Two-particle spectral function and ac conductivity of an amorphous system far below the mobility edge: A problem of interacting instantons

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We use a variational approach to calculate the two-particle spectral function $S_2(x_1,x_1',x_2,x_2',E_1E_2)$ of a Gaussian-disordered electron system in the limit of deeply localized states and small energy difference $\omega = E_1 - E_2$. The solution of the variational equations yields a two-center potential, each center in lowest order being determined by the square of an instanton function. The two instantons interact via the constraint that the Hamiltonian has to have lowest eigenvalues E_1, E_2. As the two centers approach the minimum distance allowed for given ω by the tunnel effect, we are confronted with a problem of confluent saddle points, which forces us to introduce an additional constraint. Our method is rigorous in the limit of weak disorder $|E_1+E_2| \to \infty$, $\omega/|E_1+E_2| = \text{const} \ll 1$. We also apply it to the hydrodynamic limit $\omega/|E_1+E_2| \to 0$, $|E_1+E_2|$ large. It is found that these limits cannot be interchanged. In both limits we evaluated the ac conductivity. The result $\sigma(\omega) \sim \omega^2(\ln\omega)^{d+1}$ is found in the hydrodynamic limit.

I. INTRODUCTION

It is commonly believed that the electronic states in the band tail of an amorphous solid are localized. These states are bound in strong fluctuations of the potential which are so rare that the probability for the electron to tunnel away is negligibly small. Whereas perturbative methods can be used to deal with the extended states in the middle of the band, the only theoretically consistent way to investigate the properties of the localized states would appear to be a variational approach. This method focuses on the determination of the most probable potential-yielding states of the desired energy. It has been applied to several schematic models of the amorphous system: a model in which the potential has a Gaussian distribution has been studied by several authors, Halperin and Lax,[1] Zittartz and Langer,[2] Edwards,[3] and Cardy,[4] Houghton and Schäfer,[5] and Brezin and Parisi,[6] and a model in which repulsive potentials are distributed at random has been studied recently by Friedberg and Luttinger.[7] The result has been expressions for the averaged density of states and the averaged single-particle Green's function correct in the limit of deeply localized states. To the best of our knowledge no discussion of two-particle properties has been given within this approach.

In this work we will derive the two-particle spectral function and the ac conductivity of the localized states. For fixed one-particle potential V the one- and two-particle spectral functions ρ_1 and ρ_2 are defined as

$$\rho_1(x,x',E)=\sum_\nu \phi_\nu^*(x)\delta(E_\nu - E)\phi_\nu(x') \tag{1.1}$$

and

$$\rho_2(x_1,x_1',E_1;x_2,x_2',E_2)=\rho_1(x_1,x_1',E_1)\rho_2(x_2,x_2',E_2)\,, \tag{1.2}$$

where ϕ_ν and E_ν are the eigenfunctions and eigenvalues of the Hamiltonian

$$H(V)=-\frac{\hbar^2}{2m}\nabla^2+V(x)\,. \tag{1.3}$$

We will use the Gaussian white-noise model in d-dimensional space. The potential $V(x)$ is a stochastic variable distributed according to the normalized weight

$$P[V]=N^{-1}\exp\left(-\frac{1}{g}\int d^dx\, V^2(x)\right)\,, \tag{1.4}$$

and the averaged spectral functions S_i are defined as

$$S_i=\int D(V)P(V)\rho_i\,,\quad i=1,2\,. \tag{1.5}$$

Linear-response theory connects S_2 to the averaged conductivity $\sigma(\omega,E)$ at frequency ω and energy E,

$$\sigma(\omega,E)=-\frac{\pi e^2}{2d}\omega^2 \times\int d^dx\, x^2 S_2\left(0,x,E-\frac{\omega}{2};x,0,E+\frac{\omega}{2}\right), \tag{1.6}$$

which in turn determines the conductivity at temperature T:

$$\sigma(\omega)=\frac{1}{\omega}\int dE\,\sigma(\omega,E)\left[f\left(E-\frac{\omega}{2}\right)-f\left(E+\frac{\omega}{2}\right)\right]\,. \tag{1.7}$$

Here $f(E)=[1+\exp(E-\mu)/kT]^{-1}$ is the Fermi function and e is the electrical charge.

The variational method in the formulation given by Houghton and Schäfer[5] (from now on referred to as HS) searches for the "saddle-point" potential $V=V_{sp}$ which maximizes the weight $P(V)$, Eq. (1.4), under the constraint that H has eigenvalues E in the calculation of S_1, and E_1 and E_2 in the calculation of S_2. Once V_{sp} is found, the theory proceeds as a systematic expansion in powers of the fluctuations $\delta V=V-V_{sp}$. For S_1 the saddle point V_{sp} is given by the square of the instanton function well known in other branches of physics.[8-10] It has the form of a single deep and broad well. In evaluating S_2 we find that these single-well configurations do not contribute. Rather, V_{sp} has the form of two distinct instantonlike wells which interact via quantum-mechanical tunneling; this adds considerably to the difficulties of evaluating the saddle-point contribution.

The evaluation of the two-particle spectral function is complicated by two additional problems, neither of which occurs in the calculation of the density of states. The first problem concerns the identification and treatment of the collective modes of the problem. These modes describe those variations of $V(x)$ which cannot be treated by expanding around V_{sp}, but must be taken into account rigorously in the variational equations. A simple example of such a mode is the position $\vec{a}$ of the center of mass of the potential. Since the ensemble is translationally invariant, the center $\vec{a}$ can be anywhere in d-dimensional space, and therefore for large $(\vec{a}-\vec{a}')$ the difference $V_{sp}(x-a')-V_{sp}(x-a)$ is not a small perturbation about $V_{sp}(x-a)$. There is now a standard procedure[11] for dealing with such "trivial" collective modes which correspond to broken exact symmetries of the system. In our problem we find two additional nontrivial collective modes. The first can be parametrized by the distance L between the centers of the two wells which can vary from a minimum distance L_0 to infinity. The minimum distance L_0 is determined by the requirement that tunneling between two identical wells separated by the distance L_0 gives the required energy splitting $E_2-E_1=\omega$. The second nontrivial mode is associated with the fact that for fixed direction of $\vec{L}$ we have two saddle points, characterized by either the right-hand or the left-hand well being the deeper one. As L approaches L_0 the two saddle points merge, which introduces another collective mode.

A second problem is connected to the existence of two energy scales in the conductivity problem, $E=\frac{1}{2}(E_1+E_2)$ and $\omega=E_2-E_1$. In the density of states only a single energy occurs, consequently the theory can be formulated in terms of a single dimensionless parameter

$$\gamma=g^{1/2}\left(\frac{2m}{\hbar^2}\right)^{d/4}|E|^{(d/4)-1}, \tag{1.8}$$

where the energy E is measured from the mobility edge.[1,5] Expanding around the saddle point we find an asymptotic series valid in the limit $\gamma\to 0$. In the calculation of S_2 we have the additional dimensionless parameter

$$\bar{\omega}=\omega/2E, \tag{1.9}$$

which is also taken to be small. However, the results depend upon whether we consider the limit of weak disorder $\lim\bar{\omega}\to 0$, $\lim\gamma\to 0$, $\omega/\gamma\gg 1$, or the hydrodynamic limit $\lim\gamma\to 0$, $\lim\bar{\omega}\to 0$, $\omega/\gamma\ll 1$. *A priori* the variational method applies to the limit of weak disorder. In the hydrodynamic limit we have to investigate the $\bar{\omega}$ dependence of the higher-order terms in γ, which is a complicated and lengthy task.

The central result of this paper is the expansion for the two-particle spectral function S_2 given in Eq. (3.39). This result is rigorous in the limit of weak disorder. We believe it to be correct also in the hydrodynamic limit. The expression can be evaluated in several limits of interest, which gives a qualitative picture of the behavior of the averaged two-particle spectral function. One of the most interesting results is the expression for the conductivity which, in the limit of weak disorder, reads

$$\sigma(\omega,E)=\frac{\pi e^2}{d}\gamma^2\rho\left(E+\frac{\omega}{2},\gamma\right)\times\rho\left(E-\frac{\omega}{2},\gamma\right)\left|\frac{2mE}{\hbar^2}\right|^{-1-(d/2)}. \tag{1.10a}$$

In the hydrodynamic limit we recover the result of Mott, Anderson, and Halperin[12]:

$$\sigma(\omega,E)=\frac{\pi e^2}{4d}\rho^2(E,\gamma)S_d\left|\frac{2mE}{\hbar^2}\right|^{-1-(d/2)}\omega^2\left[\ln\left(\frac{c}{\omega}\right)\right]^{d+1}. \tag{1.10b}$$

Here $S_d=2\pi^{d/2}/\Gamma(d/2)$ is the surface of the d-dimensional unit sphere and c is a constant; $\rho(E,\gamma)$ is the density of states at energy E. We see from Eq. (1.10) that the limits $\lim\bar{\omega}\to 0$ and $\lim\gamma\to 0$ cannot be interchanged.

Owing to the problems mentioned above the details of the derivation of our results is rather involved. We therefore present here only an outline of our method stressing the principle aspects. (An account of the details of the calculation is in preparation and will be available in preprint form.) However, the basic physical ideas and the consequences of our general results are discussed in detail. The organization of this article is as follows. In Sec. II we give a qualitative discussion of the spectral function S_2. This discussion, which is a slightly modified version of the argument of Mott, Anderson, and Halperin,[12] introduces some of the essential physical ideas. We then show that an attempt to put this argument on a rigorous basis meets with a problem of confluent saddle points which forces us to introduce an additional collective variable. In Sec. III, we give an outline of our method and we present our result for S_2 which together with the conductivity is evaluated and discussed in Sec. IV.

II. QUALITATIVE CONSIDERATIONS

We begin by recalling an argument[12] which gives a qualitative picture of the potentials which contribute to S_2 in the limit of small γ and $\bar{\omega}$. (It will be found that this argument applies in the hydrodynamic limit.) It is obvious from the definition of S_2, Eqs. (1.2) and (1.5), that only those potentials which have eigenstates at energies $E_i=E+(-1)^i\omega/2$, $i=1,2$, can contribute. To construct such potentials we start from two identical wells which are a distance $L=|a_1-a_2|$ apart,

$$V(x)=V_0(x-a_1)+V_0(x-a_2)\,. \tag{2.1}$$

The distance L is assumed to be large compared with the diameter D of a well. The lowest eigenfunction $\phi_0(x)$ of a single-well $V_0(x)$ for $|x|\gg D$ decays exponentially,

$$\phi_0(x)\sim\exp(-\mu|x|),\quad \mu^2=2m|E|/\hbar^2\,. \tag{2.2}$$

Diagonalizing the Hamiltonian with the two-center potential (2.1) in the space spanned by the two wave functions $\phi_0(x-a_i)$, $i=1,2$, we find that the tunnel effect causes a splitting $\omega(L)$ of the energy levels which in lowest order is determined by the amplitude of the wave function of one well at the center of the other one:

$$\omega(L)\sim c\exp(-\mu L)\,. \tag{2.3}$$

The wave functions and energies are given by

$$\begin{aligned}
&\phi_1(x)=\cos\theta_0\phi_0(x-a_1)+\sin\theta_0\phi_0(x-a_2),\\
&E_1=E-\tfrac{1}{2}\omega(L),\\
&\phi_2(x)=-\sin\theta_0\phi_0(x-a_1)+\cos\theta_0\phi_0(x-a_2),\\
&E_2=E+\tfrac{1}{2}\omega(L),
\end{aligned} \tag{2.4}$$

where for the symmetric potential considered here we have $\theta_0=\frac{1}{4}\pi$. The potential $V(x)$ contributes to $\sigma(\omega,E)$ provided $\omega(L)=\omega$. We have

$$L=L_0\sim\mu^{-1}\ln(\tau/\omega)\,. \tag{2.5}$$

For $L>L_0$ the splitting $\omega(L)$ is less than ω. We can correct for this by adjusting the depths of the wells making one well more and the other one less attractive so that the single wells allow lowest eigenstates of energies $E\pm\frac{1}{2}\delta$. This changes the probability $P(V)$ and the wave function ϕ_0 only by a term of order $\delta\leq\omega$, which is negligible. The splitting of the eigenstates increases to $\omega=[\delta^2+\omega^2(L)]^{1/2}$ and the angle θ_0 changes considerably; $\sin2\theta_0$ behaves roughly as

$$\sin2\theta_0=\omega(L)/\omega\sim\exp(-\mu|L-L_0|)\,. \tag{2.6}$$

We see that this equation allows for two solutions $\theta_0=\frac{1}{4}\pi\mp\epsilon$, $-\frac{1}{4}\pi\leq\theta_0\leq\frac{3}{4}\pi$, which correspond to either the a_1-centered or the a_2-centered well being the deeper one. For $L<L_0$ the splitting $\omega(L)$ is greater than ω, and there is no small change of the potentials which can bring the energies back to their correct position.

We now assume that for small γ and ω the maximum contribution to S_2 is given by two-center potentials. The probability of finding a potential with an eigenstate in the interval $E,E+dE$ centered in the volume element d^da is the same as that of finding the eigenstate itself, $\rho(E,\gamma)dEd^da$. The probability of finding the two-center potential is taken to be the product of the probabilities of finding the one-center potentials, i.e., $\rho_2(E,\gamma)dEd\delta d^da_1d^da_2$. Therefore S_2 satisfies

$$S_2dEd\omega\simeq\rho^2\int d^da_1d^da_2(\phi_1^*\phi_1\phi_2^*\phi_2)dEd\delta\,. \tag{2.7}$$

With $\partial\delta/\partial\omega|_L=1/\cos2\theta_0$ we find

$$S_2\left(x_1,x_1',E-\frac{\omega}{2};x_2,x_2',E+\frac{\omega}{2}\right)=\rho^2(E,\gamma)\int d^da_1d^da_2\phi_1^*(x_1)\phi_1(x_1')\phi_2^*(x_2)\phi_2(x_2')\Theta(|a_1-a_2|-L_0)/\cos2\theta_0\,, \tag{2.8}$$

where the ϕ_i also depend on a_i and θ_0, Eq. (2.4), and $\Theta(x)$ is the Θ function. Now it is easily checked that in the integral Eq. (1.6), which determines $\sigma(\omega,E)$, the leading contribution to the product of the wave functions is of the form

$$\prod_{\gamma=1}^{2}\phi_\gamma^*\phi_\gamma\simeq\tfrac{1}{4}\sin^2\theta_0[|\phi_0(-a_1)\phi_0(x-a_1)|^2-|\phi_0(-a_1)\phi_0(x-a_2)|^2+a_1\rightleftharpoons a_2]\,. \tag{2.9}$$

The interactions over x and $(a_1+a_2)/2$ are trivial and we find

$$\sigma(\omega,E)=-\frac{\pi e^2}{2d}\omega^2\rho^2(E,\gamma)\frac{S_d}{2}\times\int_{L_0}^{\infty}dL\,L^{d+1}\frac{\sin^2 2\theta_0(L,\omega)}{\cos 2\theta_0(L,\omega)}=\frac{\pi e^2}{2d}\rho^2(E,\gamma)S_d\omega^2\frac{L_0^{d+1}}{\mu}\left[1+O\left(\frac{1}{\omega}\right)\right]. \tag{2.10}$$

By virtue of Eq. (2.5) the result (1.10b) follows. We note that the integral in Eq. (2.10) is dominated by small L values $[L_0\leq L\lesssim L_0+\mu^{-1}\ln(L_0\mu)]$.

A variational procedure can be used to put this argument on a more rigorous basis. This method identifies the potential constructed above with the saddle-point potential $V=V_{sp}$ which optimizes $P(V)$ under the constraints that $H(V)$ yields the correct eigenvalues,

$$H(V)\,|\Phi_i\rangle=E_i\,|\Phi_i\rangle,\quad E_i=E+(-1)^i\omega/2,\quad i=1,2 \tag{2.11}$$

and the additional constraint that the two centers of V_{sp} are a distance L apart. It turns out that we indeed can find a solution to this problem. The Φ_i are given approximately by Eq. (2.4), with θ_0 specified by Eq. (2.6), and so far the qualitative considerations are correct.

We now consider the fluctuations around the saddle point,

$$V=V_{sp}+g^{1/2}\delta V. \tag{2.12}$$

In the qualitative argument these fluctuations were taken into account by equating the probability to find adequate one-center potentials with the density of states. The potential V has to have the correct eigenvalues E_i. To leading order this constraint can be evaluated by solving the Schrödinger equation, Eq. (2.11), in the space spanned by the eigenfunctions ϕ_i, $i=1,2$ of $H(V_{sp})$, or equivalently the space spanned by $\phi_0(x-a_i)$. Compare Eq. (2.4). We find

$$g^{1/2}(\langle 1|\delta V|1\rangle+\langle 2|\delta V|2\rangle)=O(g(\delta V)^2), \tag{2.13a}$$

$$g^{1/2}(\langle 1|\delta V|1\rangle-\langle 2|\delta V|2\rangle)=\omega(\cos 2\theta-\cos 2\theta_0)+O(g(\delta V)^2), \tag{2.13b}$$

$$g^{1/2}\langle 1|\delta V|2\rangle=\tfrac{1}{2}\omega(\sin 2\theta-\sin 2\theta_0)+O(g(\delta V)^2), \tag{2.13c}$$

where $|i\rangle$ stands for the wave function $\phi_0(x-a_i)$. The angle θ determines the approximate eigenfunctions ϕ_i of $H(V)$ according to

$$|\phi_1\rangle=\cos\theta\,|1\rangle+\sin\theta\,|2\rangle,\qquad |\phi_2\rangle=-\sin\theta\,|1\rangle+\cos\theta\,|2\rangle. \tag{2.14}$$

Equations (2.13) determine the projection of δV onto the space spanned by the functions $\phi_0^2(x-a_i)$, $i=1,2$, $\phi_0(x-a_1)\phi_0(x-a_2)$ in terms of the component $\delta V_\perp$ orthogonal to this space and of the angle θ.

We now focus on the special variation

$$\delta V=b_1V_1+b_2V_2, \tag{2.15}$$

where the functions

$$V_1(x)=N_1^{-1}[\phi_0^2(x-a_1)-\phi_0^2(x-a_2)],\qquad V_2(x)=N_2^{-1}[\phi_0(x-a_1)\phi_0(x-a_2)] \tag{2.16}$$

are normalized and orthogonal to each other. Equations (2.13b) and (2.13c) show that the eigenvalue constraints trace out an ellipse in (b_1,b_2) space,

$$b_1=\frac{\omega}{g^{1/2}N_1}[(\cos 2\theta-\cos 2\theta_0)+O(g(b_ib_j)^2)], \tag{2.17}$$

$$b_2=\frac{\omega}{2g^{1/2}N_2}[(\sin 2\theta-\sin 2\theta_0)+O(g(b_ib_j)^2)]. \tag{2.18}$$

Whereas the normalization N_1 is of order one and nearly independent of $L\equiv|a_1-a_2|$, the normalization N_2 vanishes exponentially as $L\to\infty$ and reaches $N_2\sim\omega/E$ as $L\to L_0$. Thus the half-axis a_2 diverges as $L\to\infty$. In the space spanned by $1/L$ and $\hat{b}_1=(\omega/g^{1/2}N_1)\cos 2\theta$, $\hat{b}_2=(\omega/2g^{1/2}N_2)\sin 2\theta$ the constraints are represented by the surface plotted in Fig. 1. As we noted after Eq. (2.6) for fixed $L>L_0$ there are two saddle points corresponding to $\theta_0=\pi/4\pm\epsilon$; as a function of L the saddle points follow the curve indicated in Fig. 1. Now consider $L\gg L_0$ fixed. Those potentials which can contribute significantly to S_2 are found within unit distance of the saddle points; for $L\gg L_0$ these regions do not overlap [Fig. 2(a)] and we clearly have to add the (identical) contributions of both saddle points. If, however, L approaches L_0 the

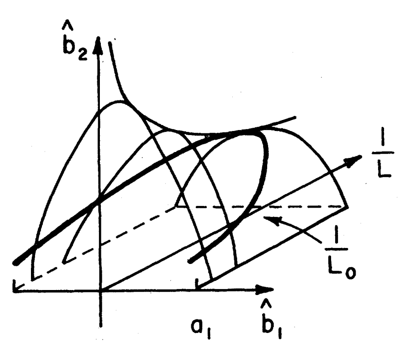

FIG. 1. Surface of the contributing potentials. Thin lines represent curves with L = const. The thick line represents the path of the saddle points as function of L.

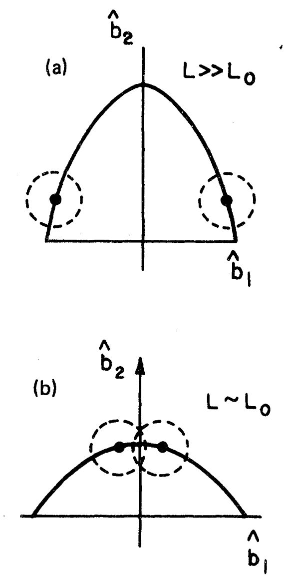

FIG. 2. Planes L = const from the previous figure. The unit circles around the saddle points are shown. In (a) we have $L \gg L_0$ whereas in (b) L is close to L_0.

two saddle points approach each other and the contributing regions overlap [Fig. 2(b)]. Adding the contributions of each saddle point clearly leads to an incorrect estimate of the integral in the important region $L \sim L_0$. This problem of confluent saddle points cannot be avoided without introducing a new collective variable which essentially fixes the angle θ and hence the point on the surface of Fig. 1. In our final result, Eq. (3.39), this effect is recovered; it can be seen that for $L \sim L_0$ the θ fluctuations are quartic rather than quadratic as in the usual fluctuation integral for $L \gg L_0$.

III. OUTLINE OF THE CALCULATION: GENERAL RESULTS

To simplify the notation we extract all dimensions in powers of an inverse length μ:

$$\mu = |2mE/\hbar^2|^{1/2} . \qquad (3.1)$$

The dimensionless width γ and dimensionless frequency $\bar{\omega}$ have been introduced in Eqs. (1.8) and (1.9), respectively. To exhibit the role of γ as a coupling constant we rescale the potential

$$V(x) = -\mu^{d/2} g^{1/2} \psi(\mu x) . \qquad (3.2)$$

The probability distribution of $\psi(z)$ reads

$$P[\psi] = N_\psi^{-1} \exp\left(-\int d^d z\, \psi^2(z)\right), \qquad (3.3)$$

and the Schrödinger equation takes the form

$$\Gamma[\psi] | \Phi_i \rangle = \lambda_i | \Phi_i \rangle , \qquad (3.4)$$

where

$$\Gamma[\psi] = -\nabla_z^2 + 1 - \gamma\psi(z) \qquad (3.5)$$

and

$$\lambda_i = 1 + (E_i/E)\ [\equiv (-1)^i \bar{\omega},\quad i = 1,2] . \qquad (3.6)$$

We use Φ_i to distinguish the wave functions for arbitrary potentials from those for the saddle-point potentials $\phi_i = \Phi_i(\psi_{sp})$. The Φ_i are normalized and real and are related to the wave functions of the original problem by the rescaling

$$\Phi_i(x) = \mu^{d/2} \Phi_i(\mu x) . \qquad (3.7)$$

The spectral function S_2 is expressed as

$$S_2(x_1, x_1', E - \tfrac{1}{2}\omega; x_2, x_2', E + \tfrac{1}{2}\omega) = \mu^{2d} |E|^{-2} S_2(\mu x_1, \mu x_1', \mu x_2, \mu x_2'; \bar{\omega}) , \qquad (3.8)$$

where

$$S_2(z_1, \ldots, z_4; \bar{\omega}) = \int D[\psi] \exp\left(-\int d^d z\, \psi^2(z)\right) \sum_{\nu\nu'} \Phi_\nu(z_1)\Phi_\nu(z_2)\delta(\lambda_\nu + \bar{\omega})\ \Phi_{\nu'}(z_3)\Phi_{\nu'}(z_4)\delta(\lambda_{\nu'} - \bar{\omega}) . \qquad (3.9)$$

Here the normalization N_ψ of $P[\psi]$ has been adsorbed into $D[\psi]$.

Following HS we express the eigenvalue constraints as

$$\mathrm{Det}\{\Gamma[\psi] - (-1)^i \bar{\omega}\} = 0, \quad i = 1,2 . \qquad (3.10)$$

To handle the problem of confluent saddle points we define the functions Φ_r and Φ_l and the angle $\theta[\psi]$ by the requirements

$$\Phi_1 = \cos\theta[\psi]\Phi_r + \sin\theta[\psi]\Phi_l ,$$
$$\Phi_2 = -\sin\theta[\psi]\Phi_r + \cos\theta[\psi]\Phi_l , \qquad (3.11)$$

$$\int d^d x\, \Phi_r^2(x)\Phi_l^2(x) = \min . \qquad (3.12)$$

Introducing a variable θ with the range $-\pi/4 \leq \theta < 3\pi/4$ symmetric with respect to $\pi/4$ we then impose the constraint

$$\theta[\psi] - \theta = 0 . \qquad (3.13)$$

This choice is suggested by the qualitative discussion of the preceding section. The condition Eq. (3.12) quarantees that the functions Φ_r and Φ_l have minimal overlap; they are therefore centered about separate centers. They play the role of the functions $|i\rangle = \phi_0(x - a_i)$ occurring in Eqs. (2.14) and the angle θ introduced there is the analog of $\theta[\psi]$.

In constructing a constraint which fixes the distance L we note that the L mode is nearly a zero-frequency mode. A change of L only changes the saddle-point contribution by a term of order $\bar{\omega} \ll 1$. Therefore for the L constraint we may use a form suitable for zero-frequency modes,[11]

$$0 = F_4(L, [\psi]) = \int d^d z\, \psi(z)\psi_4(z, L). \quad (3.14)$$

The L-dependent set of functions $\psi_4(z, L)$ will be specified below.

With these constraints the variation of $\int \psi^2(x)$ gives

$$\psi_{sp} = \frac{1}{\gamma} \sum_{i=1}^{4} \rho_i \psi_i, \quad (3.15)$$

where the ρ_i/γ are Lagrange multipliers. We can introduce a linear combination of the constraints Eqs. (3.10) and (3.13) such that the normalized functions ψ_i, $i = 1, 2, 3$ are simply expressed in terms of $\phi_\alpha = \Phi_\alpha[\psi_{sp}]$, $\alpha = r$ or l:

$$\psi_1 = N_1^{-1}(\phi_r^2 + \phi_l^2), \quad (3.16a)$$

$$\psi_2 \sim N_2^{-1}(\phi_r^2 - \phi_l^2), \quad (3.16b)$$

$$\psi_3 \sim N_3^{-1}\phi_r\phi_l. \quad (3.16c)$$

Equations (3.16b) and (3.16c) are not exact; in this outline of our method we have omitted terms of order $\bar{\omega}$ which originate from $\delta\theta/\delta\psi$. Further in the actual calculation we found it useful to orthogonalize the ψ_i. We note that the functions ψ_2 and ψ_3 correspond to the functions V_1 and V_2 introduced in Eqs. (2.16).

The functions ϕ_i and therefore the functions ϕ_r and ϕ_l are to be determined from the Schrödinger equation

$$\Gamma[\psi_{sp}]\,|\phi_i\rangle = (-1)^i\bar{\omega}\,|\phi_i\rangle, \quad i = 1, 2. \quad (3.17)$$

Equations (3.15) to (3.17) constitute a self-consistency problem. To solve this problem we introduce instanton functions $\chi_{r,l}$ centered at $\pm L/2$;

$$\chi_{r,l}(z) = \chi(z \mp L/2), \quad (3.18)$$

where $\chi(z)$ satisfies the instanton equation

$$0 = [-\nabla^2 + 1 - \chi^2(z)]\chi(z) = \Gamma[\chi^2/\gamma]\,|\chi\rangle. \quad (3.19)$$

In zeroth order we choose

$$\phi_{r,l}^0 = \chi_{r,l}\left(\int d^d z\, \chi^2(z)\right)^{-1/2}, \quad (3.20)$$

$$\rho_1^0 = N_1^0 = \left(\int d^d z[(\phi_r^0)^2 + (\phi_l^0)^2]\right)^{1/2}, \quad (3.21a)$$

$$\rho_2^{(0)} = \rho_3^{(0)} = \rho_4^{(0)} = 0. \quad (3.21b)$$

It is easily checked that this ansatz solves the self-consistency problem up to terms of order $\bar{\omega}$, $\rho_i - \rho_i^0$, and the overlap measured by integrals

$$I_{ij} = I_{ij}(L) = \int d^d z (\chi_r)^i (\chi_l)^j \quad (3.22)$$

for $i = j = 1$ or $i = 1$, $j = 3$, etc. Using the framework of degenerate-state perturbation theory we can construct the full solution as a power series in these parameters. To define the problem fully we have to specify the function ψ_4. Again by analogy with the treatment of zero-frequency modes we choose

$$\psi_4 = N_4^{-1} L \nabla[(\phi_r^0)^2 - (\phi_l^0)^2] \sim (\partial/\partial L)\psi_{sp}^0. \quad (3.23)$$

We have calculated the first-order solution explicitly. Imposing the constraints we find that ρ_4 vanishes at first order, whereas ρ_2 is of order $\bar{\omega}$ and $\rho_1 - \rho_1^0$ is proportional to ρ_3. Therefore the expansion is in powers of $\bar{\omega}$, ρ_3, and the overlap. The overlap integrals are roughly of order $\bar{\omega}$. The distance L_0 is defined by

$$I_{13}(L_0) = \bar{\omega} I_{02}, \quad (3.24)$$

and only values $L \geq L_0$ make an essential contribution which shows that $I_{13}L \leq \bar{\omega} I_{02}$. We will find that ρ_3 is of order γ and therefore our perturbative solution of the saddle-point equation can be consistently ordered in powers of the two small parameters of the problem γ and $\bar{\omega}$.

From the constraints ρ_3 is determined as

$$\rho_3 = \bar{\omega}[\sin 2\theta - I_{13}(L)/\bar{\omega} I_{02}]/N_3^0 \langle f|M|f\rangle, \quad (3.25)$$

where

$$|f\rangle = |\psi_3^0\rangle - |\psi_1^0\rangle\langle\psi_1^0|M|\psi_3^0\rangle/\langle\psi_1^0|M|\psi_1^0\rangle \quad (3.26)$$

and

$$M = 1 + 2\sum_{\alpha = r, l} \chi_\alpha \hat{Q}_\alpha[\hat{Q}_\alpha(-\nabla^2 + 1 - 3\chi_\alpha^2)\hat{Q}_\alpha]^{-1}\hat{Q}_\alpha\chi_\alpha. \quad (3.27)$$

The operator $\hat{Q}_\alpha$ projects onto the space orthogonal to $|\chi_\alpha\rangle$ and $|\partial\chi_\alpha/\partial z_i\rangle$, $i = 1, \ldots, d$. The saddle-point value of the exponent $\int \psi^2$ is found to be

$$\int \psi_{sp}^2 = \frac{1}{\gamma^2}[J(L) + \rho_3^2\langle f|M|f\rangle], \quad (3.28)$$

where

$$\begin{aligned} J(L) = {} & \{[1 + \delta(L)]^{2-(d/2)} + [1 - \delta(L)]^{2-(d/2)}\} I_{40} \\ & - 2I_{22}(L) + 4I_{11}(L)I_{13}(L)/I_{02} \\ & - 4\langle\chi_r\chi_l^2|\,G_l\,|\chi_l^2\chi_r\rangle, \end{aligned} \quad (3.29)$$

$$\left.\begin{aligned} G_\alpha &= Q_\alpha(Q_\alpha\Gamma[\chi_\alpha^2/\gamma]Q_\alpha)^{-1}Q_\alpha, \\ Q_\alpha &= 1 - I_{02}^{-1}|\chi_\alpha\rangle\langle\chi_\alpha|, \end{aligned}\right\}\ \alpha = r, l \quad (3.30)$$

and

$$\delta^2(L) = \bar{\omega}^2 - [I_{13}(L)/I_{02}]^2. \quad (3.31)$$

Since the matrix element $\langle f|M|f\rangle$ is of order one, Eq. (3.28) shows that ρ_3 is of order γ. Note the close relationship with the previous qualitative discussion. The parameter $\delta(L)$, in very much the same role, has been introduced before and $I_{13}(L)/\bar{\omega}I_{02}$ may be formally identified as $\sin 2\theta_0$ [compare Eq. (2.6)]. The first contribution to $J(L)$ just reproduces the contribution of two noninteracting instanton wells of readjusted depths $[1 \pm \delta(L)]\chi^2(0)$. The remaining parts of $J(L)$ are due to tunneling; Eq. (3.23) contains an additional contribution from the θ constraint.

We now consider the fluctuations around the saddle point. Since the functional integral is invariant under rotations and translations, it is obvious that if $\psi_{sp}(z,L,\theta)$ is a saddle point then $\psi_{sp}(z-a,L',\theta)$ is also a saddle point for any a, $|L'|=|L|$. The treatment of the broken translational and rotational symmetries is standard.[11] We introduce normalized vectors ψ_i, $i=5,\ldots,2d+3$ proportional to the derivatives of ψ_{sp} with respect to the $d-1$ angles fixing $\vec{L}/|L|$ and the d components of $\vec{a}$. The zero-order functions $\psi_i^{(0)}$, $i=1,\ldots,2d+3$ form an orthogonal set and we expand any potential ψ as

$$\psi(z)=\psi_s(z-a,L,\theta)+\sum_{i=1}^{2d+3} b_i\psi_i^{(0)}+\delta\psi_\perp\,, \qquad (3.32)$$

where $\delta\psi_\perp$ is orthogonal to all $\psi_i^{(0)}$. In the functional integral we then eliminate the variables b_1,b_2,b_3 in favor of $\lambda_1,\lambda_2,\theta$, and we eliminate $b_4,\ldots,b_{2d+3}$ in favor of $\vec{a}$ and $\vec{L}$. We find

$$S_2(z_1,\ldots,z_4;\bar{\omega})=\tfrac{1}{2}\pi^{-d-(3/2)}\int d^d a\int d^d L\int_{-\pi/4}^{3\pi/4}d\theta\int D[\delta\psi_\perp]\mathrm{Det}\left(\frac{\partial b_1\cdots b_3}{\partial\lambda_i,\theta}\right)\mathrm{Det}\left(\frac{\partial b_4\cdots b_{2d+3}}{\partial a,L}\right)$$
$$\times\,\Phi_1(z_1)\Phi_1(z_2)\Phi_2(z_3)\Phi_2(z_4)\exp\left(-\int\psi^2\right), \qquad (3.33)$$

where ψ is given by Eq. (3.32) with b_1,b_2,b_3 as determined from the λ_i,θ constraints and with $b_i=0$, $i=4,\ldots,2d+3$. The fluctuations associated with the coefficients $b_4,\ldots,b_{2d+3}$ are taken into account by integrating a and L over all space. In Eq. (3.33) the factor $\pi^{-d-(3/2)}$ is due to the normalization of the $2d+3$ integrals over the eliminated variables $b_1,\ldots,b_{2d+3}$ and the factor $\frac{1}{2}$ corrects for the symmetry under the operation $\vec{L}\to-\vec{L}$, $\theta\to(\pi/2)-\theta$.

The functional determinants and the dependence of b_1,b_2,b_3 on $\theta,\delta\psi_\perp$ have to be determined with the help of degenerate-state perturbation theory. We find

$$\mathrm{Det}\left(\frac{\partial b_1\cdots b_3}{\partial\lambda_i,\theta}\right)=2\gamma^{-3}\bar{\omega}\frac{I_{02}^2}{I_{04}N_3^{(0)}}[1+O(\bar{\omega},\gamma)]\,, \qquad (3.34)$$

$$\mathrm{Det}\left(\frac{\partial b_4\cdots b_{2d+3}}{\partial a,L}\right)=\gamma^{-2d}\hat{I}_4^d[1+O(\bar{\omega},\gamma)]\,, \qquad (3.35)$$

$$\exp\left(-\int\psi^2\right)=\exp\left(-\int\psi_{sp}^2-\langle\delta\psi_\perp|(1-2\chi_rG_r\chi_r-2\chi_lG_l\chi_l)|\delta\psi_\perp\rangle+O(\bar{\omega},\gamma)\right), \qquad (3.36)$$

where

$$\hat{I}_4=\frac{1}{d}\int(\nabla\chi^2)^2 \qquad (3.37)$$

and G_α, $\alpha=r,l$ has been defined in Eq. (3.30). The operator occurring in the matrix element on the right-hand side of Eq. (3.36) closely resembles the operator which occurs in the corresponding matrix element in the calculation of the density of states [compare HS Eqs. (3.29) and (3.25)]. Indeed the integral over $\delta\psi_\perp$ can be reduced to a product of those fluctuation integrals occurring in the density of states, the only complication arising from the fact that the space of functions $\delta\psi_\perp$ differs slightly from the corresponding space in the density-of-states calculation. Introducing the (dimensionless) density of states $\bar{\rho}(\delta)$ of energy $(1+\delta)E$,

$$\bar{\rho}(\delta)=(1+\delta)^{(d/4)(5-d)}\pi^{-(1+d)/2}\gamma^{-d-1}I_{02}I_{04}^{-1/2}\hat{I}_4^{d/2}\times\mathrm{Det}^{-1/2}(1-2\chi_\alpha G_\alpha\chi_\alpha)\times\exp\left(-(1+\delta)^{2-(d/2)}\frac{I_{04}}{\gamma^2}\right), \qquad (3.38)$$

we find the following expression for S_2:

$$S_2(z_1,z_2,z_3,z_4,\bar{\omega})=\int d^d a\int d^d L\,\bar{\rho}(\delta(L))\bar{\rho}(-\delta(L))\exp\left[-\frac{2}{\gamma^2}\left(2\frac{I_{11}I_{13}}{I_{02}}-I_{22}-2\langle\chi_r\chi_l^2|G_l|\chi_r\chi_l^2\rangle+O(\bar{\omega}^3)\right)\right]$$
$$\times\frac{\hat{z}(L,\bar{\omega})}{\sqrt{\pi}}\int_{-\pi/4}^{3\pi/4}d\theta\,\phi(z_1,\ldots,z_4;L,\theta,a)\exp\left[-\hat{z}^2(L,\bar{\omega})\left(\sin 2\theta-\frac{I_{13}(L)}{\bar{\omega}I_{02}}\right)^2\right]$$
$$\times[1+O(\gamma,\bar{\omega})]\,, \qquad (3.39)$$

where

$$\hat{z}(L,\overline{\omega})=\frac{1}{\gamma}\frac{\overline{\omega}}{N_3^{(0)}}[\langle f|M|f\rangle^{-1/2}+O(\overline{\omega},\gamma)] \tag{3.40}$$

and

$$\phi(z_1,\ldots,z_4;L,\theta,a)=I_{02}^{-2}\prod_{i=1}^{2}[\cos\theta\chi_r(z_i-a)+\sin\theta\chi_l(z_i-a)]\prod_{j=3}^{4}[-\sin\theta\chi_r(z_j-a)+\cos\theta\chi_l(z_j-a)]. \tag{3.41}$$

This is our general result. We have indicated the order to which the accuracy has been checked explicitly.

Since our methods is, in principle, an expansion in powers of γ followed by an evaluation of the coefficients of this expansion for $\overline{\omega}\ll 1$, the result certainly gives the leading behavior in the limit of weak disorder: $\lim\overline{\omega}\to 0$, $\lim\gamma\to 0$. The hydrodynamic limit $\lim\gamma\to 0$, $\lim\overline{\omega}\to 0$ poses a more difficult problem since, *a priori*, divergent higher-order terms of the form $\gamma^n\overline{\omega}^{-k}$, $k>0$ can occur. That such terms do not exist can be proven to all orders in γ. Essentially, they are eliminated by the use of the degenerate-state perturbation theory. However, we have not been able to exclude the existence of terms of the form $\gamma^n|\ln\overline{\omega}|^k, k>0$. Such terms might occur since the matrix elements $I_{11}(L_0)$, $I_{13}(L_0)$, etc. differ by powers of $L_0\sim|\ln\overline{\omega}|$. Indeed, evaluating these overlap integrals for large values of L we find

$$\overline{\omega}\sim I_{13}(L_0)\sim e^{-L_0}L_0^{(1-d)/2}, \tag{3.42a}$$

$$I_{11}(L_0)\sim I_{13}(L_0)L_0, \tag{3.42b}$$

$$I_{22}^{1/2}(L_0)\sim I_{13}(L_0)L_0^s,\quad\text{where}\begin{cases}s=0, & d>3\\ s=(3-d)/4, & d<3\\ L_0^s\equiv\ln L_0, & d=3.\end{cases} \tag{3.42c}$$

A careful analysis of the formalism shows that factors of $\ln\overline{\omega}$ can only occur in connection with the function ψ_3 [Eq. (3.16c)]. For this function the estimates are very complicated, since the normalization N_3 itself vanishes as $L\to\infty$. In leading order $\psi_3\sim\chi_r\chi_l$ and hence

$$N_3\sim I_{22}^{1/2}. \tag{3.43}$$

If such logarithmic factors do occur we believe that a modification of the θ constraint and hence ψ_3 would eliminate them. We therefore suggest that the result (3.39) gives also the correct hydrodynamic limit.

IV. EVALUATION

If we can evaluate the θ integral in our result (3.39) in the saddle-point approximation, we recover the result (2.8) of the qualitative argument except for corrections of order $\overline{\omega}$ to the saddle-point contribution $J(L)$. To show the influence of the non-Gaussian θ fluctuations we analyze the integrals

$$\theta_m(L)=\frac{\hat{z}}{\sqrt{\pi}}\int_{-\pi/4}^{3/4\pi}d\theta(\sin2\theta)^m\times\exp\left[-\hat{z}^2\left(\sin2\theta-\frac{I_{13}(L)}{\omega I_{02}}\right)^2\right]. \tag{4.1}$$

These integrals with $m=0,2$ occur in the expressions for $S_2(0,R,R,0;\overline{\omega})$ and $S_2(0,0,R,R;\overline{\omega})$. From the definition (3.40) together with the results (3.43) and (3.24) we find $\hat{z}$ as

$$\hat{z}=\hat{z}_0L_0^{-1}\gamma^{-1}e^{L-L_0}\left(\frac{L}{L_0}\right)^{[(d-1)/2]-s}. \tag{4.2}$$

We formally define an angle θ_0 via the identity

$$\sin2\theta_0=\frac{I_{13}(L)}{I_{13}(L_0)}=e^{L_0-L}\left(\frac{L}{L_0}\right)^{(1-d)/2}. \tag{4.3}$$

The saddle-point approximation holds as long as (i) $\hat{z}\to\infty$ and (ii) the two saddle points $\theta=\frac{1}{4}\pi\pm|\theta_0-\frac{1}{4}\pi|$ are well separated. It gives

$$\theta_m(L)=\frac{[\sin2\theta_0(L)]^m}{\cos2\theta_0(L)}\,\Theta(L-L_0), \tag{4.4}$$

where $\Theta(L-L_0)$ denotes the Θ function.

We first discuss the limit of small disorder and the hydrodynamic limit for $d>3$. In these cases $\hat{z}(L)$ is large compared to one for all $L\gtrsim L_0$. The saddle-point approximation breaks down only if $|\theta_0-\frac{1}{4}\pi|\lesssim\gamma^{1/2}$. The confluence of the saddle points just smooths the singularity of the approximation (4.4) and only effects $\theta_m(L)$ for $|L-L_0|\lesssim\gamma^{1/2}$, i.e., for a region small compared to one. For $d\lesssim 3$, $\hat{z}(L_0)$ vanishes in the hydrodynamic limit. It reaches values $\hat{z}(L)\gtrsim\gamma^{-1}$ only for $L\gtrsim L_1\sim L_0+s\ln L_0$. Thus the saddle-point approximation is modified in a region large compared to one. The typical shape of θ_0 and θ_2 as functions of L is given in Fig. 3. We recall that the unit of length is given by the range of the instanton function.

To conclude our analysis we need to evaluate integrals of the form

$$K_n^{(m)}=\int_0^\infty dL\left(\frac{L}{L_0}\right)^{d-1+n}F[L,\gamma^2]\theta_m(L),\begin{cases}m=2, & n=0,2\\ m=0, & n=0,\end{cases} \tag{4.5}$$

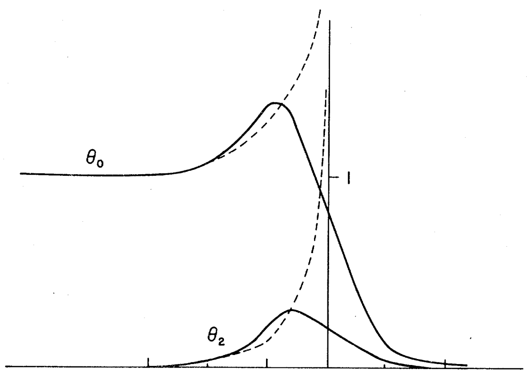

FIG. 3. Plot of the functions θ_0 and θ_2 for $L_0=10$, $\hat{z}(L_0)=1$. This case is intermediate between the limit of small disorder and the hydrodynamic limit. The broken curves give the saddle-point approximation.

where

$$F(L,\gamma^2]=\frac{\bar\rho(\delta(L))}{\bar\rho(\bar\omega)}\frac{\bar\rho(-\delta(L))}{\bar\rho(-\bar\omega)}\times\exp\left[-\frac{1}{\gamma^2}\left(\frac{4}{I_{02}}I_{11}(L)I_{13}(L)-2I_{22}(L)-4\langle\chi_r\chi_l^2|G_l|\chi_l^2\chi_r\rangle\right)\right]. \quad (4.6)$$

In the hydrodynamic limit $F[L,\gamma^2]$ tends to one. For $m=2$ we find

$$K_n^{(2)}=1+O(L_0^{-1},\gamma^{1/2}) \quad (4.7)$$

independent of n. For $m=0$ and $L\to\infty$ the total integrand in Eq. (4.5) tends to one. Extracting this asymptotic behavior and expressing it in terms of the volume Ω of the system we find

$$K_0^{(0)}=L_0^{1-d}S_d^{-1}\Omega-\frac{L_0}{d}+\begin{cases}\text{const}, & d>3\\ -s\ln L_0+\ln\gamma+\text{const}, & d\leq 3.\end{cases} \quad (4.8)$$

In the limit of small disorder the variation of $F[L,\gamma^2]$ is crucial. According to Eqs. (3.42) and (3.38) the contribution $-4I_{11}I_{13}/(\gamma^2I_{02})$ gives the dominant L dependence.

$$F[L,\gamma^2]=\exp\left(-\frac{\alpha}{\gamma^2}e^{-2L}L^{2-d}[1+O(L^{2s-1},L^{-1})]\right), \quad (4.9)$$

where α is a constant. In estimating $K_n^{(2)}$,

$$K_n^{(2)}\sim\frac{L_0^{-d+1-n}}{\omega^2I_{02}^2}\int_0^\infty dL\,L^n\exp\left(-2L-\frac{\alpha}{\gamma^2}e^{-2L}L^{2-d}\right), \quad (4.10)$$

we note that the integrand takes its maximum value for $L_2\sim-\ln\gamma-(d-2)\ln|\ln\gamma|+O(\ln\ln|\ln\gamma|)$, which gives the rough estimate

$$K_n^{(2)}\sim\frac{L_0^{-d+1-n}}{\omega^2I_{02}^2}\gamma^2, \quad (4.11)$$

correct up to factors of order $\ln\gamma$. Similarly we find

$$K_0^{(0)}\sim L_0^{1-d}S_d^{-1}\Omega-\text{const}\times L_2^d. \quad (4.12)$$

We now consider the spatial dependence of $S_2(z_1,\ldots,z_4;\bar\omega)$. If $|z_i-z_j|\ll L_0/2$ for all pairs i,j then the leading contribution is of the form

$$S^I(z_1,\ldots,z_4;\bar\omega)=\tfrac{1}{2}I_{02}^{-2}S_d\,\bar\rho(\bar\omega)\bar\rho(-\bar\omega)L_0^{d-1}K_0^{(2)}\times\int d^da\prod_{i=1}^4\chi(z_i-a), \quad (4.13a)$$

which decays rapidly for $|z_i-z_j|\gg1$. We note that in the hydrodynamic limit this expression diverges as $[\ln(\text{const}/\bar\omega)]^{d-1}$ because of the factor L_0^{d-1}. We also give here the single instanton contribution for $\omega=0$ (compare HS Sec. IV).

$$\hat S^I=\delta(\bar\omega)\bar\rho(0)I_{02}^{-2}\int d^da\prod_{i=1}^4\chi(z_i-a). \quad (4.13b)$$

Next we assume that the z_i are close together pairwise, and consider first the case $z_1=z_2$, $z_3=z_4=z_1+R$, $R\gg L_0/2$. The leading contribution to S_2 has the form

$$S^{II}(z_1,z_1,z_1+R,z_1+R'\bar\omega)=\bar\rho(\bar\omega)\bar\rho(-\bar\omega)I_{02}^{-2}\int d^da\int d^dL\,\chi^2(a)\chi^2(R-a-L)F(L)[\theta_0(L)-\tfrac{1}{2}\theta_2(L)]. \quad (4.14)$$

As $R\to\infty$ the integrand tends to one, thereby reproducing the contribution of two decoupled instantons. In the hydrodynamic limit the increase from $S^{II}\sim0$ to $S^{II}\sim\bar\rho^2$ occurs near $L\sim L_0$. For $|R-L_0|\gg1$, S^{II} behaves as

$$S^{II}\sim\bar\rho^2[\theta_0(R)-\tfrac{1}{2}\theta_2(R)]. \quad (4.15)$$

In the limit of small disorder the increase occurs near $R\sim L_2$. A schematic representation of the function $S_2(0,0,R,R;\bar\omega)$ is given in Figs. 4(a) and 4(b). The integral of $S_2(0,0,R,R,\bar\omega)/[\bar\rho(\bar\omega)\bar\rho(-\bar\omega)]$ yields the volume of the system reduced by the volume Ω_l which one well excludes for the occurrence of the other one. In the qualitative argument of Sec. II we estimated $\Omega_l\sim(1/d)S_dL_0^d$. Our calculation yields

$$\Omega_l=\Omega-L_0^{d-1}S_dK_0^{(0)} \quad (4.16)$$

or

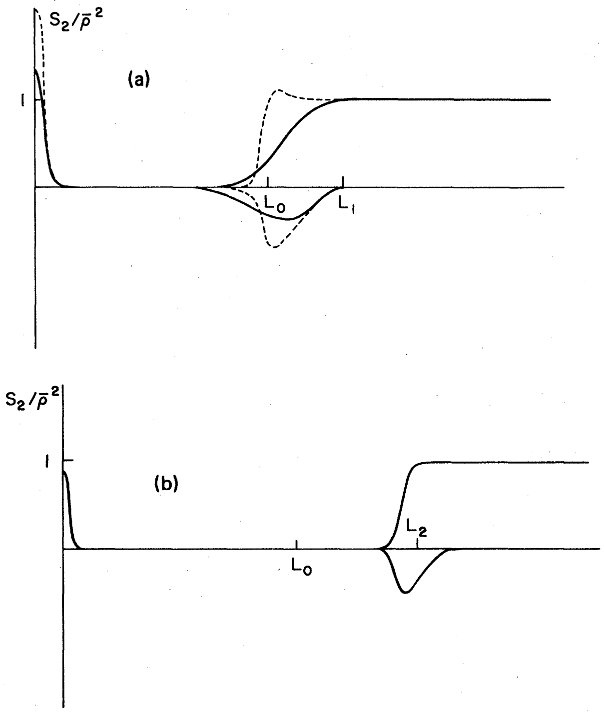

FIG. 4. Schematic plots of the functions $S_2(0,0,R,R;\bar{\omega})$ and $S_2(0,R,R,0,\bar{\omega})$ in the hydrodynamic limit [(a)] and in the limit of small disorder [(b)]. For $R \ll L_0/2$ both functions coincide. In (a) the broken curve holds for $d>3$, whereas the full curve represents $d \leq 3$.

$$\Omega_1 = \frac{1}{d} S_d \times \begin{cases} L_0^d, & d>3 \\ (L_0 + s\ln L_0)^d, & d \leq 3 \end{cases} \tag{4.17a}$$

for the hydrodynamic limit and

$$\Omega_1 = \text{const} \times L_2^d \tag{4.17b}$$

in the limit of small disorder. In all cases the correlations increase the excluded volume.

The function $S_2(0,R,R,0,\bar{\omega})$ can be discussed in a similar way. For $R \gg L_0/2$ the leading contribution is of the form

$$S^{III}(0,R,R,0;\bar{\omega}) = -\tfrac{1}{2} I_{02}^{-2} \bar{\rho}(\bar{\omega})\bar{\rho}(-\bar{\omega}) \times \int d^d a \int d^d L \, F[L,\gamma^2]\theta_2(L)\chi^2(a) \times \chi^2(R-a-L) \,. \tag{4.18}$$

A schematic plot is given in Figs. 4(a) and 4(b). The integral over all R vanishes identically.

The conductivity can be calculated from $S_2(0,R,R,0;\bar{\omega})$ according to Eq. (1.6). We find

$$\sigma(\omega,E) = -\frac{\pi e^2}{2d}\omega^2 |E|^{-2}\mu^{d-2} \int d^d R \, R^2 S_2(0,R,R,0;\bar{\omega}) = \frac{\pi e^2}{d}\bar{\omega}^2 \mu^{d-2}\bar{\rho}(\bar{\omega})\bar{\rho}(-\bar{\omega}) S_d L_0^{d+1} K_2^{(2)} \,. \tag{4.19}$$

In the hydrodynamic limit this gives

$$\sigma(\omega,E) = \frac{\pi e^2}{d}\bar{\omega}^2\mu^{d-2}\bar{\rho}(\bar{\omega})\bar{\rho}(-\bar{\omega})\left(\ln\frac{\text{const}}{\bar{\omega}}\right)^{d+1} \tag{4.20}$$

in agreement with the results of the qualitative considerations of Ref. 12 and of Sec. II. In the limit of weak disorder we find

$$\sigma(\omega,E) \simeq (\pi e^2/d)\gamma^2\mu^{d-2}\bar{\rho}(\bar{\omega})\bar{\rho}(-\bar{\omega}) I_{02}^{-2} S_d \tag{4.21}$$

correct up to factors of order $\ln\gamma$. In this case the leading frequency dependence is due to $\bar{\rho}(\bar{\omega})$.

To summarize, our analysis has shown that the simple picture of Mott, Anderson, and Halperin[12] correctly describes the main features of the two-particle spectral function. Deviations from this picture occur near the minimal distance L_0 and are due to two effects. First, in this region there are strong fluctuations of the angle which determines the superposition of the single-well wave functions. These fluctuations smooth the sharp singularity which, in the simplified picture, occurs at $L \sim L_0$. Second, the interaction of the two wells in the saddle-point contribution itself leads to deviations from the simple superposition approximation. These deviations are important only in the limit of small disorder; they essentially shift the cutoff distance from L_0 to a larger value L_2.

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