Ising gauge theory at negative temperatures and spin-glasses

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We study the $Z(2)$ coupled spin-gauge theory at negative gauge coupling. We find two frustrated phases and a disordered phase in both three and four dimensions. All critical lines are of first order in four dimensions. In three dimensions, the frustration ↔ frustration transition as well as one of the frustration ↔ disorder lines appear to be of second order. We find that the contour in the coupling parameter space along which the unit Wilson loop vanishes intersects a transition line only at infinite negative gauge coupling and infinite spin-spin coupling. This indicates that, contrary to a recent conjecture, this contour cannot describe a transition in spin-glasses.

I. INTRODUCTION

A popular model for spin-glasses[1] is a system of Ising spins S_i on lattice sites interacting via random interactions $\{U_{ij}\}$ distributed with a probability law $P(\{U_{ij}\})$. The total free energy is obtained by averaging the free energy for a given distribution $\{U_{ij}\}$ of bonds over all possible distributions. This is called the quenched sum.

An approximation that is often used is to treat the U_{ij}'s as dynamical gauge variables and thermally average over the spins and U_{ij}'s simultaneously. In this thermal average the gauge-invariant coupling of Wegner[2] and Wilson[3] is included in the Hamiltonian. This procedure, which is simpler to implement in practice, is called the annealed sum. The system is characterized by two coupling constants, h representing the spin-spin interaction through the bonds U_{ij} and β representing the pure gauge theory coupling. These will be defined more precisely in Sec. II.

The annealed sum is known to underestimate the effect of frustrations because of correlations between the U_{ij}'s. Indeed, the expectation value of the product of four U_{ij}'s around an elementary plaquette on the lattice (also called the unit Wilson loop[3]) which vanishes identically in the quenched theory (because the U_{ij}'s are random), is in general nonvanishing in the annealed theory. The authors of Ref. 1 suggest that the quenched sum may be better approximated by augmenting the annealed sum with a condition requiring that the unit Wilson loop vanish identically. The system then becomes a coupled $Z(2)$ spin-gauge theory evaluated on the contour C, in the two-dimensional coupling-constant space, along which the unit Wilson loop vanishes. This contour is reached only for negative gauge coupling β. This modified annealed model would have a transition only if this contour C intersects a transition line in the full phase diagram of the theory.

With this motivation, we have studied the $Z(2)$ spin-gauge theory both in three and four dimensions for negative gauge coupling β and positive Ising coupling h via Monte Carlo computer simulations. We find that for negative gauge coupling, this theory has three distinct phases. There is a disordered phase for small β and two frustrated phases for $\beta << 0$. In four dimensions, all observed transitions are first order. In three dimensions, only one of the transition lines representing a frustration ↔ disorder transition is clearly first order. The other two lines appear to be of second order.

In both three and four dimensions, the contour in the (β,h) plane, along which the unit Wilson loop vanishes, intersects a critical line only at $\beta=-\infty$ and $h=\infty$. This indicates that the modified annealed system has no critical point for finite coupling in either three or four dimensions and is therefore incapable of describing spin-glass transitions.

In Sec. II, we define the $Z(2)$ spin-gauge model and our order parameters. We also construct the lattice configurations that characterize the two frustrated phases we observe. Section III contains the results of our computer simulations and our conclusions.

II. MODEL

The coupled spin-gauge theory at positive gauge coupling was discussed in Refs. 4–6. Therefore, we will be brief and concentrate on the new features appearing at negative β.

The theory is defined by the action

$$\mathcal{S}(\beta,h)=\beta\sum_{p}(1-U_p^4)+h\sum_{l}(1-SUS) \qquad (2.1)$$

and the partition function

$$Z(\beta,h)=\sum_{\{U,S\}}\exp[-\mathcal{S}(\beta,h)] \ . \qquad (2.2)$$

The U's and the S's are Ising variables on the links and sites, respectively, of a d-dimensional hypercubical lattice. U_p^4 is a shorthand notation for the product of four U's around a unit plaquette. The first sum in Eq. (2.1) is over all such plaquettes while the second sum is over all links.

Two useful order parameters are the average link L and plaquette P defined by

$$L=\frac{\partial \ln Z}{N_l \partial h}=\langle 1-SUS\rangle \tag{2.3}$$

and

$$P=\frac{\partial \ln Z}{N_p \partial \beta}=\langle 1-U_p^4\rangle \ . \tag{2.4}$$

The constant 1 in Eq. (2.4) is added to agree with the definition in Ref. 6. The unit Wilson loop is just $1-P$, and thus the contour of vanishing Wilson loop is the contour $P=1$.

For $h=0$, the phase diagram of the theory must be symmetric around $\beta=0$. This is because, given a configuration of links, we can change the sign of every U_p^4 term in Eq. (2.1) by the following transformation.

Let $\vec{\gamma}=(x,y,z,t, \ldots ,)$ label a lattice site and $\mu=(1,2, \ldots , d)$ label the d directions in our space. A link variable can be labeled as $U_{\vec{\gamma},\mu}$. The required transformation is

$$U_{\vec{\gamma},1}\rightarrow U_{\vec{\gamma},1}\ , \quad U_{\vec{\gamma},2}\rightarrow(-1)^x U_{\vec{\gamma},2}\ ,$$
$$U_{\vec{\gamma},3}\rightarrow(-1)^{x+y}U_{\vec{\gamma},3}\ , \tag{2.5}$$

etc. Every link configuration at negative β is mapped by this transformation into a configuration at positive β and vice versa. The phase diagram of $Z(2)$ gauge theory for $\beta>0$ is known.[7] Hence we know all the critical points for $h=0$.

We now define two fully frustrated lattice configurations that are useful in characterizing the two new phases we find at $h\neq 0$. First, for $\beta=-\infty$, $h=0$, the lattice finds itself (apart from gauge transformations) in a configuration with random spins and the U's defined by applying Eq. (2.5) to a fully ordered set of gauge variables. We call this configuration $F1$. It is characterized by the values $P=2$, and $L=1$ of the order parameters.

As h grows at $\beta=-\infty$, the theory approaches a critical point to a preferred configuration with $P=2$ and L as small as possible. The lattice likes to have neighboring spins aligned if they interact ferromagnetically ($U=1$) and antialigned if they interact antiferromagnetically ($U=-1$). This cannot be done perfectly and it is easy to show that a lower bound on L in any number of dimensions for a fully frustrated lattice is 0.5. This is seen as follows:

In unitary gauge, where the gauge freedom is used to set all S_i to $+1$, suppose we flip a fraction f of the links. A flipped link has $U=-1$. Each link is in $2(d-1)$ plaquettes. The number of links on a lattice of N sites is dN. Therefore, the number of plaquettes frustrated by flipping fdN links must be less than or equal to $2d(d-1)fN$. Hence, a fully frustrated lattice, where all $\frac{1}{2}d(d-1)N$ plaquettes have value (-1), satisfies

$$2d(d-1)fN\geq\tfrac{1}{2}d(d-1)N\ , \tag{2.6}$$

i.e.,

$$f\geq\tfrac{1}{4}\ ,\quad L=0\times(1-f)+2\times f\geq\tfrac{1}{2}\ . \tag{2.7}$$

We denote by $F2$ the following realization of this bound in three and four dimensions. These configurations have $P=2$ and $L=0.5$.

$$U_{\vec{\gamma},1}=(-1)^{tyz+(y-1)(z-1)(t-1)}\ ,$$
$$U_{\vec{\gamma},2}=(-1)^{x(t-1)z+(x-1)t(z-1)}\ ,$$
$$U_{\vec{\gamma},3}=(-1)^{tx(y-1)+(t-1)(x-1)y}\ , \tag{2.8}$$
$$U_{\vec{\gamma},4}=(-1)^{xy(z-1)+(x-1)(y-1)z}\ ,$$

and $S=1$ on all sites. The configuration $F2$ in three dimensions is obtained by setting $t=0$ in Eq. (2.8) and omitting the link $U_{\vec{\gamma},4}$. We do not know whether the bound $L=0.5$ when $P=2$ can be realized in arbitrary dimensions.

Finally, we note that for $h=\infty$ and any finite β, both P and L must vanish. This implies that the limits $h\rightarrow\infty$ and $\beta\rightarrow-\infty$ do not commute and suggests a possible phase boundary extending to $\beta=-\infty$ and $h=\infty$. We will see that this is indeed the case.

III. PHASE STRUCTURE

The phase diagrams for positive gauge coupling for $d=3$ and $d=4$ are known.[5,6] We use Monte Carlo methods to study the theory for negative β. The data here were obtained using the heat-bath method as described in Ref. 6. Both site and link variables were varied in the algorithm.

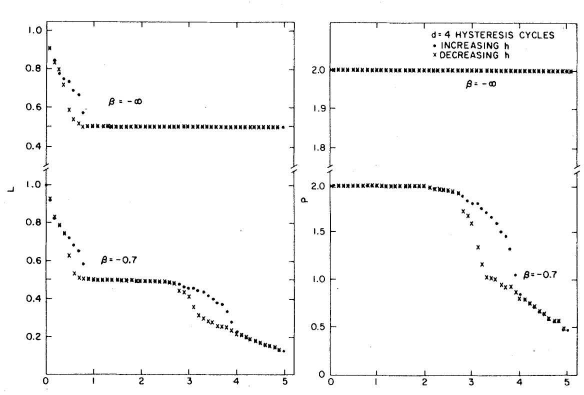

FIG. 1. Hysteresis cycles at $\beta=-\infty$ and $\beta=-0.7$ on a 6^4 lattice, 10 iterations per point. The initial lattice was $F1$ at $h=0$.

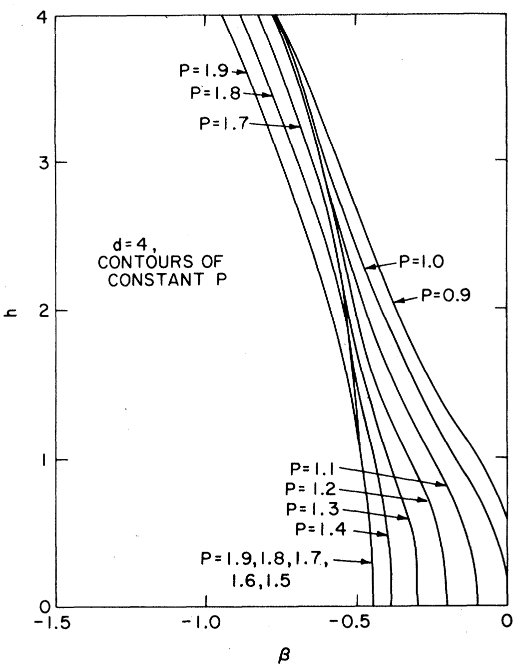

FIG. 2. Contours of constant P on 6^4 lattice.

A. $d=4$

Figure 1 shows hysteresis cycles in L and P at $\beta=-\infty$ and $\beta=-0.7$. It is clear from this figure that for $\beta=-\infty$, there is one transition that separates a phase with $L\approx 0.5$ from one where $L>0.5$. On both sides of this transition, the lattice is totally frustrated ($P=2$). For finite negative β, there are two transitions—one which is seen in both L and P and another which is seen only in L. The first of these is a frustration↔disorder transition. The second is analogous to the frustration↔frustration transition at $\beta=-\infty$.

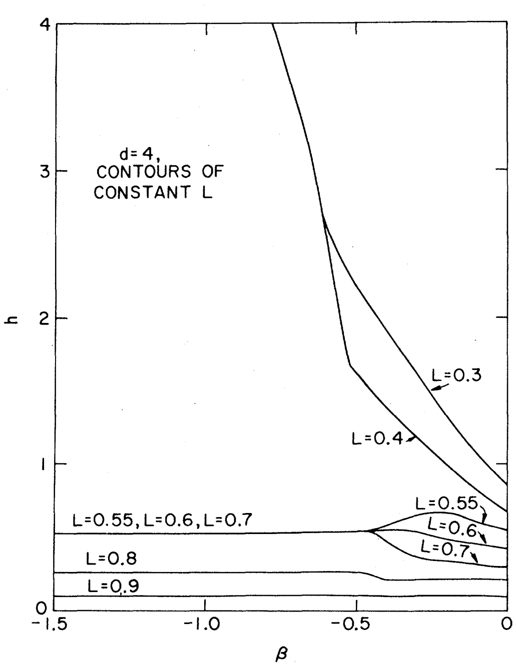

FIG. 3. Contours of constant L on 6^4 lattice.

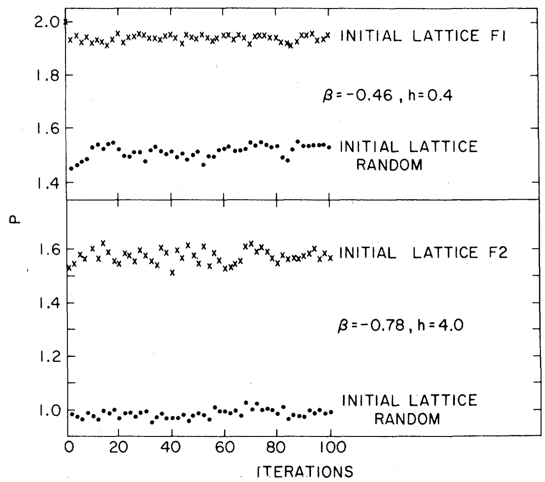

FIG. 4. Iterations from frustrated and random configurations at two points along frustration↔random transition line.

Figures 2 and 3 are contours of constant P and L, respectively. The transition lines are clearly defined as a piling up of the contours. Note that just as in Fig. 1, the disorder↔frustration transition is seen in both P and L, while the frustration↔frustration transition shows up as a singularity in L alone.

In Fig. 4 we plot P as a function of the number of iterations at two critical points along the disorder ↔frustration line. The first-order nature of the transition is evidenced by the appearance of two distinct stable phases.

Figure 5 is a similar plot of L versus the number of iterations at a critical point on the frustration↔frus-

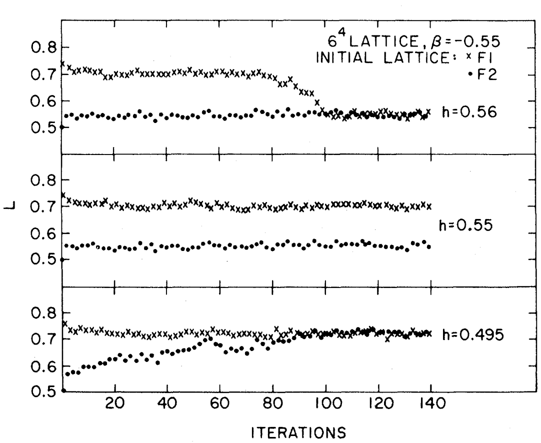

FIG. 5. Iterations from $F1$ and $F2$ near a point on the frustration↔frustration transition line.

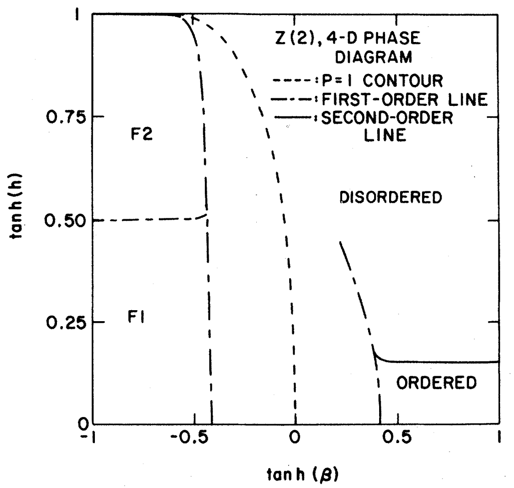

FIG. 6. Phase diagram for $d=4$ $Z(2)$ spin-gauge theory.

tration transition. The critical parameters are identified as $\beta=-0.55$ and $L=0.55$. The other two graphs in this figure show the decay of the false ground state into the true ground state on either side of the critical point. It is evident from Fig. 5 that the frustration↔frustration transition is also first order. A similar analysis shows that the $\beta=-\infty$ transition is first order as well.

Figure 6 is our estimate of the phase structure of the four-dimensional theory. For completeness, we have also included the results of the analysis of Ref. 6 for positive β.[6] We see that the contour for $P=1$ intersects the critical line only at $h-\infty$ and $\beta=-\infty$. The negative β triple point has coordinates $\beta=-0.46\pm0.01$ and $h=0.56\pm0.01$.

B. $d=3$

In three dimensions, a similar study leads to the phase diagram of Fig. 7.

Just as in four dimensions, there are two frustrated phases which we generically call $F1$ and $F2$ (see Fig. 7). However, unlike the case of four dimensions, these two phases are separated by an apparently second-order line. This line begins at ($\beta=-\infty$, $h=1.25\pm0.05$) and terminates in a triple point at ($\beta=-0.79\pm0.01$, $h=1.3\pm0.05$). For small h, there is also a second-order line separating $F1$ from a disordered phase. This line extends from ($\beta=-0.7613$, $h=0$) to the triple point. Finally,

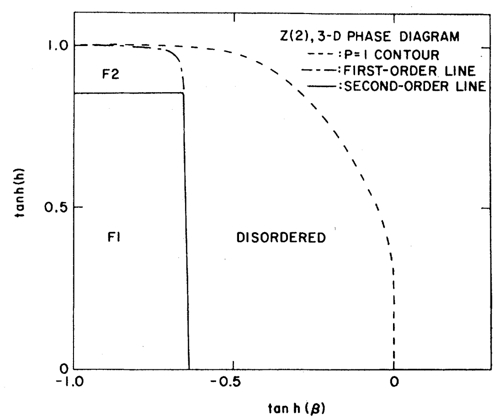

FIG. 7. Phase diagram for $d=3$ $Z(2)$ spin-gauge theory.

there is a first-order line separating $F2$ from the disordered phase. This line originates in the triple point and extends to ($\beta=-\infty, h=\infty$). The $P=1$ contour intersects a critical line only for infinite h and infinite negative β.

IV. CONCLUSION

We have uncovered a rich phase structure in the coupled Z_2 spin-gauge system at negative gauge temperature. This structure, however, avoids the contour suggested in Ref. 1 as an approximate model of spin-glasses. Hence, that model will not exhibit a spin-glass transition in three or four dimensions. Note, however, that in going from three to four dimensions the $P=1$ contour, i.e., vanishing unit Wilson loop, moves closer to the disorder $-F2$ phase transition line. Indeed, in four dimensions this contour appears asymptotically tangent to that line. This suggests that four dimensions may be critical and that with higher dimensionality this contour may indeed disappear beneath the corresponding first-order transition line.

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[1]G. Toulouse and J. Vahnimenus, Phys. Rep. (in press).
[2]F. Wegner, J. Math. Phys. 12, 2259 (1971).
[3]K. G. Wilson, Phys. Rev. D 10, 2445 (1974).
[4]E. Fradkin and S. Shenker, Phys. Rev. D 19, 3682 (1979).
[5]G. A. Jongeward, J. D. Stack, and C. Jayaprakash, Phys. Rev. D (to be published).
[6]M. Creutz, Phys. Rev. D 21, 1006 (1980).
[7]R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D 11, 2098 (1975).