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New method for a scaling theory of localization

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We base a scaling theory of localization on an expression for conductivity of a system of random elastic scatterers in terms of its scattering properties at a fixed energy. This expression, proposed by Landauer, is first derived and generalized to a system of indefinite size and number of scattering channels (a "wire"), and then an exact scaling theory for the one-dimensional chain is given. It is shown that the appropriate scaling variable is $f(\rho) = \ln(1+\rho)$ where ρ is the dimensionless resistance, which has the property of "additive mean," and that scaling leads to a well-behaved probability distribution of this variable and to a very simple scaling law not previously given in the literature.

I. INTRODUCTION

A scaling theory of the localization phenomenon based on the formalism of Thouless was recently proposed.[1] Some of the results were similar to those of a method of Wegner.[2] Simulations by Lee[3] directed at testing these theories have given results apparently contradicting this theory. Conditions for achieving one-dimensional localization have been discussed by Thouless,[4] but the basic theorem that one-dimensional systems always localize[5] has been questioned in numerical work.[6] On the other hand, the bulk of recent experimental evidence tends to support Refs. 1 (Refs. 7 and 8) and 4 (Refs. 9 and 10).

This somewhat confusing situation led us to search for a more rigorous formalism, which we believe we have developed. This formalism is based on quantum scattering theory rather than on properties of the eigenstates and their response to boundary conditions. Of course, the two formalisms must be nearly equivalent, but the present one has the advantage of proceeding from an unequivocal definition of conductance which is valid at any scale and is devoid of qualitative arguments. It also applies very naturally to the weak-scattering, high-conductance, "classical-transport" regime where most of the differences among authors appear. Our formalism proceeds from and generalizes a method of Landauer[11] for treating the simplest one-dimensional case. While Landauer's starting point is correct, certain results of his are faulty in a fairly subtle way (which Landauer himself remarked on). We correct this difficulty, which we feel is very instructive as to the subtle differences among different calculations. As actually already remarked both by Thouless and less specifically by Landauer, the probability distributions are very broad and do not converge satisfactorily unless treated with precision: It is possible to average correctly but for the average to be not representative of the distribution and thus to give an incorrect physical result.

In a subsequent paper, one of us (P.W.A.) will give a theory of the general one-dimensional case with many channels transverse to the unique dimension: a "wire" rather than a "chain." It will be argued in that paper, finally, that in the weak-scattering limit the results of the theory also apply to two and three dimensions and confirm the results of Ref. 1.

II. FORMULA FOR CONDUCTANCE OF A SYSTEM OF A SYSTEM OF SCATTERING

We introduce our general formula for conductance, following Landauer, in the simple one-dimensional chain. First, consider a random scattering region in the one-dimensional chain. We can characterize such a region by a scattering matrix S which is a matrix connecting the two incoming and two outgoing channels (see Fig. 1).

$$S = \begin{array}{c} \\ i \\ i' \end{array} \begin{array}{c} \begin{array}{cc} r & o \end{array} \\ \begin{bmatrix} r & t \\ t' & r' \end{bmatrix} \end{array} . \tag{1}$$

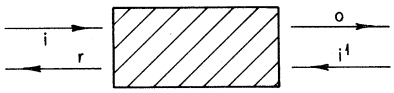

FIG. 1. "Black box" representing a section of wire containing random scatterers.

In the absence of magnetic fields, S is time-reversal invariant, which implies symmetry: $t=t'$. The reflection and transmission coefficients are $R=|r|^2=|r'|^2$ and $T=|t|^2=|t'|^2$ (the equalities are enforced by unitarity as is $r/r'^*=-t/t'^*$).

We propose, with Landauer,[11] that the conductance is

$$G=\frac{e^2}{2\pi\hbar}\frac{T}{R}\,. \tag{2}$$

The derivation is very simple, analogous to the derivation of the tunneling conductance of a tunnel junction. We imagine that the Fermi level of the reservoir on the left from which electrons are appearing in the i channel and to which they disappear in the r channel is raised by $e\delta V$ relative to that on the right. The extra density of electrons in the levels near the Fermi surface will be

$$\delta n=\frac{dn}{dE}\,e\delta V\,,$$

and it will also be equal to the sum of the magnitudes of the currents in the left channels divided by the speeds, less that of the right:

$$\delta n=\frac{j_i+j_r}{v_i}-\left(\frac{j_o+j_{i'}}{v_r}\right)$$
$$=\frac{j_i+j_r-(j_o+j_{i'})}{\partial E/\partial P_x}=\frac{2R(j_i-j_{i'})}{\partial E/\partial P_x}\,.$$

The current will be $I/e=j_i-j_r=j_o-j_{i'}=(j_i-j_{i'})T$. Then the conductance G is

$$G=\frac{I}{\delta V}=\frac{T}{2R}\,e^2\frac{dn}{dE}\frac{\partial E}{\partial P_x}\,.$$

Taking into account that $dn/dE=(1/\pi\hbar)(\partial P_x/\partial E)$, we note that the velocity and density-of-states factors cancel exactly (as in the usual tunneling problem) so that

$$G=\frac{e^2}{2\pi\hbar}\frac{T}{R}\,.$$

In any physical conductor of electrons, we should take into account spin and time-reversal (T) invariance. If the scattering is T-invariant, as it will be in the absence of magnetic fields or magnetic scatterers, there is always a precise degeneracy of time-reversed channels, which conduct in parallel, and no scattering between them. Thus the physical G including spin is

$$G(\text{incl. spin})=\frac{e^2}{\pi\hbar}\left(\frac{T}{R}\right)_{\text{1 channel}} \tag{3}$$

In what follows, we shall use the dimensionless conductance g and resistance ρ defined as

$$g=\frac{1}{\rho}=\frac{T}{R}\,, \tag{4}$$

and the conversion factor for a nonmagnetic system is $e^2/\pi\hbar$.

This formula can be generalized immediately, because of this very simple form, to any wire which is topologically one-dimensional. Such a system has a set of ingoing channels α and of outgoing ones β, and in general the transmission and reflection are matrices in the channel indices

$$\underline{r}=(r_{\alpha\beta}),\quad \underline{t}=(t_{\alpha\beta}) \tag{5}$$

where, by T invariance, $r=\tilde{r}$, $t'=\tilde{t}$, and $r'=-(t^*)^{-1}r^*t$. It may be shown that $\underline{r}$ and $\underline{t}$ may be simultaneously diagonalized by separate unitary transformations on the incoming and outgoing channels so that the S matrix in general takes the form

$$S=\left(\begin{array}{cc|cc} r_{11} & & 0 & t_{11} & & 0 \\ & r_{22} & & & t_{22} & \\ 0 & & r_{\beta\beta} & 0 & & t_{\alpha\alpha} \\ \hline & t' & & & r' & \end{array}\right)\,, \tag{6}$$

with each pair of incoming and outgoing channels satisfying the unitarity requirements separately. It is clear that since the voltage difference is the same for every channel, while the current adds, the conductance is just the sum of all channel conductances

$$G=\frac{e^2}{2\pi\hbar}\sum_\alpha\frac{|t_{\alpha\alpha}|^2}{|r_{\alpha\alpha}|^2} \tag{7}$$

(again, the channels will divide into two noncommunicating sets belonging to the two different eigenvalues of the time-reversal operator $T=\pm1$). In general, if we try to use the *same* representation for left and right (i and o) channels, $t_{\alpha\beta}$ cannot be diagonalized, so it is more symmetrical to write

$$G=\frac{e^2}{2\pi\hbar}\sum_\alpha\frac{1}{|r_{\alpha\alpha}|^2}\sum_\beta(t_{\alpha\beta})^2$$
$$=\frac{e^2}{2\pi\hbar}\sum_{\alpha\beta}\frac{|t_{\alpha\beta}|^2}{1-\sum_\beta|t_{\alpha\beta}|^2}\,. \tag{8}$$

Consider the limit of a big wire where n, the number of channels, becomes large. It is clear that

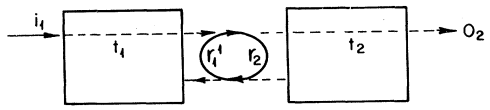

FIG. 2. Elementary scaling process: Composition of two wire sections involves eigenvalues of backscattering operator $r_1' r_2$.

for G to be finite, $t_{\alpha\beta} \sim 1/n$, and $|r_{\alpha\alpha}|^2 = 1 - \sum_\beta |t_{\alpha\beta}|^2 \simeq 1 - O(1/n)$. Therefore,

$$G_{n\to\infty} \cong \frac{e^2}{2\pi\hbar} \mathrm{Tr}\underline{t}\underline{t}^\dagger. \tag{9}$$

It is easy to check that in the two limiting cases where conductivity may be calculated by perturbation theory, namely, weak random scatterers in a pure matrix (with $T \sim 1$, R small) and high tunneling barriers ($R \sim 1$, T small), these formulas are precise. It is also very important to recognize that they are a *general, rigorous, abstract* description of the problem of resistance due to potential fluctuations: The concept of channel is completely general and is the mathematical description of what we mean by attaching a reservoir to a resistor. In the absence of interference effects, it may be shown that T/R adds in parallel and R/T in series, as must be true to give Ohm's laws for such resistors.

III. THE ONE-CHANNEL CASE: COMPLETE SOLUTION OF ONE-DIMENSIONAL CONDUCTIVITY

The first steps in carrying out this calculation were re-invented by us independently but are identical to those of Landauer in 1970. The plan is to place two scatterers in series, calculate the net overall conductance of the pair, and then put our compound systems together, etc., until we have scaled up to a long wire. For definiteness we like to think of a system of rather weak scatterers along the wire, well separated by random intervals relative to the average wavelength so that their phasing is already random at a very early stage. There are two important length scales: the mean free path l which is the length at which $|r|^2 \sim 1/e$, and what one might call the "coherence length" or "randomness length l_p" at which the phases are randomized. This length l_p is the scale at which we might hope to begin to have universality in the scaling process, and since the most interesting localization phenomena take place near l, they can be discussed in a universal fashion when $l_p \ll l$.

The law of composition of two scatterers is easily calculated (see Fig. 2):

$$t = t_1 \frac{1}{1 - r_1' r_2} t_2. \tag{10}$$

The basic assumption of this approach is that r_1' and r_2 are the reflectivities of two systems which are stochastically unrelated to each other. They belong to the same population so that $P(r_1) = P(r_2)$, where P is the probability distribution. But eventually (once $L > l_p$), the phase of r_1 is completely uncorrelated with that of r_2, and in fact $P(|r|e^{i\phi}) = P(|r|)$, the phase being arbitrary.

If this is the case, then taking averages of t is very easy. In particular, the average resistance after compounding two scatterers obeys

$$\rho = \left\langle \frac{|r|^2}{|t|^2} \right\rangle = \left\langle \frac{1}{|t|^2} \right\rangle - 1 = \left\langle |1 - r_1 r_2'|^2 \frac{1}{|t_1|^2} \frac{1}{|t_2|^2} \right\rangle - 1. \tag{11}$$

We average over the random phase of $r_1 r_2'$,

$$\rho = \frac{1 + |r_1|^2 |r_2|^2}{|t_1|^2 |t_2|^2} - 1 = 2\left(\frac{1}{|t_1|^2} - 1\right)\left(\frac{1}{|t_2|^2} - 1\right) + \frac{1}{|t_1|^2} - 1 + \frac{1}{|t_2|^2} - 1. \tag{12}$$

Thus

$$\langle R \rangle = \frac{2\pi\hbar}{e^2} \frac{|r|^2}{|t|^2} = \langle R_1 \rangle + \langle R_2 \rangle + 2\langle R_1 \rangle \langle R_2 \rangle \left(\frac{e^2}{2\pi\hbar}\right). \tag{13}$$

This is the answer found by Landauer: It leads to an apparently plausible localization behavior. Let us revert to the dimensionless resistance and conductance, normalized by the universal factor as in Eq. (4),

$$\rho = \frac{|r|^2}{|t|^2} = \frac{e^2}{2\pi\hbar} R = \frac{1}{g}.$$

Then (13) reads

$$\langle \rho \rangle = \langle \rho_1 \rangle + \langle \rho_2 \rangle + 2\langle \rho_1 \rangle \langle \rho_2 \rangle. \tag{14}$$

As Landauer did, we may convert this into a scaling equation by letting $\rho_2 = d\rho \ll \rho_1 \cong \rho$. We assume that the starting resistance at scale L_0 is classical, $\rho_0 \ll 1$, and that $d\rho = \alpha dL$ where α is the classical resistivity ρ_0/L_0. Then, from Eq. (14)

$$\ln \frac{1 + 2\rho}{1 + 2\rho_0} = 2\rho_0 \left(\frac{L}{L_0} - 1\right) = 2\alpha(L - L_0). \tag{15}$$

Since $\rho_0 \ll 1$, we have

$$\rho \cong \tfrac{1}{2}(e^{2\alpha L} - 1). \tag{16}$$

Equation (16) gives exponential localization, as expected, for $\rho > 1$.

This is highly satisfactory but it is not correct except in a strict sense, because while it is correct for the mean resistance, it is not at all so for a typical or scale resistance. To see the difficulty, note what happens if we average conductance instead. We again perform the phase average so that

$$\langle g \rangle = \frac{\langle |t|^2 \rangle}{1 - |t|^2} = \frac{t_1^2 t_2^2}{|t_1^2 - t_2^2|} \tag{17}$$

and

$$\rho = \frac{1}{\langle g \rangle} = |\rho_1 - \rho_2|. \tag{18}$$

This leads to no rational scaling law. More reasonable is the result of averaging $|t|^2$, which gives

$$\rho = \frac{1}{\langle |t|^2 \rangle} - 1 = \rho_1 + \rho_2, \tag{19}$$

which scales to perfect classical additivity.

It is clear that a unique answer to the scaling problem cannot be obtained by simply averaging, and that we must take the full distribution into account. One can calculate the effect on the distribution function of combining two scatterers (see the Appendix), but we would like to illustrate the more general method we will use later.

The idea of this new method is to find a variable in which the probability distribution is not singular and is more or less Gaussian, or at least obeys a weakened form of the law of large numbers. A requirement which aids in the choice of such a variable is that physically the resistance should not depend on the order in which the scaling is done—whether we add equal segments or unequal ones, for instance—so that our appropriate scaling variable should have an additive mean. These requirements are not met for the resistance. Instead, we shall see that the inverse localization length is appropriate.

To proceed, we reexpress the probability distribution

$$P(\rho) d\rho = W(f(\rho))\, df \tag{20}$$

such that $\langle f(\rho) \rangle$ grows linearly with scale:

$$\langle f(\rho) \rangle = \langle f(\rho_1) \rangle + \langle f(\rho_2) \rangle. \tag{21}$$

Here, the angular brackets indicate an average over phases and magnitudes. This assures that $\langle f(\rho) \rangle \propto L$. In practice, (21) is best solved by finding an f for which the phase average $\langle f(\rho) \rangle_{\text{av}}$ satisfies

$$\langle f(\rho) \rangle_{\text{av}} = f(\rho_1) + f(\rho_2), \tag{22}$$

although there seems no necessity that this be the case in general. The general procedure is unavailing unless

$$\langle [\Delta f(\rho)]^2 \rangle < L^{2-\epsilon}, \tag{23}$$

so that the distribution $W(f)$ has a breadth going slower than linearly:

$$\langle [(\Delta f)^2] \rangle^{1/2} / \langle f \rangle \to 0 \text{ as } L \to \infty. \tag{24}$$

Of course, when Eq. (22) holds, this requirement is trivially met.

When Eq. (24) is satisfied, the distribution in f will converge to a universal distribution, which may have diverging moments in ρ but reflects a Gaussian or other rational distribution in $f(\rho)$.[12] $f(\rho)$ is easily found in this case. Consider

$$\ln(t^2) = \ln(t_1^2) + \ln(t_2^2) - \ln(1 + r_1^2 r_2^2 - 2 r_1 r_2 \cos\theta).$$

Since

$$\int d\theta \ln(a + b\cos\theta) = \pi \ln \tfrac{1}{2}[a + (a^2 - b^2)^{1/2}]$$

is zero for our a, b, we have

$$\langle \ln t^2 \rangle_{\text{av}} = \ln t_1^2 + \ln t_2^2. \tag{25}$$

This means that on averaging over any distribution of t_1, t_2,

$$\langle \ln(1+\rho) \rangle = \langle \ln(1+\rho_1) \rangle + \langle \ln(1+\rho_2) \rangle. \tag{26}$$

This scaling variable reduces to ρ for small resistance, reflecting classical additivity, and to $\ln\rho$ for large, which gives exponential localization. The scaling which (26) implies is, of course

$$\ln(1+\rho) = \alpha L. \tag{27}$$

Thus, we are proposing α, the inverse localization length as the meaningful scaling variable. From (27), we have

$$\rho = e^{\alpha L} - 1 \tag{28}$$

instead of (16), αL being the "classical" additive resistance. This differs by a factor of 2 from Landauer's result.

Next we check that the variance does not scale singularly. This means that we must calculate

$$\langle (\Delta f)^2 \rangle = \langle [\ln(1+\rho) - \langle \ln(1+\rho) \rangle]^2 \rangle, \tag{29}$$

and for this calculation to take on a simple scaling form, it is essential that we have found the linearizing, in the sense of (21), function f. $\langle \Delta f^2 \rangle$ may be separated out as follows:

$$\begin{aligned} \langle (\Delta f)^2 \rangle &= \langle (f - f_1 - f_2)^2 \rangle + (f_1 - \langle f_1 \rangle)^2 + (f_2 - \langle f_2 \rangle)^2 \\ &= \langle \Delta^2 \rangle + \langle (\Delta f_1)^2 \rangle + \langle (\Delta f_2)^2 \rangle, \end{aligned} \tag{30}$$

where Δ^2 is added by the phase-averaging pro-

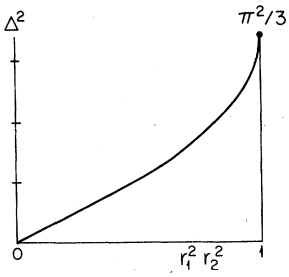

FIG. 3. Sketch of added variance function Δ^2 as function of reflectance product $r_1^2 r_2^2$.

cesses ($\langle\ \rangle_{av}$) and $\langle(\Delta f_{1,2})^2\rangle$ is the variance one would have in a simple addition of stochastic variables. The essential thing is that $\langle\Delta^2\rangle$ as defined in this way must not be singular as $L\to\infty$. This is trivially so in this case:

$$\langle\Delta^2\rangle=\frac{1}{2\pi}\int d\theta[\ln(1+r_1^2r_2^2+2r_1r_2\cos\theta)]^2. \tag{31}$$

It is easy to see that this takes on its maximum value as a function of r_1r_2 when $r_1r_2=1$:

$$\langle\Delta^2\rangle_{(r_1r_2=1)}=\tfrac{1}{3}\pi^2=\frac{1}{2\pi}\int d\theta\left(\ln 4\sin^2\frac{\theta}{2}\right). \tag{32}$$

A calculation shows that if we define $r_1^2r_2^2=1-\epsilon$,

$$\langle\Delta^2_{\epsilon\to 0}\rangle=\tfrac{1}{3}\pi^2+2\epsilon\ln\epsilon, \tag{33}$$

so it has infinite negative slope at $r_1r_2=1$, while for small r_1r_2, it is easily evaluated as

$$\langle\Delta^2_{r_1r_2\to 0}\rangle\cong 2r_1^2r_2^2. \tag{34}$$

A sketch of this function is shown in Fig. 3.

The crudest estimate of the solution of Eq. (30) suffices to demonstrate convergence in the sense of (23). Let us write (30) as

$$\Delta^2(L_1+L_2)=\langle\Delta^2\rangle+\Delta^2(L_1)+\Delta^2(L_2), \tag{35}$$

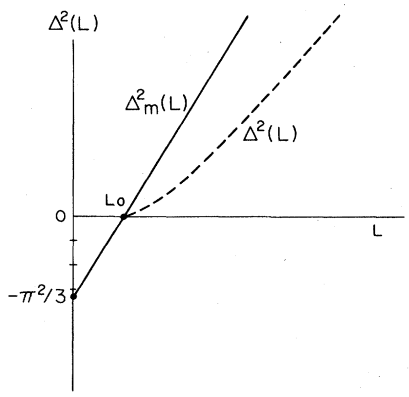

FIG. 4. Solution of scaling equation for variance: upper limit Δ_m^2 and probable cause of real solution $\Delta^2(L)$.

where $\langle\Delta^2\rangle$ is a function of L_1 and L_2 via its dependence on $r_1^2r_2^2=\rho_1\rho_2/(\rho_1+1)(\rho_2+1)$. But the only really important property it has is that $\langle\Delta^2\rangle<\pi^2/3$; it is less than a constant. The solution of

$$\Delta_m^2(L_1+L_2)=\tfrac{1}{3}\pi^2+\Delta_m^2(L_1)+\Delta_m^2(L_2) \tag{36}$$

is everywhere greater than the solution of (35), and (36) is solved by

$$\Delta_m^2(L)=KL-\tfrac{1}{3}\pi^2.$$

So,

$$\Delta^2(L)<KL-\tfrac{1}{3}\pi^2=\tfrac{1}{3}\pi^2(L/L_0-1). \tag{37}$$

Figure 4 shows this solution and an approximate estimate of the real solution for $\Delta^2(L)$ starting with $\Delta^2(L_0)=0$.

Equation (37) approaches a simple additive behavior for the variances, so that the distribution in the variable $\ln(1+\rho)$ remains quite well behaved, except perhaps in the tails. With this demonstration of that fact, we complete our solution of the one-dimensional chain problem.

The real solution referred to above may be estimated in two ways. First, we note that for sufficiently small L,

$$r_1^2\cong\rho_1\cong\rho_c=\alpha L_1,$$

where α is the nominal resistivity. Thus for small L,

$$\Delta^2(L_1+L_2)\cong\Delta^2(L_1)+\Delta^2(L_2)+2\alpha^2L_1L_2,$$

which clearly may be solved by

$$\Delta^2(L)\,|_{L\to 0}=\alpha^2L^2+\text{const}. \tag{38}$$

Thus the initial rise of Δ^2 is quadratic in Fig. 4. If we make the very crude approximation that (38) holds until (37) takes over, we get $2\alpha(\pi^2/3)^{1/2}$ for the slope K in (37).

A second estimate makes use of the "false" scaling law (16) for the mean resistance. How is this compatible with the presumed Gaussian distribution of $f(\rho)$? In fact, this is no problem at all:

$$\langle\rho\rangle(L)=\langle\exp[f(L)]-1\rangle$$

$$=\frac{\int(e^f-1)\exp[-(f-\langle f\rangle)^2/2\Delta^2]\,df}{\int\exp[-(f-\langle f\rangle)^2/2\Delta^2]\,df}$$

$$=\exp(\langle f\rangle+\tfrac{1}{2}\Delta^2)-1=\exp(\alpha L+\tfrac{1}{2}\Delta^2)-1.$$

For this to agree with (16) we require

$$\Delta^2=2\ln\cosh\alpha L, \tag{39}$$

which agrees both with (38) and (37), and may be precise; it need *not* be, however, since $\langle\rho\rangle$ depends heavily on the tail of the distribution which is not necessarily Gaussian-like.

IV. CONCLUDING REMARKS

It is interesting first to derive the β function $d\ln g/d\ln L$ for comparison with other scaling work, specifically Ref. 1. By direct differentiation, we obtain for $g=1/\rho_s$, where ρ_s is the scaling resistance $[\exp(\alpha L)-1]$ of Eqs. (27) and (28),

$$\beta=\frac{d\ln g}{d\ln L}=-(1+g)\ln\left(1+\frac{1}{g}\right)$$

$$=-1-\frac{1}{2g}+\frac{1}{6g^2}+\cdots,$$

so it has precisely the form hypothecated in Ref. 1. The coefficient of $1/g$, however, does not agree with the Langer-Neal weak-scattering perturbation theory used in Ref. 1 which presumably gives $\langle 1/\rho\rangle$, not $1/\rho_s$.

A second important question is what the physical resistance of a long chain would actually be. As we have already pointed out, the mean resistance follows a quite different scaling law from the typical or "scale" resistance ρ_s; even worse, the mean conductance $\langle g\rangle$ scales extremely slowly: For very long wires, if we use the Gaussian approximation, it behaves like

$$\langle g\rangle\simeq e^{-\langle f\rangle+\Delta^2/2}\simeq\text{const}.$$

This means that the distribution of g is very skew with a few extremely large values: There are energies and samples where there is almost no reflection.

These very large fluctuations have important consequences both for numerical computations and for estimates of the localization length. For example, the perturbation-theory analysis of Abrikosov and Ryzhkin[13] and the numerical work of Czycholl and Kramer[6] are based on a study of the average conductivity. The work of Borland[5] and of Papatriantafillou and Economou[14] studies the geometric mean. Andereck and Abrahams[15] have performed numerical and analytic calculations based on the ideas of this paper. They study both the average resistance and the geometric mean for chains of varying length. Each of these three approaches should give a different rate of exponential change for long chains. The slow exponential decrease of the average conductivity for weakly disordered chains could not be seen in Ref. 9, but later work by Kirkpatrick and Thouless[16] has established that it does indeed decrease as expected.

Finally, we remark on what might be expected in experimental observations. Two important points must be recognized. First, measurements involve a range of energies and, usually as in polymers or molecular chains, a set of chains in parallel. Second, the scale length L is in general set, as explained in Ref. 4, by inelastic processes which represent a series resistance coupling the sets of parallel chains. Therefore, if the sample length is sufficiently short, one might expect to see the mean conductance, not the typical or scale conductance $(1/\rho_s)$ unless the probability distribution has such a long tail that the mean is dominated by very high values of the conductance which is improbable even in macroscopic samples. The high values of the conductance, corresponding to near transparency, only occur at special energies, and as the length increases the inelastic scattering limits the occupation of these channels. Thus, for long chains, the typical, or scale resistance ρ_s will be observed.

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APPENDIX

Thus far in this paper, we have dealt only with averaged quantities; the properties of the scatterers have been characterized by, for example, an average and a mean-square resistance. In order to be more complete, however, it is necessary to consider the scatterers to be characterized by a probability distribution which is a function of one magnitude (e.g., the dimensionless resistance ρ) and two phases. Let us consider two scatterers with resistances ρ_1 and ρ_2 put together.

Then the combined scatterer with resistance ρ can be conveniently parametrized by

$$\frac{1}{t}=\sqrt{\rho+1}\,e^{-i\theta}=e^{-i\theta_1-i\theta_2}\sqrt{\rho_1+1}\sqrt{\rho_2+1}$$

$$+\sqrt{\rho_1}\sqrt{\rho_2}\,e^{-i\phi_1+i\phi_2}\equiv K_1 \qquad \text{(A1)}$$

and

$$\frac{r}{t}=\sqrt{\rho}\,e^{i\phi}=e^{i\phi_1-i\theta_2}\sqrt{\rho_1}\sqrt{\rho_2+1}$$

$$+\sqrt{\rho_1+1}\sqrt{\rho_2}\,e^{i\phi_2+i\theta_1}\equiv K_2, \qquad \text{(A2)}$$

where the phases θ_1,ϕ_1, etc., include a contribution from propagation of the wave between the scatterers. We take the magnitudes and phases of the "1" and "2" scatterers to be distributed with probability distributions $P_1(\rho_1,\theta_1,\phi_1)$ and $P_2(\rho_2,\theta_2,\phi_2)$, respectively; the distribution of magnitude and phases for the combined scatterer is

$$P(\rho,\theta,\phi)=\int_0^{2\pi} d\theta_1 d\phi_1 d\theta_2 d\phi_2 \int_0^\infty d\rho_1 \int_0^\infty d\rho_2 P_1(\rho_1,\theta_1,\phi_1)P_2(\rho_2,\theta_2,\phi_2)$$

$$\times \delta(\theta - i\ln(K_1/\sqrt{\rho}+1))\delta(\phi + i\ln(K_2/\sqrt{\rho}))\delta(\rho - |K_1|^2+1)\,. \tag{A3}$$

If we assume, as we have throughout this paper, that the length scale is such that the phases of each scattering block (1 and 2) are random then we just have

$$P_1(\rho_1,\theta_1,\phi_1)=[1/(2\pi)^2]P_1(\rho_1) \tag{A4}$$

and similarly for P_2.

Three of the phase integrals can then be done immediately after the substitutions

$$\begin{aligned}\alpha &= -\theta_1-\theta_2+\theta\,,\\ \beta &= \phi_1-\theta_2-\phi\,,\\ \gamma &= +\theta_1+\theta_2-\phi_1+\phi_2\,,\\ \delta &= \phi_1+\theta_2+\phi_2+\theta_1\,,\end{aligned} \tag{A5}$$

by noting that in these variables

$$\begin{aligned}K_1 &= e^{-i\theta}e^{i\alpha}(\sqrt{\rho_1+1}\sqrt{\rho_2+1}+\sqrt{\rho_1}\sqrt{\rho_2}\,e^{i\gamma})\,,\\ K_2 &= e^{i\phi}e^{i\beta}(\sqrt{\rho_1}\sqrt{\rho_{2+1}}+\sqrt{\rho_2}\sqrt{\rho_1+1}\,e^{i\gamma})\,,\end{aligned} \tag{A6}$$

and hence the α and β integrals can be used to satisfy the angular delta functions (and the δ integral done trivially) to yield a P independent of θ and ϕ. The integral over γ is of the form

$$\int_0^{2\pi}\frac{d\gamma}{2\pi}\,\delta(A-B\cos\gamma)=\frac{1}{\pi(B^2-A^2)^{1/2}}\,, \tag{A7}$$

where A and B are functions of ρ, ρ_1, and ρ_2. This yields, finally,

$$P(\rho)=\int_0^\infty d\rho_1\int_0^\infty d\rho_2\frac{1}{\pi\sqrt{H}}\Theta(H)P_1(\rho_1)P_2(\rho_2)\,, \tag{A8}$$

where Θ is a Heaviside function and

$$H=4\rho\rho_1\rho_2+2(\rho_1\rho_2+\rho_1\rho+\rho_2\rho)-(\rho^2+\rho_1^2+\rho_2^2)\,. \tag{A9}$$

We note that for a given ρ_1,ρ_2 the resistance ρ lies in the range

$$\rho_-<\rho<\rho_+\,,$$

where

$$\rho_\pm=\rho_1+\rho_2+2\rho_1\rho_2\pm 2[\rho_1\rho_2(1+\rho_1)(1+\rho_2)]^{1/2}\,, \tag{A10}$$

and in terms of $\rho_\pm$,

$$H=(\rho_+-\rho)(\rho-\rho_-)\,.$$

Once the probability distribution of ρ is known in terms of those for ρ_1 and ρ_2, all the results of this paper can be rederived. In particular, we consider calculating

$$\langle f(\rho)\rangle=\int_0^\infty d\rho f(\rho)P(\rho)\,. \tag{A11}$$

From Eq. (A8) we have

$$\langle f(\rho)\rangle=\int_0^\infty P_1(\rho_1)P_2(\rho_2)d\rho_1 d\rho_2$$

$$\times\int_{\rho-}^{\rho+}d\rho\frac{f(\rho)}{[(\rho_+-\rho)(\rho-\rho_-)]^{1/2}}\,. \tag{A12}$$

If performing the integral over ρ in Eq. (A12) yields a result of the form

$$g(\rho_1)+g(\rho_2)+h(\rho_1)h(\rho_2)\,, \tag{A13}$$

then (and only then) the integrals over ρ_1 and ρ_2 can be simply interpreted as averages over the distributions of ρ_1 and ρ_2,

$$\langle f(\rho)\rangle=\langle g(\rho_1)\rangle+\langle g(\rho_2)\rangle+\langle h(\rho_1)\rangle\langle h(\rho_2)\rangle\,. \tag{A14}$$

We thus clearly see why expressions like (14) and (26) are useful while those like (18) are not.

As noted by Landauer,[11] typical probability distributions $P(\rho)$ will be rather badly behaved and, as discussed earlier, averages of (for example) ρ or ρ^2 are not very useful in characterizing them. These difficulties are due to the large ρ (localized) "tail" of $P(\rho)$. This can easily be seen if we consider putting together two scatterers with

$$P_i(\rho_i)=(1/a_i)e^{-\rho_i/a_i} \tag{A15}$$

(for $i=1,2$), which are characterized by typical "scale" resistances a_1 and a_2. The integrals in Eq. (A8) can be carried out exactly for this specific case to yield

$$P(\rho)=[(a_1+a_2)^2+4\rho a_1a_2]^{-1/2}$$

$$\times\exp\left\{-\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)-\left[\frac{1}{4}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)^2+\frac{\rho}{a_1a_2}\right]^{1/2}\right\}, \tag{A16}$$

which is a considerably different distribution from P_i. In particular, for ρ large

$$P(\rho)\sim\frac{1}{2(\rho a_1a_2)^{1/2}}\exp[-(\rho/a_1a_2)^{1/2}]\,, \tag{A17}$$

which is to be compared with (A15). We see in this example that in contrast to $P_1(\rho_1)$ and $P_2(\rho_2)$, $P(\rho)$ cannot be characterized by a simple "scale" resistance. In addition, the high-resistance tail for a wire consisting of many scatterers will be non-Gaussian and may depend on the detailed form of the probability distribution of the individual scatterers.

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