dropped out, so that Eq. (3.20) is directly related to the percolation equation of state [Eq. (2.7)].

Equation (3.20), together with Eqs. (2.1)–(2.7), now enable us to calculate the universal ratios (1.6)–(1.9). Direct algebra yields

$$\tilde{C}^+ = \lim_{\tilde{x}\to\infty} [\tilde{x}^\gamma/\tilde{h}(\tilde{x})] = 2^{\delta-2} + O(\epsilon^3) , \quad (3.21)$$

$$\tilde{C}^- = \beta/\tilde{h}'(-1) = 2^{\delta-2}\beta/\gamma + O(\epsilon^3) , \quad (3.22)$$

$$\tilde{A}^+ = -\beta\int_0^\infty dy\, y^\alpha h'''(y) = 2^{-\delta}\beta(1 + \tfrac{6}{7}\epsilon) + O(\epsilon^2) , \quad (3.23)$$

$$\tilde{A}^- = \beta\left[\alpha\tilde{h}'(-1) + \tilde{h}''(-1) + \int_0^1 dy\, y^\alpha h'''(-y)\right] = -2^{-\delta}\beta(5 + \tfrac{4}{7}\epsilon) + O(\epsilon^2) . \quad (3.24)$$

Together with Eqs. (2.18)–(2.21), these yield the final results [Eqs. (1.6)–(1.9)].

In addition to these results, a few more comments about $\tilde{h}(\tilde{x})$ are appropriate:

(i) In the vicinity of the "coexistence curve," $\tilde{x} \to -1$, the function $\tilde{h}(\tilde{x})$ is completely well behaved

$$\tilde{h}(\tilde{x}) \simeq 2^{2-\delta}\gamma(\tilde{x}+1), \quad \tilde{x}+1 \ll 1 . \quad (3.25)$$

One does not encounter here the problem of positivity of $\tilde{h}$, arising in the usual n-component spin problem.[24]

(ii) For large $\tilde{x}$, one expects

$$\tilde{h}(\tilde{x}) = \sum_{n=1}^{\infty} \eta_n \tilde{x}^{\gamma-(n-1)\beta} . \quad (3.26)$$

This expansion involves *all* integer powers n, and not only the even ones which appear in the (time-reversal symmetric) magnetic case.[19] The coefficients η_n are universal, given by

$$\eta_1 = 2^{2-\delta} + O(\epsilon^3) ,$$
$$\eta_2 = 2^{2-\delta}(\delta - 1) + O(\epsilon^3) = 2^{2-\delta}(2\gamma - 1) + O(\epsilon^3) , \quad (3.27)$$
$$\eta_3 = \frac{2^{3-\delta}}{7}\epsilon\left(1 + \frac{817}{2^2 3^2 7^2}\epsilon\right) + O(\epsilon^3) ,$$

etc.

(iii) For small $\tilde{x}$, $\tilde{h}(\tilde{x})$ is analytic, with

$$\tilde{h}(\tilde{x}) = \sum_{i=0}^{\infty} h_i x^i . \quad (3.28)$$

The coefficients h_i are again universal, e.g.,

$$h_0 = 1 ,$$
$$h_1 = \gamma 2^{-(\gamma-\beta)/2} + O(\epsilon^3) . \quad (3.29)$$

IV. AMPLITUDES OF CORRELATION LENGTHS

We now turn to the pair connectedness function, to derive the correlation length. The equivalence of the percolation problem to the q-state Potts model relates this function to the Potts order-parameter correlation function.[1] In the notation of Sec. III, this becomes [up to factors of $(q-1)$]

$$\hat{\chi}(k,t,Q) = \langle \mathcal{L}(\vec{k})\mathcal{L}(-\vec{k})\rangle = G_L(\vec{k}) + \text{diagrams} . \quad (4.1)$$

The necessary diagrams are again those shown in Fig. 2(a), yielding

$$2[\chi(k,t,Q)]^{-1} = r_L + k^2 + 36w^2c^2\left[(q-2)\int G_L(\vec{k}^1)[G_L(\vec{k}^1+\vec{k}) - G_L(\vec{k}^1)] + \int G_T(\vec{k}^1)[G_T(\vec{k}^1+\vec{k}) - G_T(\vec{k}^1)]\right] + \cdots . \quad (4.2)$$

In the limit $q\rightarrow 1$ we can again combine the integrals in the large parentheses,

$$\Big(\;\Big)=(q-1)\int G(\vec{k}^1)[G(\vec{k}+\vec{k})-G(\vec{k})]+(r_T-r_L)\frac{\partial}{\partial r}\int G(\vec{k}^1)[G(\vec{k}^1+\vec{k})-G(\vec{k}^1)]+O[(q-1)^2]\;, \quad (4.3)$$

so that in this limit [see Eq. (3.17)]

$$2[\chi(k,t,Q)]^{-1}=r+k^2+36w^2\Big(\int G(\vec{k}^1)\int G(\vec{k}^1)[G(\vec{k}+\vec{k}^1)-G(\vec{k}^1)]$$
$$+12wcQ\frac{\partial}{\partial r}\int G(\vec{k}^1)[G(\vec{k}^1+\vec{k})-G(\vec{k}^1)]\Big)+\cdots\;. \quad (4.4)$$

This can now be used to evaluate the full correlation function in scaling form, as done by Stephen.[11] For our purposes we need only the small k behavior of $\hat{\chi}$ [Eq. (2.23)]. Expanding the integrals in power of $\vec{k}$ and using Eq. (A4) and a similar result for $\int k^2G^4$, this becomes

$$\int G(\vec{k}^1)[G(\vec{k}+\vec{k}^1)-G(\vec{k}^1)]=k^2\Big(-\tfrac{1}{2}\int G^3+\tfrac{1}{24}\int (k^1)^2G^4\Big)+O(k^4)=k^2K_d[\tfrac{2}{3}\ln r+\tfrac{5}{9}+O(r)]+O(k^4) \quad (4.5)$$

and thus

$$2[\hat{\chi}(k,t,Q)]^{-1}=r\{1+k^2r^{-1}\times[1+36K_dw^2(\tfrac{2}{3}\ln r+\tfrac{5}{9}+8wcQ/r)]+\cdots\}\;. \quad (4.6)$$

Comparison with Eq. (2.23) therefore identifies ξ,

$$\xi^2=r^{-1}[1+36K_dw^2(\tfrac{2}{3}\ln r+\tfrac{5}{9}+8wcQ/r)]\;. \quad (4.7)$$

Substituting for w [Eq. (3.16)] we find

$$\xi^2=\left[1+\frac{\epsilon}{28}\left(\frac{5}{9}+\frac{8wcQ}{r}\right)\right]r^{-2/(2-\eta)}\;, \quad (4.8)$$

where we identify[9–11]

$$\eta=2-\gamma/\nu=-\epsilon/21+O(\epsilon^2)\;. \quad (4.9)$$

For $H=0$, r was defined as the inverse susceptibility [Eqs. (1.3) and (3.11)]; i.e.,

$$r=2[\hat{\chi}(0,t,Q)]^{-1}=2(C|t|^{-\gamma})^{-1}\;. \quad (4.10)$$

Therefore, Eq. (4.8) reduces to Eq. (1.4) with

$$\xi_0^2=\left[1+\frac{\epsilon}{28}\left(\frac{5}{9}+\frac{8wcQ}{r}\right)\right]\left(\frac{C}{2}\right)^{2/(2-\eta)}\;. \quad (4.11)$$

For $t>0$, $Q=0$ and we have

$$\xi_0^+=\left(1+\frac{5\epsilon}{2^33^27}\right)\left(\frac{C^+}{2}\right)^{1/(2-\eta)}\;. \quad (4.12)$$

For $t<0$, to lowest order, $r=6wcQ$ [Eqs. (A5) and (B1)], and thus

$$\xi_0^-=\left(1+\frac{17\epsilon}{2^33^27}\right)\left(\frac{C^-}{2}\right)^{1/(2-\eta)}\;. \quad (4.13)$$

Note that in fact Eq. (4.8) is quite general and will also yield the correlation length at p_c, when $r=12wcQ$ +diagrams is given by

$$2/r=S(p_c,H)=\partial P_\infty(p_c,H)/\partial H=C^cH^{-\gamma/\beta\delta}\;, \quad (4.14)$$

with

$$C^c=E/\delta\;. \quad (4.15)$$

Therefore,

$$\xi=\xi^cH^{-\nu/\beta\delta}\;, \quad (4.16)$$

with

$$\xi^c=\left(1+\frac{11\epsilon}{2^33^27}\right)\left(\frac{C^c}{2}\right)^{1/(2-\eta)}\;. \quad (4.17)$$

We have thus related ξ^c and C^c directly to E, and Eq. (1.8) can be used to relate these to other amplitudes. Dividing Eq. (4.12) by Eq. (4.13), and using Eq. (1.7), finally yields the result Eq. (1.10).

We finally turn to a discussion of *two-scale-factor universality*, i.e., Eq. (1.11). Combining Eqs. (1.11), (2.11), (2.13), and (4.12) one finds

$$(R_\xi^+)^d=\left(1+\frac{5\epsilon}{2^33^27}\right)^d\tilde{A}^+\left(\frac{\tilde{C}^+}{2}\right)^{d/(2-\eta)}h_0^{-2/(\delta-1)}\;, \quad (4.18)$$

where one uses the scaling relations $2-\alpha=d\nu=d\gamma/(2-\eta)$ to cancel the powers of x_0 and

$$(d-2+\eta)/(2-\eta)=2\beta/\gamma=-2/(\delta-1)$$

to obtain the power of h_0. Substituting h_0 from Eq. (B4), and using Eqs. (3.21) and (3.23), one thus finds

$$(R_\xi^+)^d=\tfrac{1}{2}(12wc)^{-2}[1+\tfrac{41}{84}\epsilon+O(\epsilon^2)]\;. \quad (4.19)$$

As in similar calculations,[3] one must now replace $(12wc)^2$ by its fixed point value [Eq. (3.16)]. This fact is further explained in Appendix C. Noting that the "free energy" F defined in Eq. (1.2) is related to that of the corresponding Potts model[5] via a factor $(q-1)$, which cancels the factor c^{-2} in Eq. (4.19), this finally yields Eq. (1.11).

In addition to the universal amplitudes already discussed, there are also amplitudes which appear in the pair connectedness function at $t=H=0$, when one expects[21]

$$\tilde{Z}(0,y)=y^{-(2-\eta)}[D_0^{\infty}+D_1^{\infty}y^{-(1-\alpha)/\nu}+D_2^{\infty}y^{-1/\nu}+\cdots] \quad (4.20)$$

Since the function $\tilde{Z}(0,\tilde{y})$ is explicitly given by Stephen [Eq. (5.4) of Ref. 11] we do not elaborate on these amplitudes here.

V. CORRECTIONS TO SCALING AND $d=6$

Appendix C reviews the results of an order $-\epsilon$ renormalization-group calculation[26] for the Potts model Hamiltonian.[23] The final results are summarized in Eqs. (C11)–(C14) and (C19)–(C23). Expanding Eq. (C11) for small t we find

$$\xi=\xi_0^+t^{-\nu}(1+a_\xi t^{\omega\nu}) \ , \quad (5.1)$$

with $a_\xi^+=5a/42$, $a=[(w^*/w)^2-1]$. Similar expansions for $\chi(\propto S)$ and F yield $a_S^+=\frac{2}{7}a$, $a_F^+=\frac{2}{7}a$. Below T_c one finds the same results for a_ξ^-, a_S^-, and a_F^- and also $a_P=\frac{2}{7}a$. (Note that $P_\infty\propto Q$.) Thus, one sees that indeed the nonuniversal amplitude a drops out of any ratio of two such amplitudes, and the remaining universal ratios are as quoted in Eq. (1.25).

As noted in Ref. 17, we mention that to this (zeroth) order in ϵ one has

$$a_P/a_F^{\pm}=-(\beta-1)/(\alpha+1) \ ,$$
$$a_P/a_\chi^{\pm}=-(\beta-1)/(\gamma-1) \ , \quad (5.2)$$
$$a_P/a_\xi^{\pm}=-(\beta-1)/(\nu-\tfrac{1}{2}) \ .$$

Since the next order in ϵ is not yet available, the two estimates (1.25) or (5.2) (in which the actually measured exponents α, β, γ, and ν are substituted) give a measure of the error involved in extrapolation to $d<6$.

It should also be mentioned that ratios like Eq. (1.25) are directly related to ratios of *effective critical exponents.*[27] If Eq. (5.1) is approximated by $\xi_{\rm eff}t^{-\nu_{\rm eff}}$ then[17]

$$\nu_{\rm eff}\simeq\nu-\omega\nu a|\bar{t}|^{\omega\nu} \ , \quad (5.3)$$

with $\bar{t}$ being some average value. If $\gamma_{\rm eff}$ is measured over a similar range of t then

$$(\gamma_{\rm eff}-\gamma)/(\nu_{\rm eff}-\nu)=a_S/a_\xi\simeq(\gamma-1)/(\nu-\tfrac{1}{2}) \ . \quad (5.4)$$

In the limit of six dimensions $[\epsilon\to0]$ Eq. (C7) becomes

$$\tilde{W}=1+(7)72K_dw^2|\ln t| \ . \quad (5.5)$$

Substitution in Eqs. (C6), (C8), and (C9) then yields Eqs. (1.20), (1.19), and (1.18) with the amplitudes $\hat{\xi}_0^+$, $\hat{C}^+$, and $\hat{A}_F^+$ obtainable from Eq. (C14) by replacing $(w/w^*)^2$ by $(7)72Kw^2$. Similar substitution in Eqs. (C20) and (C22) [using Eq. (C21)] yields Eq. (1.17) and[12]

$$H=\hat{D}Q^2|\ln Q|^{-4/7} \ . \quad (5.6)$$

Inversion of this now yields Eq. (1.21), with $\hat{D}=\hat{E}^{-2}$ again given by Eq. (C23) with $(w/w^*)^2\to(7)72\times K_dw^2$. The factor $\frac{1}{2}$ multiplying the log in Eq. (1.21) has been included so that all the amplitudes in Eqs. (1.17)–(1.21) obey the $\epsilon\to0$ limit of Eqs. (1.6)–(1.10). This may be verified by a direct multiplication of the appropriate amplitudes, in which all the factors involving the nonuniversal parameter w drop out.

There is however an exception. When one multiplies Eq. (1.18) by the sixth power of Eq. (1.20) one finds

$$F(p)\xi(p)^6=\hat{A}_F^+(\hat{\xi}_0^+)^6|\ln t| \ . \quad (5.7)$$

Equations (C14) with $(w/w^*)^2\to(7)72K_dw^2$ then show a cancellation of w^2, leading to Eqs. (1.22) and (1.23).[14]

VI. NUMERICAL VALUES

The results at $d=6$ [Eqs. (1.6)–(1.10) with $\epsilon=0$ and Eqs. (1.22) and (1.23)] are *exact,* and therefore should offer a direct check for the universality, scaling, and renormalization-group theories. Extrapolation of the results [Eqs. (1.16)–(1.11)] to finite values of ϵ is not unambiguous. Houghton *et al.*[16] undertook a detailed study of the extrapolation of Eq. (1.24) down to $d=2$ $(\epsilon=4)$. Clearly, the direct substitution of $\epsilon\geq2$ in Eq. (1.24) is nonsensical, as it yields negative values. A simple Padé analysis gave $\omega=0.568$, 0.793, 0.914, and 0.989 for $d=5,4,3,2$, while two variations of the Padé-Borel summation procedure gave $\omega=0.614$, 0.925, 1.13, 1.28 or $\omega=0.582$, 0.831, 0.973, and 1.07. This gives a feeling for the uncertainties involved.

The situation with other critical exponents involves similar uncertainties. For example, Eq. (1.14) may be written as

$$\gamma = 1 + \frac{1}{7}\epsilon + \frac{565}{2^2 3^2 7^3}\epsilon^2 = \left[1 - \frac{313}{2^2 3^2 7^2}\epsilon\right] \Big/ \left[1 - \frac{565}{2^2 3^2 7^2}\epsilon\right] = \left[1 - \frac{1}{7}\epsilon - \frac{313}{2^2 3^2 7^3}\epsilon^2\right]^{-1} . \tag{6.1}$$

The second and the third expressions gives $\gamma = 1.21$, 1.80, 12.0, and -1.0 or $\gamma = 1.20$, 1.63, 2.91, and 43.5 for $d = 5, 4, 3, 2$. Only the first expression, i.e., a direct substitution of ϵ in Eq. (1.14), gives results which compare well with other sources. Equation (1.14) yields

$$\gamma_{\epsilon \exp} = 1.189, \quad 1.469, \quad 1.840, \quad 2.304 \quad \text{for } d = 5, 4, 3, 2 \ , \tag{6.2}$$

to be compared with

$$\gamma_{\text{other}} \simeq 1.2, \quad 1.5, \quad 1.7, \quad 2.43 \quad \text{for } d = 5, 4, 3, 2 \ , \tag{6.3}$$

which Stauffer[1] collected and averaged from series and Monte Carlo data.

Results for β are somewhat worse. The [2,0] and [1,1] Padés give similar values

$$\beta_{\epsilon \exp} = 0.85, \quad 0.70, \quad 0.52, \quad 0.34, \quad \text{for } d = 5, 4, 3, 2 \ , \tag{6.4}$$

while Stauffer[1] finds

$$\beta_{\text{other}} \simeq 0.7, \quad 0.5, \quad 0.4, \quad 0.14 \quad \text{for } d = 5, 4, 3, 2 \ , \tag{6.5}$$

with a somewhat better agreement with values extracted from scaling relations like $\beta = \gamma/(\delta - 1) = \frac{1}{2}(d\nu - \gamma)$. The [0,2] Padé gives worse results.

The accuracy of the exptrapolation of δ is similar. Equation (1.15) yields

$$\delta = 2\gamma + O(\epsilon^3) \ . \tag{6.6}$$

This relation is certainly not true generally, as at $d = 3, 2$ Stauffer[1] quotes $\delta = 5$ and 18 while $2\gamma = 3.7$ and 4.6.

The correlation length is found to have

$$\nu_{\epsilon \exp} = \begin{cases} 0.575, \quad 0.683, \quad 0.822, \quad 0.992 \quad \text{for } d = 5, 4, 3, 2 \ , & (6.7\text{a}) \\ 0.581, \quad 0.755, \quad 1.40, \quad -3.0 \quad \text{for } d = 5, 4, 3, 2 \ , & (6.7\text{b}) \\ 0.579, \quad 0.723, \quad 1.03, \quad 2.07 \quad \text{for } d = 5, 4, 3, 2 \ , & (6.7\text{c}) \end{cases}$$

from the three possible Padés, to be compared with[1]

$$\nu_{\text{other}} = 0.6, \quad 0.7, \quad 0.8, \quad 1.35 \quad \text{for } d = 5, 4, 3, 2 \ . \tag{6.8}$$

Finally

$$\alpha_{\epsilon \exp} = \begin{cases} -0.893, \quad -0.858, \quad -0.894, \quad -1.00 \quad \text{for } d = 5, 4, 3, 2 \ , & (6.9\text{a}) \\ -0.713, \quad -0.522, \quad -0.385, \quad -0.337 \quad \text{for } d = 5, 4, 3, 2 \ , & (6.9\text{b}) \\ -0.887, \quad -0.817, \quad -0.776, \quad -0.755 \quad \text{for } d = 5, 4, 3, 2 \ . & (6.9\text{c}) \end{cases}$$

TABLE I. Estimates for C^+/C^- [Eq. (1.7)].

d	Direct ϵ expansion	Exponents from Eqs. (6.2) and (6.4)	Exponents from Eqs. (6.3) and (6.5)	Series and Monte Carlo
6	1.0	1.0	1.0	~ 1
5	1.4	1.4	1.7	~ 4
4	1.9	2.1	3.0	~ 5
3	2.7	3.5	4.3	~ 10
2	3.6	6.6	17.0	~ 200

TABLE II. Estimates for $R' = R_X^{1/\delta}$ [Eq. (6.11)].

d	Direct ϵ expansion	$\delta = \gamma/\beta + 1$, using Eqs. (6.2) and (6.4)	$\delta = \gamma/\beta + 1$, using Eqs. (6.3) and (6.5)	Using ϵ expansion for $(1-2/\delta)$	Series and Monte Carlo
6	1	1	1	1	~1.4
5	1.2	1.2	1.2	1.1	~1.5
4	1.4	1.3	1.3	1.3	~1.2
3	1.8	1.5	1.4	1.6	1.1–1.6
2	2.3	1.7	1.9	2.0	1.1–1.3

The only direct calculation of α gives[28]

$$\alpha_{\text{other}} = -0.688 \pm 0.004 \quad \text{for } d=2 \ . \qquad (6.10)$$

The differences among all these estimates should be used to estimate the uncertainties involved in extrapolating the ϵ expansion. When the unreasonable Padés are ignored, the results seem to have accuracies of 10–30% for $d = 5-3$ and up to factors of 3–4 at $d=2$.

We can now turn to our new ϵ expansions Eqs. (1.6)–(1.11). It was chosen to write Eqs. (1.7) and (1.8) in terms of the exponents γ, β, and δ instead of direct ϵ expansions in order to draw attention to the similarity with the related critical phenomena results. In the Ising-model case, Brézin *et al.*[29] find $C^+/C^- = 2^{\gamma-1}\gamma/\beta + O(\epsilon^3)$. The power of 2 arises simply from the different mean-field temperatures scales in that problem [in which one chooses[26] $t(l^*)=1$ and $t(l^*) = -\frac{1}{2}$ for $t>0$ and $t<0$, compared to the present $t(l^*)=1$ and $t(l^*)=-1$, Eqs. (C5) and (C16)]. Aside from this factor, the two results look identical. This raises the interesting possibility that in fact this result is quite general.

Having written Eqs. (1.7) and (1.8) in these forms, one can now choose various ways to estimate the final values for C^+/C^-. One can either use a direct ϵ expansion, to order ϵ^2 [i.e., the ϵ expansion of δ, Eq. (1.15)], or use the ϵ-expansion estimates of γ and β, or use the exponents available from other sources.[1] All of these estimates are shown in Table I.

Similar considerations apply to R_X [Eq. (1.8)]. Since δ is rather a large number, earlier publications[30] discussing R_X preferred to look at $R' = R_X^{1/\delta}$. Equation (1.8) thus becomes

$$R' = R_X^{1/\delta} = (C^+)^{1/\delta} E^{-1} B^{1-1/\delta} = 2^{1-2/\delta} + O(\epsilon^3) \ , \qquad (6.11)$$

and Table II presents estimates based on various extrapolations of the ϵ expansion.

Finally, the extrapolation of Eqs. (1.6), (1.9), (1.10), and (1.11), which we have only to order ϵ, has been done simply by substituting the appropriate value of ϵ in these equations. The results are summarized in Table III.

VII. COMPARISON WITH ALTERNATIVE CALCULATIONS

The ratio C^+/C^- was first estimated from series in $d=2$ to be of order unity.[31] Monte Carlo calculations later gave[32] a value of ~20, but more careful analyses[33,34] finally seem to converge to ~200.

TABLE III. Estimates for other amplitude ratios.

d	A_F^+/A_F^-	R_C	$\xi_0^+/\xi_0^{-\,a}$	$R_\xi^{+\,b}$
6[c]	−1/5	1/4	1	Eqs. (1.22) and (1.23)
5	−0.35	0.43	1.12(1.15)	0.38
4	−0.50	0.61	1.23(1.4)	0.34
3	−0.65	0.79	1.36(1.64)	0.30
2	−0.79	0.96	1.50(2.0)	0.21

[a]The numbers in brackets represent 2ν. See Eq. (1.10).
[b]Calculated from $(7K_d/2\epsilon)^{1/d}[1-397\epsilon/(2^3 3^2 7^2 d)]$, with $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$. See Ref. 3.
[c]These exact results refer to the amplitudes in Eqs. (1.17)–(1.21).

References 33 and 34 also contain a discussion on the reasons for the earlier difficulties. At $d=3$, Refs. 33 and 34 find $C^+/C^-=11$ and 8. The other numbers quoted in Table I are from the Monte Carlo calculation of Ref. 34. We should mention, however, that Kirkpatrick has extended his earlier Monte Carlo calculations in $2 \leq d \leq 6$,[35] and these yield somewhat smaller values of C^+/C^-, in closer agreement with the ϵ-expansion estimates.[36]

We next turn to the ratio $R'=R_\chi^{1/\delta}$ [Eq. (6.11)]. This ratio was first considered in $d=2$ by Marro,[30] who checked series data and found that indeed, $R'=R_\chi^{1/\delta}$ is universal and of order 1.25. In $d=3$, series results exist only for the fcc site problem, for which Gaunt[37] quotes $R_\chi \simeq 6.5 \pm 3$; i.e., $R' \simeq 1.3-1.6$. It is useful to note here that in many numerical calculation one does not calculate the amplitude E directly. Rather, one "measures" the amplitude q_0 [Eq. (2.5)]

$$n_s(p_c)=q_0 s^{-\tau} \ . \tag{7.1}$$

A direct algebra shows[1] that for the bond problem one has

$$E=q_0\delta\Gamma(1-1/\delta) \tag{7.2}$$

and therefore a related universal ratio is[30,34]

$$K=(C^+)^{1/\delta}q_0^{-1}B^{1-1/\delta} \ . \tag{7.3}$$

From the values of K given in Ref. 34 one can extract the values quoted in Table II for $d=4,5$, and 6.

We are not aware of any previous discussion of the amplitude ratios which are listed in Table III. Assuming that $\alpha=-\frac{2}{3}$ in $d=2$, Domb and Pearce[28] estimate that[1]

$$A_F^+/A_F^- \simeq -1.0, \quad d=2 \ , \tag{7.4}$$

which compares reasonably well with the extrapolated -0.79 (Table III). Using A_F^+ from Ref. 28 and other amplitudes as listed in Ref. 1 one also finds

$$R_C \simeq \begin{cases} 4.2, & \text{square lattice, random bond} \ , \\ 4.1, & \text{triangular lattice, random site} \ . \end{cases} \tag{7.5}$$

This confirms the universality of R_C and of A_F^+/A_F^- in $d=2$.

We are not aware of any published amplitudes of the correlation length and therefore a comparison with the results on ξ_0^+/ξ_0^- or R_ξ^+ is not yet possible.

VIII. CONCLUSIONS

The numbers in Secs. VI and VII have been quoted without error bars. The discussion in Sec. VI should shed light on the uncertainties involved in the ϵ expansion. When extrapolated with caution, these uncertainties seem to be less than 30% for $d \geq 3$, and up to factors of 3 to 4 at $d=2$. Similarly, the uncertainties in the alternative calculations, especially for $d>3$, is open to debate. The reader should go to the sources for a feeling of the uncertainties, which definitely become larger as d increases. With all of this in mind, we feel that the ϵ expansions for the amplitude ratios compare very well with existing alternative calculations. It would be nice to have more accurate series or Monte Carlo calculations, especially at the higher dimensionalities.

A crucial check for the present theory will concern the exact predictions at $d=6$. An extension of the series work of Ref. 12 or of Monte Carlo calculations[34,36] to check Eqs. (1.22) and (1.23) or the $\epsilon \to 0$ limit of Eqs. (1.6)–(1.10) will therefore be of great interest.

It should be noted that one does not need to extract amplitudes and then to calculate amplitude ratios. The result Eq. (1.6), for example, could also be interpreted as stating that[2]

$$F(-t)=(A_F^+/A_F^-)^{-1}F(t), \quad t \to 0+ \ . \tag{8.1}$$

This particular ratio may be rather difficult to "measure," since the singular part of the mean number of clusters is rather small because $(2-\alpha)$ is rather large. Similarly, Eq. (1.8) can be written as $(t \to 0+)$

$$S(t)=R_\chi H^{-1}P_\infty(-t,H=0)^{1-\delta}P_\infty(t=0,H)^\delta \ , \tag{8.2}$$

and Eq. (1.11) can be written as

$$\xi(t)=R_\xi^+[\alpha(1-\alpha)(2-\alpha)F(t)]^{-1/d} \ , \tag{8.3}$$

etc. Such a direct combination of the "raw" data [e.g., plotting $\xi(t)$ vs $F(t)^{-1/d}$] may give reasonable estimates for the amplitude ratios.

As a last comment we mention again that although our results to order ϵ prove both scaling and universality, we have assumed the scaling relations among the exponents to obtain the universal equation of state at order ϵ^2 (Appendix B). A direct confirmation that the integrals in Eq. (A13) are indeed given by Eq. (B7) will confirm the scaling relations as well.

Hopefully, this paper will stimulate further series, Monte Carlo and real experimental[38] work, to study amplitude relations.

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APPENDIX A: EVALUATION OF VARIOUS INTEGRALS

The expressions for r_L and r_T are needed only to order ϵ. The integral combination appearing in r_L [Eq. (3.14)] is

$$\begin{aligned} I &= (q-2)\int (G_L^2 - 4/k^4) + \int (G_T^2 - 4/k^4) \\ &= (q-1)\int (G_L^2 - 4/k^4) - \int (G_L^2 - G_T^2) \\ &= (q-1)\int (G_L^2 - 4/k^4) - \tfrac{1}{2}(r_T - r_L)\int (G_L + G_T) G_L G_T \\ &= (q-1)\left[\int (G^2 - 4/k^4) - 12wcQ \int G^3\right] + O((q-1)^2) , \end{aligned} \tag{A1}$$

where we used Eq. (3.18). Thus, in the limit $q \to 1$

$$r_L = r_T = r = t + 12wcQ + 36w^2\left[\int (G^2 - 4/k^4) - 12wQ\int G^3\right] . \tag{A2}$$

To lowest order in ϵ, using $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$,

$$\int (G^2 - 4/k^4) = 4K_d \int_0^1 k^5\, dk\, [(r+k^2)^{-2} - k^{-4}] = 2K_d r [2\ln r + 1 + O(r)] , \tag{A3}$$

$$\int G^3 = -2K_d[2\ln r + 3 + O(r)] , \tag{A4}$$

and therefore, ignoring unimportant terms

$$r_L = r_T = r = t + 12wcQ = 72K_d w^2 [r(2\ln r + 1) + 12wcQ(2\ln r + 3)] . \tag{A5}$$

Also

$$\int G^4 = \tfrac{8}{3}K_d/r + O(1) , \tag{A6}$$

and therefore

$$(r_T - r_L)/(q-1) \to 12wcQ[1 + 144K_d w^2(2\ln r + 3 + 2wcQ/r)] . \tag{A7}$$

We now turn to Eq. (3.13). The integral $\int (G_L - G_T)$ is needed to order ϵ. One finds

$$\int (G - 2/k^2) = -2K_d r \int_0^1 k^{3-\epsilon} dk\, (r+k^2)^{-1} = -K_d r^2(\ln r - \tfrac{1}{4}\epsilon \ln^2 r) - 2K_d r/(2-\epsilon) , \tag{A8}$$

$$\int (G_L - G_T) = (r_T - r_L) K_d [r(2\ln r + 1) - \tfrac{1}{2}\epsilon r(\ln r + \ln^2 r) + 1/(1-\epsilon/2)] . \tag{A9}$$

The last term here will cancel exactly when we subtract r_{0c} [Eq. (3.15)] from $2H/Q$ [Eq. (3.13)]. Thus, the contribution of Fig. 1(a) to $2H/Q$ in the limit $q \to 1$ is

$$72K_d w^2 Q[1 + 144K_d w^2(2\ln r + 3 + 2wcQ/r)][r(2\ln r + 1) - \tfrac{1}{2}\epsilon r(\ln r + \ln^2 r)] . \tag{A10}$$

In explicit calculations one must remember to use Eq. (A5), i.e., expand $r(2\ln r + 1)$ to order $K_d w^2$

$$r(2\ln r + 1) = \tilde{r}(2\ln\tilde{r} + 1) + 72K_d w^2(2\ln\tilde{r} + 3)[\tilde{r}(2\ln\tilde{r} + 1) + 12wcQ(2\ln\tilde{r} + 3)], \quad \tilde{r} = t + 12wcQ . \tag{A11}$$

Finally, we turn to the last term in Eq. (3.13), resulting from Fig. 1(b). Expanding about $r_T = r_L$ we find

$$\begin{aligned} &(q-2)^2 G_L^2 G_L G_L + (q-2) G_L^2 G_T G_T - 2G_T^2 G_L G_T - q(q-3) G_T^2 G_T G_T \\ &= (r_T - r_L)(q-1)\left[(q-2)\int G^3 G G + (q-3)\int G^2 G^2 G\right] \\ &\quad + \tfrac{1}{2}(r_T - r_L)^2\left[\tfrac{3}{2}(q-1)(q-2)\int G^4 G G + [3(q-2)^2 - 1]\int G^3 G^2 G + \tfrac{1}{2}(q-2)^2 \int G^2 G^2 G^2\right] + O((r_T - r_L)^3) \\ &= -12wc(q-1)^2 Q\left[\int G^3 G G + 2\int G^2 G^2 G - 6wcQ\left[2\int G^3 G^2 G + \tfrac{1}{2}\int G^2 G^2 G^2\right]\right] + O((q-1)^3) . \end{aligned} \tag{A12}$$

Here, each integral actually includes a subtraction, e.g.,

$$\int G^2G^2G \to \int\int G^2(\vec{k})G^2(\vec{k}^1)[G(\vec{k}+\vec{k}^1)-G(\vec{k}^1)] \ .$$

This is to be multiplied by $-54w^3c^3(q-2)$ to yield a contribution to $2H/Q$ equal to $-1296w^4[\cdots]$, with the square brackets which appear on the right-hand side of Eq. (A12), subtracting their value at $r=Q=0$ to account for r_{0c}.

A simple power counting now shows that

$$\int G^3GG+2\int G^2G^2G - \text{subtractions} = K_d^2 r(a_1\ln^2 r+b_1\ln r+c_1) \ ,$$

$$2\int G^3G^2G+\tfrac{1}{2}\int G^2G^2G^2 - \text{subtractions} = -K_d^2(a_2\ln^2 r+b_2\ln r+c_2) \ , \tag{A13}$$

with coefficients of order unity. The constants c_1 and c_2 turn out to fall out of our final result, and therefore must not be evaluated. The other coefficients can be identified as described in Appendix B and the final result yields Eq. (3.19).

APPENDIX B: SPECIAL LIMITS

Combining Eqs. (3.13), (A8), (A9), (A13), and (3.16), the equation of state becomes (to order ϵ^2)

$$\frac{2H}{Q} = t+6wcQ+\frac{1}{2}\left[\frac{1}{7}\epsilon+\frac{629}{(2)3^27^3}\epsilon^2\right]\tilde{r}(2\ln\tilde{r}+1)$$
$$+\frac{1}{(4)7^2}\epsilon^2[3\tilde{r}(2\ln\tilde{r}+1)(2\ln\tilde{r}+3)+12wcQ(2\ln\tilde{r}+3)^2+4wcQ(2\ln\tilde{r}+1)-7\tilde{r}(\ln\tilde{r}+\ln^2\tilde{r})]$$
$$-\frac{1}{(16)7^2}\epsilon^2[\tilde{r}(a_1\ln^2\tilde{r}+b_1\ln\tilde{r}+c_1)+6wcQ(a_2\ln^2\tilde{r}+b_2\ln\tilde{r}+c_2)] \ , \tag{B1}$$

with $\tilde{r}=t+12wcQ$.

In the limit $t=0$ we expect the critical isotherm to have the form given in Eq. (1.5). Using the exponent δ which results via scaling from Priest and Lubensky's results[9] [Eq. (1.15)] this implies

$$H \sim Q^2\left[1+\tfrac{2}{7}\epsilon\ln Q+\frac{565}{(2)3^27^3}\epsilon^2\ln Q+\frac{2}{7^2}\epsilon^2\ln^2 Q\right] \ . \tag{B2}$$

This form is consistent with Eq. (B1) and with Eq. (3.16) only if

$$2a_1+a_2=40, \quad 2b_1+b_2=\tfrac{880}{3} \ . \tag{B3}$$

With the identification we find [Eqs. (1.5), (2.8), and (2.15)]

$$E^{-\delta}=h_0=\tfrac{1}{4}(12wc)^{\delta-1}(1+\tfrac{1}{7}\epsilon+W_h\epsilon^2) \ ,$$

$$W_h=\frac{1}{7^2}\left[\frac{55}{6}-\frac{(2c_1+c_2)}{16}\right]+\frac{629}{(2)3^27^3} \ . \tag{B4}$$

In the limit $H=0$ we expect the order parameter in the ordered phase to be given by Eq. (1.1). Using the exponent β from scaling and comparing with Eq. (B1), we find

$$a_1+a_2=28, \quad b_1+b_2=\tfrac{568}{3} \tag{B5}$$

and [Eqs. (1.1), (2.8), and (2.10)]

$$B^{-1/\beta}=x_0=(6wc)^{1/\beta}\left[1+\frac{1}{(2)7}\epsilon+W_x\epsilon^2\right] \ ,$$

$$W_x=\frac{1}{7^2}\left[\frac{83}{12}-\frac{(c_1+c_2)}{16}\right]+\frac{629}{2^23^27^3} \ . \tag{B6}$$

Solving Eqs. (B3) and (B5) we therefore find

$$a_1=12, \quad a_2=16, \quad b_1=104, \quad b_2=\tfrac{256}{3} \ . \tag{B7}$$

These are the values inserted in Eq. (3.19). In fact, the coefficients a_1 and a_2 are also consistent with those derived from the corresponding integrals of Priest and Lubensky[9] using the equivalence of $r\to 1/b^2$. One further checks that assuming the values in Eq. (B7) all other exponents come out correctly. For example, for $t>0$ one can set $Q=0$, $\tilde{r}=t$, and recover Eq. (1.3) with Priest and Lubensky's value for γ [Eq. (1.14)].

APPENDIX C: RENORMALIZATION-GROUP APPROACH

The renormalization-group approach of Rudnick and Nelson[26] has been used by Pytte[23] to analyze both the Hamiltonian Eq. (3.1) in the "disordered" phase and Eq. (3.8) in the "ordered" one. The differential recursion relation for w turns out to be[9,23] (setting $q=1$)

$$\frac{dw(l)}{dl}=\left[\frac{\epsilon}{2}-\frac{3}{2}\eta(l)\right]w(l)+\frac{288K_d7w^3(l)}{[1+r(l)]^3} \ , \tag{C1}$$

with

$$\eta(l)=-48K_dw^2(l) \ . \tag{C2}$$

The solution of Eq. (C1) is

$$w^2(l)=w^2e^{\epsilon l}/W(l) \ , \tag{C3}$$

where

$$W(l) = 1 + (7)144 K_d w^2 (e^{\epsilon l} - 1)/\epsilon$$
$$= 1 + (w/w^*)^2 (e^{\epsilon l} - 1) \ , \tag{C4}$$

with w^* given in Eq. (3.16). Thus, $w(l) \to w^*$ as $l \to \infty$. In the "disordered" phase one integrates the recursion relations up to l^*, defined via

$$t(l^*) = t e^{2l^*} W(l^*)^{-5/21} = 1 \ ,$$
$$t = r_0 + 72 K_d w^2 + O(r_0^2) \ . \tag{C5}$$

This implies that the correlation length is

$$\xi = e^{l^*} = t^{-1/2} \tilde{W}^{5/42} \ , \tag{C6}$$

where now

$$\tilde{W} = W(l^*) = 1 + (w/w^*)^2 (t^{-\epsilon/2} - 1) \ . \tag{C7}$$

Similarly, the susceptibility and the free energy are given by[23,26]

$$\chi^{-1} = \tfrac{1}{2} t \tilde{W}^{-2/7} \ , \tag{C8}$$

$$(q-1)^{-1} F = \tfrac{1}{2} K_d \int_0^{l^*} \ln[1 + r(l)] e^{-dl} dl$$
$$\simeq \frac{t^3}{12^3 w^2} [W(l^*)^{2/7} - 1] \ . \tag{C9}$$

For small t, Eq. (C7) can be rewritten as

$$\tilde{W} = (w/w^*)^2 t^{-\epsilon/2} (1 + a t^{\omega\nu}) \ , \tag{C10}$$

where ω was defined in Eq. (1.24) while $a = [(w^*/w)^2 - 1]$. The parameter a measures how far w is from its fixed point value, while the exponent ω, responsible for the corrections to scaling, measures how fast this fixed point value will be approached.

Thus,

$$\xi = \xi_0^+ t^{-\nu} (1 + a t^{\omega\nu})^{5/42} \ , \tag{C11}$$

$$\chi = C^+ t^{-\gamma} (1 + a t^{\omega\nu})^{2/7} \ , \tag{C12}$$

$$(q-1)^{-1} F = A_F^+ t^{2-\alpha} (1 + a t^{\omega\nu})^{2/7} - B t^3 \ , \tag{C13}$$

with

$$\xi_0^+ = (w/w^*)^{5/21}, \quad C^+ = 2(w/w^*)^{4/7},$$
$$A_F^+ = (w/w^*)^{4/7} / 12^3 w^2 \ . \tag{C14}$$

Combining Eqs. (C11), (C13), and (C14) we find

$$(q-1)^{-1} F \xi^d = \frac{1}{12^3 (w^*)^2} (1 + a t^{\omega\nu}) \ . \tag{C15}$$

The factor w^2 in the denominator of A_F^+ is thus replaced by $(w^*)^2$. This justifies the substitution of Eq. (3.16) in Eq. (4.19) to obtain Eq. (1.11). To lowest order in ϵ, Eqs. (C15) and (1.11) indeed give the same result.

In the "ordered" phase ($t < 0$ or $H \neq 0$) one must integrate separate recursion relations for r_L, r_T, and H. Defining l^* via

$$r_L(l^*) = t(l^*) + 12 c w(l^*) Q(l^*) = 1 \ , \tag{C16}$$

where

$$Q(l) = Q e^{(2-\epsilon/2)l} W(l)^{-1/42} \ , \tag{C17}$$

one then finds

$$H = \tfrac{1}{2} t Q W(l^*)^{-2/7} + 3 w c Q^2 W(l^*)^{-4/7} \ . \tag{C18}$$

For $H = 0$, $W(l^*)$ becomes equal to $\tilde{W}$ [Eq. (C10)] with t replaced by $|t|$. One can then find equations similar to Eqs. (C11) and (C12) for ξ and χ. The "free energy" is[23]

$$(q-1)^{-1} F = (q-1)^{-1} \left(\tfrac{1}{2} r Q^2 - (q-2) c w Q^2 + \tfrac{1}{2} K_6 \int_0^{l^*} \{ \ln[1 + r_L(l)] + (q-2) \ln[1 + r_T(l)] \} e^{-dl} dl \right)$$
$$= \frac{|t|^3}{12^3 w^2} (5 \tilde{W}^{2/7} - 1) = A_F^- |t|^{2-\alpha} (1 + a |t|^{\omega\nu})^{2/7} - B |t|^3 \tag{C19}$$

and the order parameter is

$$Q = B |t|^{\beta} (1 + a |t|^{\omega\nu})^{2/7}, \quad B = (w/w^*)^{4/7} / 6 w c \ . \tag{C20}$$

At $t = 0$, Eq. (C16) implies

$$W(l^*) = 1 + (w/w^*)^2 [(12 w c Q)^{-\epsilon/2} - 1] \ , \tag{C21}$$

so that

$$H = 3 w c Q^2 \tilde{W}^{-4/7} = D Q^{\delta} [1 + a (12 w c Q)^{\omega\nu}]^{-4/7} \ , \tag{C22}$$

with

$$D = 3 w c (w/w^*)^{-8/7} = E^{-\delta} \ . \tag{C23}$$

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