Renormalization group for first-order phase transitions: Equation of state of the compressible Ising magnet

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Thermodynamic functions of an n-component magnetic system coupled to an isotropic elastic medium in $d=4-\epsilon$ dimensions are calculated to leading order in ϵ. Solutions to renormalization-group recursion relations correct to $O(\epsilon)$ along with trajectory-integral methods are employed in these calculations. For $n \geq 2$ the finite compressibility modifies the critical-point behavior producing new corrections to scaling in the thermodynamic functions which are given. In the $n=1$ case renormalization-group predictions of a first-order transition are verified. Thermodynamic functions are given and analyzed in this case. In particular the transition temperature and discontinuity in order parameter are determined and the phase diagram is constructed.

I. INTRODUCTION AND SUMMARY

The critical behavior of magnetic systems with couplings to the elastic lattice has been the subject of considerable study. Rice[1] was the first to point out that a coupling between elastic and magnetic degrees of freedom could qualitatively change the character of the magnetic phase transition (from second to first order). This prediction was later verified by Larkin and Pikin[2] in the case where the coupling was to an isotropic elastic medium. Some years later, using modern renormalization-group methods, Sak[3] and Wegner[4] obtained similar results. Subsequent to these calculations a number of refinements were added to the model Hamiltonian: for example, Bergman and Halperin,[5] in a very comprehensive paper, considered an Ising model coupled to an elastic lattice with cubic anisotropy. Nattermann,[6] in his study of the n-vector model, included the effects of cubic anisotropy not only in the elastic part of the Hamiltonian but also in the magnetic and coupling parts. These and other[7] calculations, while different in some respects, all led to qualitatively similar conclusions regarding the character of the magnetic transition: If the critical exponent α associated with the specific heat of the rigid magnet were positive then the compressible system would undergo a first-order transition[7(a)] while if it were negative, a second-order transition with the rigid-system exponents would be possible. The dominant role played by the specific-heat exponent in these determinations was due to the fact that in most cases studied the magnetic piece of the term coupling elastic and magnetic degrees of freedom was a spin-energy density. Hence α, which is the exponent governing fluctuations in spin-energy density, was directly related to the effects finite compressibility had on magnetic behavior. A partial generalization to this case is seen in Nattermann's work[6] where α lost its predominant role because magnetic operators other than spin-energy density were included in the elastic-magnetic interaction part of the Hamiltonian. This had the effect of making the conditions for a second-order transition more stringent since conditions on more than one exponent had to be satisfied. Similarly, in the work of de Moura *et al.*,[8] the inclusion of elastic anisotropy in the problem had a predictable destabilizing effect: The conditions under which a second-order transition would occur became more restrictive. Nevertheless the model of a compressible magnetic system with a coupling to an isotropic elastic medium, while less acceptable than those models reflecting physically present anisotropies, is useful in that it contains the essential features which are responsible for driving the transition from second to first order.

The purpose of this paper is to examine in detail the properties of a completely isotropic compressible magnet. In particular, we construct expressions for the free energy of this system from which we obtain the susceptibility, specific heat, equation of state, and various universal scaling functions. Some of the methods used in the calculations are those first employed by Nelson and Rudnick[9] for the rigid magnetic system. The differential renormalization-group recursion relations for the thermodynamic fields in the problem are determined and solved to leading order in $\epsilon(=4-d$; d is dimensionality of space). These solutions are then used to map the system near its critical point (or near a weakly first-order transition) onto a system far from criticality whose thermodynamic functions are calculable by perturbative methods.[10]

As mentioned earlier, the sign of α for the rigid system will greatly affect the qualitative conclusions and hence the details of calculations to be presented. The best estimates[11] for the sign of α in three dimen-

sions for rigid magnetic systems from the ϵ expansion, high-temperature series, and experiments are as follows. In systems which are suitably described by a scalar order parameter ($n=1$) it is found that α is positive. On the other hand, for systems described by order parameters with two or more components ($n \geq 2$) α is negative. Thus, in $d=3$, a compressible Ising magnet will undergo a first-order transition while the transition in compressible X-Y and Heisenberg systems can remain of the second order. Our main interest lies in the use of renormalization-group ideas for the study of the weakly first-order transition. These methods have recently met with considerable success in the work by Rudnick[12] on the X-Y model with cubic anisotropy and in the works of Halperin, Lubensky, and Ma[13] and Chen, Lubensky, and Nelson[14] on the weak first-order transition in superconductors. Thus our emphasis will be on Ising systems (where $\alpha > 0$). We will however construct thermodynamic functions for $n \geq 2$ systems valid in the disordered phase. This will allow us to calculate physical properties of systems where the finite compressibility does not affect the order of the transition but merely produces new corrections to scaling behavior.

The organization of the paper is as follows. In Sec. II we review the construction of an effective magnetic Hamiltonian useful for describing the compressible magnet. Section III contains Landau-theory predictions for this system while Sec. IV summarizes previously obtained renormalization-group results. These three sections, while not representing any new results, are needed because ideas presented in them will be called upon in Secs. V and VI. At this point in the paper we begin treating the Ising case separately from the n-vector case ($n \geq 2$). Section V contains leading-order solutions to the differential recursion relations and calculations of thermodynamic functions for the $n \geq 2$ case where a critical point exists. Nonlinear scaling fields appropriate to the problem are also given with universal scaling functions for various quantities. In Sec. VI we begin an analysis of the Ising magnet by setting up and solving to leading order in ϵ a system of differential recursion relations appropriate for the ordered phase. We then calculate various thermodynamic functions including the equation of state and discuss their range of validity. For the region of thermodynamic fields where these solutions are not valid (which includes the temperature at which a first-order phase transition occurs) we reanalyze the problem. This leads to a calculation of the free energy in a region of the thermodynamic fields where fluctuation-corrected Landau theory is appropriate. This free energy is then used in conjunction with mapping formulas to construct the free energy of the actual system near the transition temperature. The expression we obtain is then analyzed, features of the first-order transition are determined, and the phase diagram is constructed. Section VII contains discussion and concluding remarks.

II. EFFECTIVE MAGNETIC HAMILTONIAN

A model Hamiltonian useful for describing a magnetic system coupled to an isotropic elastic medium is given by[2] ($\mathcal{H} \equiv H/T$)

$$\mathcal{H} = \mathcal{H}_m + \mathcal{H}_e + \mathcal{H}_{em} \ ,$$

where

$$\mathcal{H}_m = \int d^d x \left[\frac{1}{2} r \sum_{i=1}^{n} S_i^2(\vec{x}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{\alpha=1}^{d} \left(\frac{\partial S_i(\vec{x})}{\partial x_\alpha} \right)^2 + u_0 \left(\sum_{i=1}^{n} S_i^2(\vec{x}) \right)^2 - h S_1(\vec{x}) \right] , \tag{2.1a}$$

$$\mathcal{H}_e = \int d^d x \left[\left(\frac{1}{2} K - \frac{1}{d} \mu \right) [\vec{\nabla} \cdot \vec{u}(\vec{x})]^2 + \mu \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \left(\frac{\partial u_\alpha(\vec{x})}{\partial x_\beta} \right)^2 \right] , \tag{2.1b}$$

$$\mathcal{H}_{em} = g \int d^d x \left[\left(\sum_{i=1}^{n} S_i^2(\vec{x}) \right) \vec{\nabla} \cdot \vec{u}(\vec{x}) \right] . \tag{2.1c}$$

The magnetic part given by Eq. (2.1a) is the Ginzburg-Landau-Wilson[11] Hamiltonian for a rigid n-component magnetic system: r and u_0 are analytic functions of the thermodynamic fields, h is proportional to the external magnetic field (chosen to lie along the "1" direction), and $S_i(\vec{x})$ is an n-component spin density. The elastic part given by Eq. (2.1b) is the usual[15] expression for the energy of a deformed isotropic elastic medium in the harmonic approximation where K and μ are, respectively, the bulk and shear moduli of the underlying lattice divided by absolute temperature T. We assume that in the region of thermodynamic fields interesting for critical magnetic behavior, these elastic constants have no anomalous behavior except for that due specifically to the coupling with magnetic degrees of freedom. The field $\vec{u}(\vec{x})$ is a d-component vector-displacement field. Both the spin density and displacement fields are constrained from having spatial variations on a scale smaller than a lattice constant (the cutoff is introduced in $\vec{k}$ space below). Finally, the coupling between elastic and magnetic degrees of freedom is given by Eq. (2.1c) where the spin-energy density multiplies the local dilatation with a coupling strength g. While Nattermann[6] has shown that other couplings can affect critical behavior, this coupling is

enough to alter the qualitative character of the magnetic transition in the Ising case ($n=1$).

Since our main interest is in magnetic behavior, we can define an effective magnetic Hamiltonian by

$$e^{-\mathcal{H}_{\text{eff}}} \equiv \mathrm{Tr}_{\vec{u}(\vec{x})} e^{-\mathcal{H}} , \tag{2.2}$$

where the trace is carried over all elastic configurations of the system. Before performing the trace we must separate a given elastic configuration into two pieces: the homogeneous part where the relative extensions are spatially constant and the "phonon" part where the strains are periodic functions of position. Explicitly this becomes

$$\frac{\partial u_\alpha(\vec{x})}{\partial x_\beta} = e_{\alpha\beta} + L^{-d} \sum_{\vec{k}} i k_\beta u_{\alpha,\vec{k}} e^{i\vec{k}\cdot\vec{x}} , \tag{2.3}$$

where we have for the phonon contribution to a given elastic configuration

$$u_\alpha^{\text{ph}}(\vec{x}) = L^{-d} \sum_{\vec{k}} u_{\alpha,\vec{k}} e^{i\vec{k}\cdot\vec{x}} . \tag{2.4}$$

The set of constants $e_{\alpha\beta}$ given in Eq. (2.3) describe the homogeneous part of a given deformation. Let us also Fourier-transform the spin-density

$$S_i(\vec{x}) = L^{-d} \sum_{\vec{k}} S_{i,\vec{k}} e^{i\vec{k}\cdot\vec{x}} . \tag{2.5}$$

The allowed values of $\vec{k}$ in the sums are determined by the imposition of periodic boundary conditions on $u_\alpha(\vec{x})$ and $S_i(\vec{x})$ where the variable $\vec{x}$ is measured in a coordinate system which deforms with the solid. The wave vectors are confined to a spherical Brillouin zone of unit radius which provides the short-distance cutoff. It should be pointed out that Eq. (2.3) does not represent the most general deformation of a solid with free surfaces since the $e_{\alpha\beta}$ necessarily leave the surfaces planar. This however is not crucial since Wegner[4] has shown that the inclusion of surface-warping deformations will not affect the critical behavior.

After evaluating the Gaussian integrals of Eq. (2.2) we obtain an effective magnetic Hamiltonian given by

$$\mathcal{H}_{\text{eff}} = \int d^d x \left[\tfrac{1}{2} r \sum_{i=1}^{n} S_i^2(\vec{x}) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{\alpha=1}^{d} \left(\frac{\partial S_i(\vec{x})}{\partial x_\alpha} \right)^2 + u \left(\sum_{i=1}^{n} S_i^2(\vec{x}) \right)^2 + v L^{-d} \left(\int d^d y \sum_{i=1}^{n} S_i^2(\vec{y}) \right) \sum_{j=1}^{n} S_j^2(\vec{x}) - h S_1(\vec{x}) \right] , \tag{2.6}$$

where

$$u = u_0 - \frac{g^2}{2K} \left(1 + \frac{2\mu(d-1)}{dK} \right)^{-1} ,$$

$$v = \frac{g^2}{2K} \left[\left(1 + \frac{2\mu(d-1)}{dK} \right)^{-1} - 1 \right] . \tag{2.7}$$

The fourth term in Eq. (2.6) is an interaction of a new type between magnetic degrees of freedom which is mediated by the lattice. It is of a long-range nature and constitutes an energy-density–energy-density coupling. The effect finite compressibility has on the other fields given in Eq. (2.7) is merely to shift u_0 from its rigid value. It should also be noted that the shear modulus μ cannot be set to zero in Eq. (2.6) (for instance, in the hope to produce Baker-Essam[16] model). This is because some of the integrals performed in Eq. (2.2) resulted in spin-independent terms (which have been suppressed in $\mathcal{H}_{\text{eff}}$ since they are analytic in T) that diverge when μ vanishes.

We are now in a position to study the critical behavior of the magnetic system described by Eq. (2.6). The first step in this analysis will be to review the predictions of Landau theory.[17] For a majority of phase-transition problems the conclusions of this theory are qualitatively correct. There are however exceptions[13,14] and, as will become evident in Sec. IV, the compressible Ising magnet is a case in point.

III. LANDAU THEORY

The free-energy density of the system described by Eq. (2.6) is given by

$$\mathcal{F}(r,u,v,h) = -L^{-d} \ln\left(\mathrm{Tr}_{\vec{S}(\vec{x})} e^{-\mathcal{H}_{\text{eff}}[\vec{S}(\vec{x})]} \right) , \tag{3.1}$$

where the trace is carried over all magnetic configurations. In the Landau theory one assumes that only one configuration (that one which minimizes the free energy) contributes appreciably to Eq. (3.1). In this vein we replace $S_i(\vec{x})$ by $M\delta_{i,1}$ where M is a constant and we have chosen the "1" direction to be the direction of ordering. Then we obtain

$$\mathcal{F}(r,u,v,h) = \tfrac{1}{2} r M^2 + (u+v) M^4 - hM . \tag{3.2}$$

The equilibrium value of M is given by

$$\frac{\partial \mathcal{F}}{\partial M} = 0 \tag{3.3}$$

and we obtain for the equation of state

$$M[r + 4(u+v)M^2] = h . \tag{3.4}$$

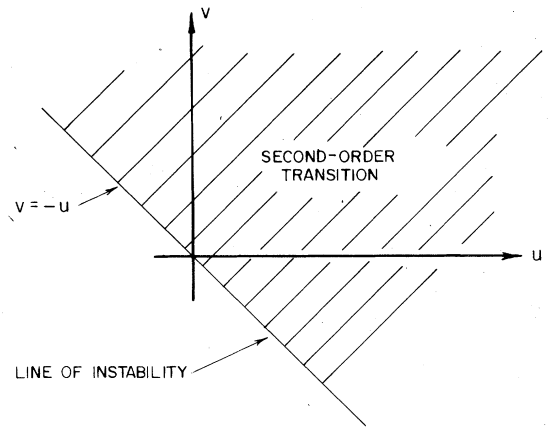

FIG. 1. Schematic view of Landau theory. The line $v=-u$ is a line of instability for Eq. (3.2).

This has the usual solutions: In the absence of external fields for $r>0$ we obtain $M_0=0$ (disordered phase) while for $r<0$ we obtain $M_0=\pm[|r|/4(u+v)]^{1/2}$ (the ordered phase). These results are identical to the rigid-magnet results[18] where the effective four-point coupling is $(u+v)$. As in the rigid-magnet analysis we assume this to be positive otherwise a stable minimum to Eq. (3.2) would not exist and one would have to add to Eq. (3.2) higher powers of M such as $u_6M^6(u_6>0)$ to restore stability. The predictions of Landau theory for the compressible system are shown in Fig. 1. For values of u, v in the shaded region we obtain a second-order transition with classical exponents while points outside this shaded region require u_6M^6 and the result can be a first-order transition for some $r=r_c>0$.

The next step in the study of this system is a renormalization-group[11,19] analysis where one systematically includes the effects of fluctuations in the calculation of thermodynamic properties.

IV. RENORMALIZATION GROUP

A renormalization-group (RG) analysis in $4-\epsilon$ dimensions of the system described by Eq. (2.6) begins with the determination of a set of recursion relations for the thermodynamic fields r, u, v, and h. This is most easily carried out by first rewriting Eq. (2.6) in terms of the set $S_{i,\vec{k}}$ given in Eq. (2.5). We then obtain for $\mathcal{H}_{\rm eff}$

$$\mathcal{H}_{\rm eff}=\frac{1}{2}L^{-d}\sum_{i,\vec{k}}(r+k^2)S_{i,\vec{k}}S_{i,-\vec{k}}+uL^{-3d}\sum_{\vec{k}_1\vec{k}_2\vec{k}_3,i,j}S_{i,\vec{k}_1}S_{i,\vec{k}_2}S_{j,\vec{k}_3}S_{j,-(\vec{k}_1+\vec{k}_2+\vec{k}_3)}+vL^{-3d}\left[\sum_{i,\vec{k}}S_{i,\vec{k}}S_{i,-\vec{k}}\right]^2-hS_{1,\vec{0}}\ , \tag{4.1}$$

where the $S_{i,\vec{k}}$ constitute a set of nN collective coordinates, N being the number of spins in volume L^d. We employ the momentum shell technique[11,19] to determine how the fields renormalize when a fraction of the degrees of freedom, e.g., all $S_{i,\vec{k}}$ with $b^{-1}<|\vec{k}|<1$; $b>1$, are integrated out of the partition function. The resulting set of discrete recursion relations[3] can be put into differential form by setting $b=e^{\delta l}$ and taking the $\delta l\to 0$ limit. One then obtains

$$\frac{dr(l)}{dl}=2r(l)+\frac{Au(l)}{1+r(l)}+\frac{Bv(l)}{1+r(l)}\ , \tag{4.2a}$$

$$\frac{du(l)}{dl}=\epsilon u(l)-\frac{Cu^2(l)}{[1+r(l)]^2}\ , \tag{4.2b}$$

$$\frac{dv(l)}{dl}=\epsilon v(l)-\frac{2Au(l)v(l)}{[1+r(l)]^2}-\frac{Bv^2(l)}{[1+r(l)]^2}\ , \tag{4.2c}$$

$$\frac{dh(l)}{dl}=(3-\tfrac{1}{2}\epsilon)h(l)\ , \tag{4.2d}$$

where $A=(n+2)/2\pi^2$, $B=n/2\pi^2$, and $C=(n+8)/2\pi^2$. The fixed points of these equations and properties of solutions around the fixed points[3] are given in Table I. The first four columns identify the fixed point while the next four columns give the various critical exponents at the respective fixed points. For example, near fixed point $F4$, the field v scales as $b^{(4-n)\epsilon/(n+8)}$ where b is the spatial rescaling factor. The specific-heat exponent α which is given in the last column of Table I is simply related to λ_v via $\alpha=\lambda_v\nu$. This can be shown by counting dimensions in a trivial rescaling of the v term in Eq. (4.1) and using the hyperscaling relation $d\nu=2-\alpha$. This connection between λ_v, ν, and α is very useful because our knowledge of α and ν allows us to make reasonable predictions for the exponent λ_v in three dimensions.

In general, critical behavior is governed by the fixed point of the RG equations which is only unstable with respect to the temperaturelike field r and the magnetic field h (since these are the only fields which need to be controlled to observe critical behavior). A naive look at Table I would then suggest that for $n<4$ only fixed point $F3$ satisfies the above criterion: v and u being irrelevant fields at this fixed point. One must, however, be very careful with such extrapolations since the exponent α actually changes sign between $d=4$ and 3 for $n=3$ and probably also for $n=2$,. Thus the fixed point which controls critical behavior depends on n: For $n=1$ fixed point $F3$ is most stable (since $\alpha>0$ in $d=3$) while

TABLE I. The four fixed points of the set (4.2) are indicated along with the critical exponents associated with the linear scaling fields at the respective fixed points. The exponents are only correct to $O(\epsilon)$ except for those associated with the Gaussian fixed point $F1$ and λ_u of $F2$: These are exact.

Fixed point	r^*	u^*	v^*	$\lambda_r(\equiv\nu^{-1})$	λ_u	λ_v	$\alpha(=\nu\lambda_v)$
$F1$	0	0	0	2	ϵ	ϵ	$\frac{1}{2}\epsilon$
$F2$	$-\frac{1}{2}\epsilon$	0	$\frac{2\pi^2\epsilon}{n}$	$2-\epsilon$	ϵ	$-\epsilon$	$-\frac{1}{2}\epsilon$
$F3$	$-\frac{3\epsilon}{(n+8)}$	$\frac{2\pi^2\epsilon}{(n+8)}$	$\frac{2\pi^2(4-n)\epsilon}{n(n+8)}$	$2-\frac{6\epsilon}{(n+8)}$	$-\epsilon$	$-\frac{(4-n)\epsilon}{(n+8)}$	$-\frac{(4-n)\epsilon}{2(n+8)}$
$F4$	$-\frac{1}{2}\frac{(n+2)\epsilon}{(n+8)}$	$\frac{2\pi^2\epsilon}{(n+8)}$	0	$2-\frac{(n+2)\epsilon}{(n+8)}$	$-\epsilon$	$\frac{(4-n)\epsilon}{(n+8)}$	$\frac{(4-n)\epsilon}{2(n+8)}$

for $n \geq 2$ fixed point $F4$ becomes the stable fixed point.

With the important fixed points for critical behavior in the two cases having been determined, the next step in this analysis is to analyze their domains of attraction. Before doing this, however, it is important first to note from Eq. (2.7) that the bare value of v can only be negative (for K, $\mu > 0$). Furthermore, as can be seen from Eq. (4.2c), v can never become positive. This is because as v goes to zero so does dv/dl. Thus, starting from negative values of v, continued RG iterations could at most make v go to zero (if $\alpha < 0$) and the entire region $v > 0$ is physically inaccessible. With this in mind the RG predictions for this system become clear. In Ising systems where $\alpha > 0$ we have a situation where, although a stable fixed point exists (fixed point $F3$), the system can never be found within its domain of attraction. Instead, starting from a bare value of v less than zero, v becomes more negative under repeated RG transformations. The domain of attraction of fixed point $F3$, however, only includes the region $v > 0$. This runaway situation, which is displayed schematically in Fig. 2, is often interpreted[20] as the signal of a first-order transition. Indeed, we will show that this is the case in Sec. VI. On the other hand in X-Y and Heisenberg systems where $\alpha < 0$ fixed point F4 becomes stable. This case is in-

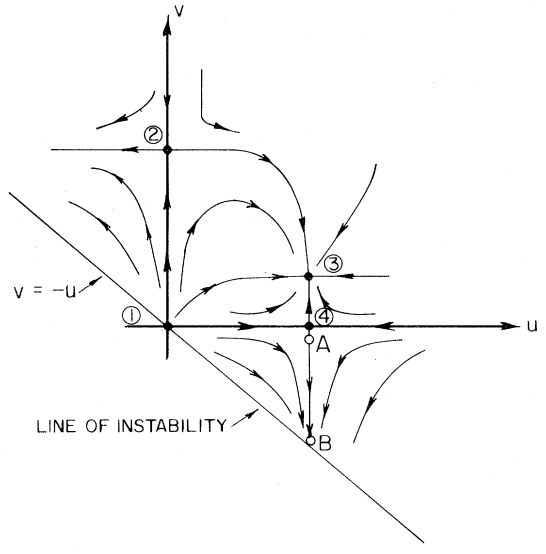

FIG. 2. Projection of RG flows and fixed points in plane of constant r for $n = 1$ system. The fixed points are numbered as in Table I.

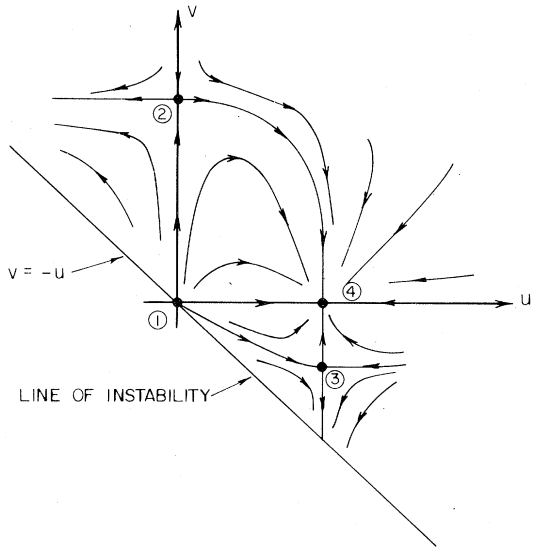

FIG. 3. Projection of RG flows and fixed points in plane of constant r for $n > 4$ systems (which in $d = 3$ are qualitatively similar to $n \geq 2$ systems). The fixed points are numbered as in Table I.

dicated schematically in Fig. 3 where it is clear that negative values of v are within the domain of attraction of $F3$ and a second-order transition may occur with scaling behavior modified by the presence of v (v being less irrelevant than u at this fixed point). As is evident from these remarks the Landau theory and RG predictions are qualitatively different in the Ising case and henceforth we will treat the two cases separately, first analyzing systems with $n \geq 2$. It should also be pointed out that since we will only be able to calculate thermodynamic functions with their appropriate singularities exponentiated to $O(\epsilon)$ it will sometimes become necessary in this analysis to replace an expression like $b^{[(4-n)/(n+8)]\epsilon}$ with $b^{2\alpha}$, tacitly making the previously explained assertion that for $n=2$ or 3 α is negative in three dimensions.

V. CRITICAL BEHAVIOR OF $n \geq 2$ SYSTEMS ($T > T_c$)

A. Solutions to recursion relations

The idea behind the recursion-relation approach to calculating thermodynamic functions is as follows. The system near its critical point is characterized by its "bare" thermodynamic fields r, u, v, and h. A direct attempt at calculating the system's partition function for these values of the fields is very difficult because many configurations included in the trace over magnetic degrees of freedom become equally important. In particular, magnetic configurations characterized by any length scale between a lattice spacing and the correlation length ξ cannot be ignored—a somewhat hopeless situation as ξ grows. The recursion relations given by Eq. (4.2) tell us how the thermodynamic fields are modified when a fraction $\delta l/d$ of the degrees of freedom are integrated out of the partition function and the system is properly rescaled. In this process the correlation length decreases to $\xi' = e^{-\delta l}\xi$. Continued iterations of the RG eventually map the system onto one whose correlation length is of $O(1)$. At this point, since u is still small [somewhere between 0 and $u^* = O(\epsilon)$] perturbation theory which in this context is sometimes called fluctuation corrected Landau theory can be used to calculate thermodynamic functions. One can then use the solutions of the set (4.2) in conjunction with mapping formulas to express the critical free energy in terms of the noncritical one and a trajectory integral.

The first step in implementing this scheme is to solve the set (4.2). Since u and v are almost marginal fields at fixed point $F4$ we only need the solutions to Eqs. (4.2b) and (4.2c) correct to $O(\epsilon)$. On the other hand, the solution to the r equation should clearly be accurate in the entire region $0 < r(l) < 1$. Generalizing the calculations of Rudnick and Nelson[9] we obtain[21]

$$r(l) = t(l) - \tfrac{1}{2}\lambda(l) + \tfrac{1}{2}t(l)\lambda(l)\ln[1+t(l)] + O(\epsilon^2) \ , \tag{5.1a}$$

$$u(l) = \frac{ue^{\epsilon l}}{Q(l)} + O(\epsilon^2) \ , \tag{5.1b}$$

$$v(l) = \frac{ve^{\epsilon l}}{Q(l)^{A/C} f(l)} + O(\epsilon^2) \ , \tag{5.1c}$$

$$h(l) = he^{(3-\epsilon/2)l} \ , \tag{5.1d}$$

where $t(l) \equiv te^{2l}/f(l)$, $\lambda(l) \equiv Au(l) + Bv(l)$. Similarly the functions $f(l)$ and $Q(l)$ are given by

$$f(l) = Q(l)^{A/C}\{1 + G[Q(l)^{1-2A/C} - 1]\} \ , \tag{5.1e}$$

$$Q(l) = 1 + (Cu/\epsilon)(e^{\epsilon l} - 1) \ . \tag{5.1f}$$

In the above expressions $t \equiv r + \frac{1}{2}\lambda$ and $G \equiv Bv/Cu(1-2A/C)$ where λ, r, u, v, and h are the $l=0$ values of these quantities.

B. Mapping formula

The free energy of the system can be expressed[18,22]

$$\mathcal{F}(r,u,v,h) = e^{-dl}\mathcal{F}(r(l),u(l),v(l),h(l)) + \int_0^l dl'\, e^{-dl'} G_0(l') \ , \tag{5.2}$$

where $\mathcal{F}(r(l),u(l),v(l),h(l))$ is the free energy of a system with thermodynamic fields $r(l),u(l),\ldots,$ etc., while the second term in Eq. (5.2), the trajectory integral, represents contributions to the free energy produced by the constants generated in repeated iterations of the RG transformation. Physically these terms represent contributions from spin fluctuations occurring over length scales smaller than $e^l a$ ($\equiv ba$) where a is the lattice constant. The kernel of the trajectory integral is given[22,23] by

$$G_0(l) = \tfrac{1}{2}nK_4\{\ln[1+r(l)] - \tfrac{1}{2}\} \ .$$

The first term in Eq. (5.2), apart from the e^{-dl} factor, is the free energy of a system whose correlation length $\xi[r(l),u(l),v(l)]$ is given by $e^{-l}\xi(r,u,v)$ where we have for convenience set the external field to zero. If we now choose $l = l^*$ such that $e^{-l^*}\xi(r,u,v)$ is of $O(1)$ then the calculation of $\mathcal{F}(r(l^*),u(l^*),v(l^*))$ is simplified. In particular, one obtains in Landau theory $\mathcal{F}_L(r(l^*),u(l^*),v(l^*)) = 0$ so we need only to calculate the first fluctuation correction to this result. Using the definition

$$\mathcal{F}(r(l^*),u(l^*),v(l^*)) = -L^{-d}\ln\left[\mathrm{Tr}_{\vec{s}\,\vec{k}}\, e^{-\mathcal{H}_{l^*}[\vec{s}\,\vec{k}]}\right] \ , \tag{5.3}$$

where $\mathcal{H}_{l^*}[\vec{s}_{\vec{k}}]$ is simply Eq. (4.1) with the bare fields r, u, and v replaced by their renormalized values $r(l^*)$, $u(l^*)$, and $v(l^*)$ we obtain

$$\mathcal{F}(r(l^*),u(l^*),v(l^*)) = \tfrac{1}{2}nK_4\int_0^1 k^3\,dk\,\ln[k^2+r(l^*)]+O(\epsilon)\ , \quad (5.4)$$

where the integral has been evaluated in four dimensions. Using the kernel $G_0(l)$ given earlier, the trajectory integral in Eq. (5.2) can be done and combined with the results of the integral in Eq. (5.4). One then obtains an expression for the free energy consisting of two parts: a singular part $\mathcal{F}_s$ and a regular part $\mathcal{F}_R$. Ignoring contributions to the regular part we obtain for $\mathcal{F}_s$

$$\mathcal{F}_s(r,u,v) = -\tfrac{1}{4}nK_4\int_0^{l^*} dl'\,e^{-dl'}t^2(l') + \tfrac{1}{8}nK_4e^{-dl^*}t^2(l^*)\ln[t(l^*)]\ . \quad (5.5)$$

It can be demonstrated[24] that this result is insensitive in leading order to our precise choice for l^*. Thus, results obtained from it will be correct to leading order in ϵ. Using Eq. (5.5) we can determine the singular parts of other thermodynamic functions such as the energy and specific heat which are given by[9]

$$C_s(r,u,v) = -\frac{\partial^2\mathcal{F}_s(r,u,v)}{\partial r^2}\ ; \qquad E_s(r,u,v) = -\frac{\partial\mathcal{F}_s(r,u,v)}{\partial r}\ . \quad (5.6)$$

We will use these relations later on for this purpose.

The magnetic susceptibility of the system can be calculated in the same way as the free energy. We begin with a scaling relation satisfied by the susceptibility[25]

$$\chi(r,u,v) = e^{2l}\chi(r(l),u(l),v(l))\ , \quad (5.7)$$

where we have taken $\eta=0$ in this calculation [since $\eta=O(\epsilon^2)$]. Again, choosing $l=l^*$, we can calculate $\chi(r(l),u(l),v(l))$ perturbatively making use of the relation $\chi=G(\vec{k}=0)$. The Feynman diagrams contributing in this calculation are shown in Fig. 4.

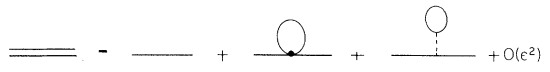

FIG. 4. Feynman diagrams contributing to the calculation of the susceptibility. The dot represents u while the broken line corresponds to v (the broken line carries no momentum).

Analytically we obtain to $O(\epsilon)$

$$\chi^{-1}(r(l^*),u(l^*),v(l^*)) = r(l^*)+\lambda(l^*)\int\frac{k^3\,dk}{k^2+r(l^*)}\ . \quad (5.8)$$

After doing the integral in Eq. (5.8) and combining the results with Eq. (5.7) we obtain for the susceptibility

$$\chi(r,u,v) = \frac{e^{2l^*}}{t(l^*)}[1-\tfrac{1}{2}\lambda(l^*)\ln t(l^*)]\ . \quad (5.9)$$

The results obtained thus far, specifically Eqs. (5.5), (5.6), (5.9) depend explicitly upon l^*. As mentioned earlier our choice of l^* is dictated by the requirement that $r(l^*)$ is of $O(1)$. Thus we choose $t(l^*)=1$. This becomes

$$te^{2l^*}/f(l^*)=1 \quad (5.10)$$

This equation can be solved iteratively for e^{l^*} resulting in

$$e^{l^*} = t^{-\nu}R^{(n+2)/2(n+8)}[1+G(R^{(4-n)/(n+8)}t^{-\alpha}-1)]^{1/2}\ , \quad (5.11)$$

where

$$\nu = \frac{1}{2}+\frac{1}{4}\frac{(n+2)}{(n+8)}\epsilon+O(\epsilon^2)\ ,$$

$$\alpha = \frac{1}{2}\frac{(4-n)}{(n+8)}\epsilon+O(\epsilon^2)\ ,$$

and

$$R \equiv \frac{Cu}{\epsilon}+\left[1-\frac{Cu}{\epsilon}\right]t^{\epsilon/2}\ .$$

Performing the integration in Eq. (5.5) and making use of these results we obtain for the free energy

$$\mathcal{F}_s(t,u,v) = -\tfrac{1}{16}n\frac{1}{(4-n)}\frac{t^{2-\alpha}}{u}\left[\frac{R^{(4-n)/(n+8)}}{1+G(R^{(4-n)/(n+8)}t^{-\alpha}-1)}\right]+\tfrac{1}{16}n\frac{1}{(4-n)}\frac{t^2}{u}\left[\frac{1}{1+G(R^{(4-n)/(n+8)}t^{-\alpha}-1)}\right]\ . \quad (5.12)$$

Using this result along with Eqs. (5.6), we obtain for the energy

$$E_s(t,u,v) = -2\mathcal{F}_s(t,u,v)/t \quad (5.13)$$

and for the specific heat

$$C_s(t,u,v) = -2\mathcal{F}_s(t,u,v)/t^2\ . \quad (5.14)$$

Similarly, the susceptibility given by Eq. (5.9) becomes

$$\chi(t,u,v) = t^{-\gamma}R^{(n+2)/(n+8)}[1+G(R^{(4-n)/(n+8)}t^{-\alpha}-1)]\ , \quad (5.15)$$

where

$$\gamma = 1+\tfrac{1}{2}\frac{(n+2)}{(n+8)}\epsilon+O(\epsilon^2)$$

is the susceptibility exponent at fixed point $F4$.

These results agree with those obtained by Rudnick and Nelson[9] for the rigid system when v is set to zero.

C. Scaling functions

It is interesting to express these thermodynamic functions in terms of nonlinear scaling fields[26] which satisfy the equations

$$g_t(l)=g_t e^{\lambda_t l}\ ,$$
$$g_u(l)=g_u e^{\lambda_u l}\ , \tag{5.16}$$
$$g_v(l)=g_v e^{\lambda_v l}\ ,$$

where g_t, g_u, and g_v are functions of the physical fields t, u, and v. The exponents λ_t, λ_u, and λ_v are the exponents associated with the t, u, and v linear scaling fields at fixed point $F4$ and they are listed in Table I. We obtain for these functions [which are defined only up to a multiplicative factor by Eqs. (5.16)]

$$g_t=\frac{t}{(Cu/\epsilon)^{A/C}(1-G)}\ ,$$
$$g_u=\frac{Cu/\epsilon-1}{Cu/\epsilon}\ , \tag{5.17}$$
$$g_v=\frac{v/\epsilon}{(Cu/\epsilon)^{2A/C}(1-G)}\ .$$

In terms of these nonlinear scaling fields the solutions to the recursion relations can be written

$$t(l)=\frac{g_t(l)}{[1-g_u(l)]^{A/C}\left[1+\frac{B}{(1-2A/C)}g_v(l)[1-g_u(l)]^{1-2A/C}\right]}\ ,$$
$$v(l)=\frac{\epsilon g_v(l)}{[1-g_u(l)]^{2A/C}\left[1+\frac{B}{(1-2A/C)}g_v(l)[1-g_u(l)]^{1-2A/C}\right]}\ , \tag{5.18}$$
$$u(l)=\frac{\epsilon/C}{[1-g_u(l)]}\ .$$

One can now substitute these results into expressions obtained for thermodynamic functions to obtain the functions in scaling form. Consider for instance the susceptibility given in Eq. (5.9). With $t(l^*)=1$ this becomes

$$\chi(r,u,v)=e^{2l^*}\ . \tag{5.19}$$

However, the condition $t(l^*)=1$ can be expressed in terms of the nonlinear scaling fields to obtain

$$g_t(l^*)=[1-g_u(l^*)]^{A/C}\times\left[1+\frac{Bg_v(l^*)}{(1-2A/C)}[1-g_u(l^*)]^{1-2A/C}\right]\ . \tag{5.20}$$

This equation for e^{l^*} can be solved iteratively and the results used in Eq. (5.19) to obtain for the susceptibility

$$\chi(g_t,g_u,g_v)=g_t^{-\gamma}\Phi(x,y)\ , \tag{5.21}$$

where the universal scaling function is given by

$$\Phi(x,y)=(1-x)^{(n+2)/(n+8)}\times\left[1+\frac{n(n+8)}{2\pi^2(4-n)}y(1-x)^{(4-n)/(4+8)}\right]\ . \tag{5.22}$$

In the above, $x\equiv g_u g_t^{-\phi_u}$, $y=g_v g_t^{-\phi_v}$ and $\phi_u(\equiv\nu\lambda_u)$, $\phi_v(\equiv\nu\lambda_v=\alpha)$ are the crossover exponents associated with the fields u and v at fixed point $F4$. As expected, when $y=0$, Eq. (5.22) reduces to the Rudnick-Nelson[9] result $\Phi(x,y=0)=(1-x)^{(n+2)/(n+8)}$. Similar expressions in scaling form can be obtained for the other thermodynamic functions.

The calculations done thus far are straightforward extensions of the Rudnick-Nelson[9] calculations on rigid magnetic systems—the only difference being the presence of an additional irrelevant field v. Ordered-phase calculations for this $n\geq 2$ case can also be done; however, the coexistence curve behavior becomes more difficult to analyze. This is because one has two distinct correlation lengths in the problem. One correlation length, r_L, is associated with the longitudinal (parallel to external field) magnetic modes while the other, r_T, is associated with transverse modes. When the recursion relations are integrated until r_L becomes of $O(1)$, transverse modes retain long-range correlations and remain critical. Thus singularities in thermodynamic functions reflecting transverse fluctuations remain unexponentiated. A complete analysis then involves integrating transverse-spin components out of the problem completely and will be deferred to a later paper. Instead we now pursue the Ising case where α is positive, v is relevant, and the runaway situation depicted in Fig. 2 must be dealt with.

VI. FIRST-ORDER TRANSITION IN ISING SYSTEMS

A. Disordered phase

The difficulty with the solutions to the recursion relations obtained in Sec. V in the case when $n=1$ stems from the following. Let the point A shown in Fig. 2 represent the thermodynamic state of the system. For convenience we have chosen point A so that $u=u^*$ although this is not necessary. The value of v at point A is very small and negative. This would represent a system with a stiff lattice and/or weak spin-lattice coupling. Under repeated RG transformations the system would eventually be mapped onto a system with thermodynamic fields given by point B. It was mentioned earlier that points in the u, v plane for which $u+v$ was negative were not permissible. This of course is true only because we explicitly ignore the higher-order interactions generated by the RG, confining the system to the r, u, v subspace. Crossing the line $v=-u$ then results in a vanishing of the effective four-point coupling and a loss of stability in the free energy. A manifestation of this difficulty can be seen in the solution to the recursion relation for v given by Eq. (5.1c). With $u=u^*$, $Q(l)$ becomes $e^{\epsilon l}$ and $v(l)$ becomes

$$v(l)=\frac{ve^{\lambda_v l}}{1+(v/3u^*)(e^{\lambda_v l}-1)} \quad . \tag{6.1}$$

Since $\lambda_v(=\frac{1}{3}\epsilon)$ is positive and v negative, the denominator in this expression can vanish for large enough l. Of course these solutions were obtained self-consistently—when $v(l)$ grows larger than $O(\epsilon)$ the set (5.1) breaks down (this difficulty does not arise in the $n \geq 2$ case in $d=3$ since λ_v is negative). It can be seen from Eq. (6.1) that for each bare value v there corresponds a maximum value of $l=l_{\max}$ above which the solutions to the recursion relations break down. For example near point A (where $|v| \ll u^*$) we obtain

$$l_{\max}=\frac{1}{\lambda_v}\ln(3u^*/|v|) \quad . \tag{6.2}$$

Fortunately we will never have need of solutions to recursion relations for values of $l > l_{\max}$ as will be demonstrated later on.

The motivation behind integrating the recursion relations lies in the ability to then map the system onto another system with a smaller correlation length. The question then arises: What is the correlation length associated with system B? The answer clearly depends not only on v and u but also upon the value of r. Figures 2 and 3 suppress the r coordinate only displaying the projections of the RG flows and fixed points on a plane of constant r. Nevertheless, it is clear that if the bare value of r is sufficiently large, then because of the rapid growth rate of r, the system will be mapped onto one with ξ of $O(1)$ before ever reaching point B. In this case problems associated with crossing the line of instability disappear and solutions obtained in Sec. V (with n set equal to 1) can be carried over unaltered. On the other hand, if r is sufficiently small then it is possible that the system would reach point B before $r(l)$ is of $O(1)$. In this case we have a problem since: (a) we cannot cross the line of instability and (b) it is not clear that thermodynamic functions can easily be calculated at point B where the correlation length may still be large. We will return to this case later on but first we will find the value of bare t, say $t_{\min}^{+}$, above which the results from Sec. V remain valid. This can be determined in the following way. Let us define l_1 to be that value of l necessary to make ξ of $O(1)$, i.e.,

$$t(l_1)\equiv 1 \quad . \tag{6.3}$$

Similarly we define l_2 as the value of l which maps the system onto the line of instability, i.e.,

$$v(l_2)\equiv -u(l_2) \quad . \tag{6.4}$$

For any value of the bare fields r,u,v (or equivalently t,u,v) we wish to map the system onto one with $t(l)\sim O(1)$. If however we reach the line of instability before $t(l)$ is $O(1)$ we must stop the integration. The value of l, say l^*, where we stop our integration is then given by

$$l^*=\min(l_1,l_2) \quad . \tag{6.5}$$

Using Eqs. (6.3) and (6.4) we find (again using simplifications appropriate to point A)

$$l_1=v\ln(1/t), \quad l_2=\frac{1}{\lambda_v}\ln(3u^*/4|v|) \quad . \tag{6.6}$$

It can be seen that for fixed v, l_2 is determined and if t gets too small l_1 will eventually grow equal to or larger than l_2. Then t will equal $t_{\min}^{+}$ when l_1 and l_2 are equal. Combining Eqs. (6.6) we obtain

$$t_{\min}^{+}=(4|v|/3u^*)^{1/\alpha} \quad , \tag{6.7}$$

where we have used $\alpha=\lambda_v\nu$. Thus for values of $t > t_{\min}^{+}$ the results of Sec. V remain valid in the Ising case. We now turn our attention to the ordered phase where similar restrictions on solutions to the recursion relations will develop.

B. Ordered phase

A useful perturbation scheme can be set up in an ordered-phase calculation by first shifting the spin field in Eq. (4.1) (with $n=1$) by the exact magnetization[27]

$$S(\vec{x})\equiv M+\sigma(\vec{x}) \quad , \tag{6.8}$$

where the average value of σ vanishes

$$\langle\sigma\rangle = 0 \ . \tag{6.9}$$

After performing this shift the effective Hamiltonian becomes

$$\mathcal{H}_{\text{eff}} = [\tfrac{1}{2}rM^2 + (u+v)M^4 - hM]L^d + \tfrac{1}{2}L^{-d}\sum_{\vec{k}}(\tilde{r}+k^2)\sigma_{\vec{k}}\sigma_{-\vec{k}} + y_2 L^{-2d}\sigma_{\vec{0}}^2 + wL^{-2d}\sum_{\vec{k}_1\vec{k}_2\vec{k}_3}\sigma_{\vec{k}_1}\sigma_{\vec{k}_2}\sigma_{\vec{k}_3}\delta_{\vec{k}_1+\vec{k}_2+\vec{k}_3,\vec{0}}$$
$$+ yL^{-d}\sigma_{\vec{0}}\sum_{\vec{k}}\sigma_k\sigma_{-k} + uL^{-3d}\sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4}\sigma_{\vec{k}_1}\sigma_{\vec{k}_2}\sigma_{\vec{k}_3}\sigma_{\vec{k}_4}\delta_{\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4,\vec{0}} + vL^{-3d}\left(\sum_{\vec{k}}\sigma_{\vec{k}}\sigma_{-\vec{k}}\right)^2 - \tilde{h}\sigma_{\vec{0}} \ , \tag{6.10}$$

where $\tilde{r} \equiv r + 12uM^2 + 4vM^2$, $y_2 \equiv 4vM^2$, $w \equiv 4uM$, $y \equiv 4vM$, and $\tilde{h} \equiv h - rM - 4(u+v)M^3$. The sums over $\vec{k}$ in Eq. (6.10) are restricted to the interior of a spherical Brillouin zone of unit radius. A system of differential recursion relations for the fields in Eq. (6.10) can be set up in the usual way.[11] The diagrams contributing to the renormalizations of the various fields are shown in Fig. 5 while the equations are given by

$$\frac{d\tilde{r}(l)}{dl} = 2\tilde{r}(l) + \frac{\lambda(l)}{1+\tilde{r}(l)} - \frac{18K_4 w(l)^2}{[1+\tilde{r}(l)]^2} \ , \tag{6.11a}$$

$$\frac{dy_2(l)}{dl} = 2y_2(l) - \frac{K_4 y(l)^2}{[1+\tilde{r}(l)]^2} - \frac{6K_4 w(l)y(l)}{[1+\tilde{r}(l)]^2} \ , \tag{6.11b}$$

$$\frac{dw(l)}{dl} = (1+\tfrac{1}{2}\epsilon)w(l) - \frac{36K_4 u(l)w(l)}{[1+\tilde{r}(l)]^2} + \frac{36K_4 w(l)^3}{[1+\tilde{r}(l)]^3} \ , \tag{6.11c}$$

$$\frac{dy(l)}{dl} = (1+\tfrac{1}{2}\epsilon)y(l) - \frac{4K_4 y(l)v(l)}{[1+\tilde{r}(l)]^2} - \frac{12K_4 u(l)y(l)}{[1+\tilde{r}(l)]^2} - \frac{12K_4 w(l)v(l)}{[1+\tilde{r}(l)]^2} + \frac{36K_4 y(l)w(l)^2}{[1+\tilde{r}(l)]^3} \ , \tag{6.11d}$$

$$\frac{du(l)}{dl} = \epsilon u(l) - \frac{36K_4 u(l)^2}{[1+\tilde{r}(l)]^2} - \frac{162K_4 w(l)^4}{[1+\tilde{r}(l)]^4} \ , \tag{6.11e}$$

$$\frac{dv(l)}{dl} = \epsilon v(l) - \frac{4K_4 v(l)^2}{[1+\tilde{r}(l)]^2} - \frac{24K_4 u(l)v(l)}{[1+\tilde{r}(l)]^2} + \frac{72K_4 v(l)w(l)^2}{[1+\tilde{r}(l)]^3} \ , \tag{6.11f}$$

$$\frac{d\tilde{h}(l)}{dl} = (3-\tfrac{1}{2}\epsilon)\tilde{h}(l) - \frac{K_4 y(l)}{[1+\tilde{r}(l)]} - \frac{3K_4 w(l)}{[1+\tilde{r}(l)]} \ . \tag{6.11g}$$

In these expansions we have taken the fields $\tilde{r}(l)$ and $y_2(l)$ of $O(1)$, $w(l)$ and $y(l)$ of $O(\epsilon^{1/2})$, $u(l)$ and $v(l)$ of $O(\epsilon)$, and $\tilde{h}(l)$ of $O(\epsilon^{-1/2})$. These order-of-magnitude estimates are obtained by analyzing an expression for the free-energy-like Eq. (3.2) where r is of $O(1)$ and Landau theory is appropriate. We now search for leading-order solutions to the set (6.11). Using iterative methods with the disordered-phase solutions as starting points[9] we obtain

$$\tilde{r}(l) = T(l) - \tfrac{1}{2}\lambda(l) + \tfrac{1}{2}\lambda(l)T(l)\ln[1+T(l)] + 144K_4 u(l)^2 M^2(l) \times \left[\ln[1+T(l)] + \frac{T(l)}{1+T(l)}\right] + O(\epsilon^2) \ , \tag{6.12a}$$

$$y_2(l) = 4v(l)M^2(l) + [8K_4 v(l)^2 M^2(l) + 48K_4 v(l)u(l)M^2(l)] \times \left[\frac{T(l)}{1+T(l)} + \ln[1+T(l)]\right] + O(\epsilon^2) \ , \tag{6.12b}$$

$$w(l) = 4u(l)M(l) + O(\epsilon^{3/2}) \ , \tag{6.12c}$$

$$y(l) = 4v(l)M(l) + O(\epsilon^{3/2}) \ , \tag{6.12d}$$

$$u(l) = \frac{ue^{\epsilon l}}{Q(l)} + O(\epsilon^2) \ , \tag{6.12e}$$

$$v(l) = \frac{ve^{\epsilon l}}{Q^{2/3}(l)f(l)} + O(\epsilon^2) \ , \tag{6.12f}$$

$$\tilde{h}(l) = h(l) - M(l)t(l) - 4[u(l)+v(l)]M^3(l) + \tfrac{1}{2}\lambda(l)M(l) - \tfrac{1}{2}\lambda(l)M(l)T(l)\ln[1+T(l)] + O(\epsilon^{3/2}) \ , \tag{6.12g}$$

where

$$M(l) \equiv Me^{(1-\epsilon/2)l} \ ,$$

$$T(l) \equiv t(l) + 12u(l)M^2(l) + 4v(l)M^2(l) \ ,$$

and the functions $f(l)$ and $\lambda(l)$ were given earlier. Although these solutions are correct to $O(\epsilon)$ we will only need the leading-order pieces of them since perturbation theory is only applied in the noncritical re-

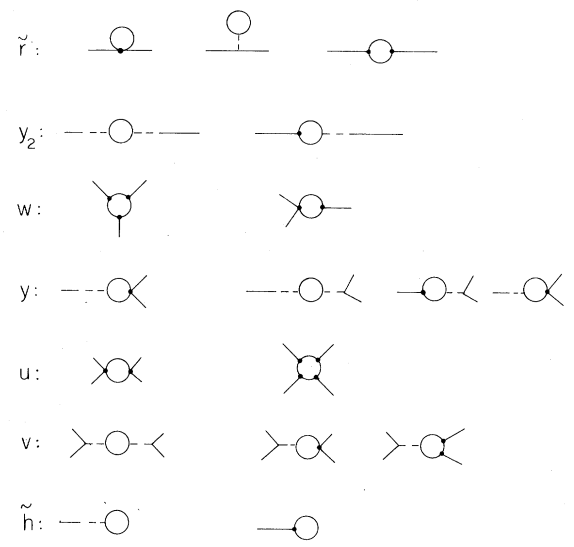

FIG. 5. Feynman diagrams contributing to the calculation of the ordered-phase recursion relations.

gion. Using these solutions we now map out the critical region to find various thermodynamic functions. The free energy is given by

$$\mathfrak{F}(r,u,v,h) = e^{-dl}\mathfrak{F}(r(l),u(l),v(l),h(l)) + \int_0^l e^{-dl'} G_0(l')\,dl' + \tfrac{1}{2}rM^2 + (u+v)M^4 - hM \quad , \tag{6.13}$$

where the kernel is

$$G_0(l) = \tfrac{1}{2}K_4\{\ln[1+\tilde{r}(l)] - \tfrac{1}{2}\} \tag{6.14}$$

and

$$\mathfrak{F}(r(l),u(l),v(l),h(l)) = -L^{-d}\ln(\mathrm{Tr}_{\sigma_{\vec{k}}} e^{-\mathcal{H}_l[\sigma_{\vec{k}}]}) \tag{6.15}$$

In the above $\mathcal{H}_l[\sigma_{\vec{k}}]$ is the spin-dependent part of Eq. (6.10) with the fields replaced by their renormalized values. Once again, as in the $n \geq 2$ calculation, we are able to calculate the trace in Eq. (6.15) because l is chosen equal to l^*; a value of l which renders $\tilde{r}(l)$ of $O(1)$. We thus obtain for the noncritical free energy

$$\mathfrak{F}(r(l^*),u(l^*),v(l^*),h(l^*)) = \tfrac{1}{4}K_4\{\tfrac{1}{2}[1-T^2(l^*)]\ln[1+T(l^*)] + \tfrac{1}{2}T(l^*) + \tfrac{1}{2}T(l^*)\ln[T(l^*)] - \tfrac{1}{4}\} \quad . \tag{6.16}$$

After combining this with the results of the trajectory integral in Eq. (6.13) we obtain for the singular part of the free energy

$$\mathfrak{F}(r,u,v,h) = \mathfrak{F}_D + e^{-dl^*}[\tfrac{1}{2}t(l^*)M^2(l^*) + [u(l^*)+v(l^*)]M^4(l^*) - h(l^*)M(l^*)] + \{e^{-dl^*}\tfrac{1}{8}K_4T^2(l^*)\ln[T(l^*)]\} \quad , \tag{6.17}$$

where

$$\mathfrak{F}_D = -\frac{t^2Q^{1/3}(l^*)}{48u}\left(\frac{Q^{1/3}(l^*)-1}{f(l^*)}\right)$$

is the result appropriate for the disordered phase which was given earlier in Eq. (5.5) (with $n=1$).

The equation of state can also be obtained by imposing the condition that the average value of the shifted spin field in the lth Hamiltonian vanish. Choosing $l = l^*$ so that the perturbation expansion makes sense we set

$$\tilde{h}(l^*) - [3K_4w(l^*) + K_4y(l^*)]\int_0^1 \frac{k^3\,dk}{k^2+\tilde{r}(l^*)} = 0 \quad . \tag{6.18}$$

The Feynman diagrams contributing in this expression are shown in Fig. 6(a). After evaluating the integral in Eq. (6.18) this becomes

$$h(l^*)/M(l^*) = t(l^*) + 4[u(l^*)+v(l^*)]M^2(l^*) + \{\tfrac{1}{2}\lambda(l^*)T(l^*)\ln[T(l^*)]\} \quad . \tag{6.19}$$

The susceptibility which obeys the scaling relation

$$\chi(r,u,v,h) = e^{2l^*}\chi(r(l^*),u(l^*),v(l^*),h(l^*)) \tag{6.20}$$

given in Sec. V can also be calculated perturbatively. Expanding in $u(l^*)$ and $v(l^*)$ we obtain for the non-

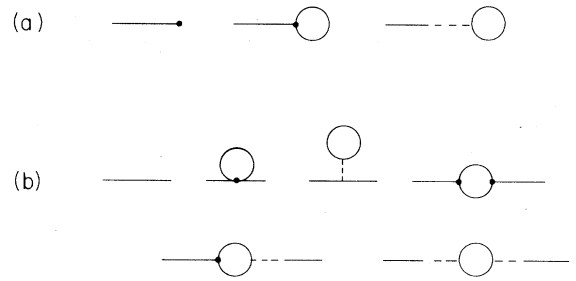

FIG. 6. Feynman diagrams contributing to the calculations of (a) the equation of state and (b) the susceptibility.

critical susceptibility

$$\chi^{-1}(r(l^*),u(l^*),v(l^*),h(l^*)) = [T(l^*)+2y_2(l^*)]\left[1+\left(\frac{\lambda(l^*)}{T(l^*)+2y_2(l^*)}\int_0^1 \frac{k^3\,dk}{T(l^*)+k^2} - \frac{288u^2(l^*)+32v^2(l^*)+192u(l^*)v(l^*)}{T(l^*)+2y_2(l^*)} \times K_4 M^2(l^*)\int_0^1 \frac{k^3\,dk}{[T(l^*)+k^2]^2}\right)\right] . \quad (6.21)$$

The graphs contributing to this expression are shown in Fig. 6(b). After performing the integrals in Eq. (6.21) and making use of the scaling relation (6.20) we obtain

$$\chi(r,u,v,h) = \frac{e^{2l^*}}{T(l^*)+2y_2(l^*)}\left\{1-\left[\frac{144u^2(l^*)M^2(l^*)}{T(l^*)+2y_2(l^*)} + \tfrac{1}{2}\lambda(l^*)\frac{\{t(l^*)+[36u(l^*)+12v(l^*)]M^2(l^*)\}}{T(l^*)+2y_2(l^*)}\ln[T(l^*)] + y_2(l^*)\frac{[\lambda(l^*)+12K_4u(l^*)]}{T(l^*)+2y_2(l^*)}\left[\frac{1}{1+T(l^*)}-\ln[1+T(l^*)]\right]\right]\right\} . \quad (6.22)$$

The basic results (6.17), (6.19), and (6.22) may be differentiated with respect to l^* to demonstrate their independence of the precise choice for l^* to leading order in ϵ. It should also be clear that the pieces of these results which are enclosed by curly brackets in Eqs. (6.17) and (6.19) and large square brackets in Eq. (6.22) are not leading order—instead they are $O(\epsilon)$ lower than their preceding terms in the noncritical region.

We are now in a position to make an explicit choice for l^*. The condition that $\tilde{r}(l^*)$ is of $O(1)$ can be written

$$T(l^*) = t(l^*)+12u(l^*)M^2(l^*)+4v(l^*)M^2(l^*) = 1 . \quad (6.23)$$

This can also be expressed in terms of the nonlinear scaling fields given in Sec. V. This results in

$$\frac{g_t e^{\lambda_t l^*}+4\epsilon M^2 g_v e^{\lambda_v l^*} e^{(2-\epsilon)l^*}}{(1-g_u e^{\lambda_u l^*})^{1/3}[1+12K_4 g_v e^{\lambda_v l^*}(1-g_u e^{\lambda_u l^*})^{1/3}]} + \frac{\epsilon M^2 e^{(2-\epsilon)l^*}}{3K_4(1-g_u e^{\lambda_u l^*})} = 1 . \quad (6.24)$$

The solution to this equation defines an l^* for each set (g_t, M^2, g_u, g_v) and it can be put into the form

$$e^{l^*} = L(g_t, M^2, g_u, g_v) , \quad (6.25)$$

where $L(x,y,z,v)$ satisfies the homogeneity relation

$$L(x,y,z,v) = bL(b^{\lambda_t}x, b^{2-\epsilon}y, b^{-\epsilon}z, b^{\epsilon/3}v) = y^{-1/(2-\epsilon)}\hat{L}(y^{-\lambda_t/(2-\epsilon)}x, y^{\epsilon/(2-\epsilon)}z, y^{-(\epsilon/3)/(2-\epsilon)}v) \quad (6.26)$$

and $\hat{L}(x,z,v) \equiv L(x,1,z,v)$. We get

$$e^{l^*} = M^{-(\delta+1)/d}\hat{L}(g_t/M^{1/\beta}, g_u M^{\epsilon}, g_v M^{-\epsilon/3}) , \quad (6.27)$$

where $\beta = \frac{1}{2}-\frac{1}{6}\epsilon+O(\epsilon^2)$ and $\delta = 3+\epsilon+O(\epsilon^2)$ are the Ising (fixed point $F4$) exponents given in Table I. This function $\hat{L}(x,z,v)$ can be determined (numerically if necessary) and the results used in Eqs. (6.17), (6.19), and (6.22) to obtain the thermodynamic functions in scaling form.

This situation becomes somewhat simpler to analyze when the external field is set to zero. In this case the equation of state becomes (to leading order in ϵ)

$$t(l^*)+4[u(l^*)+v(l^*)]M^2(l^*) = 0 . \quad (6.28)$$

To discuss the coexistence curve behavior it is better to choose a different l^* than the one given by Eq. (6.23). The only requirement placed on l^* is that it makes $T(l^*)$ of $O(1)$. Thus we may choose the condition

$$t(l^*) = -\tfrac{1}{2} . \quad (6.29)$$

With this l^* we are ensured that $T(l^*) \sim O(1)$ as long as $|v(l^*)|$ remains less than $u(l^*)$. Furthermore this choice is identical to the one used by Rudnick and Nelson[9] for rigid systems thus allowing for easy comparisons in the $v \to 0$ limit. Using the expression given in Eq. (6.29) along with Eq. (6.28) we

obtain for the magnetization on the coexistence curve

$$M_0 = \pm \frac{|t|^{\beta}}{2\sqrt{u}} \frac{R^{1/3}}{[1+|v|/3u-(4|v|/3u)R^{1/3}|t|^{-\alpha}]^{1/2}} , \tag{6.30}$$

where $R \equiv u/u^* + (1-u/u^*)|t|^{\epsilon/2}$ and $\beta = \frac{1}{2} - \frac{1}{6}\epsilon + O(\epsilon^2)$. Similarly, using Eq. (6.29), the susceptibility given in Eq. (6.22) becomes to leading order in ϵ

$$\chi(r,u,v) = \tfrac{1}{2}|t|^{-\gamma} R^{1/3}\left(1 + \frac{|v|}{3u} - \frac{|v|}{3u} R^{1/3}|t|^{-\alpha}\right) , \tag{6.31}$$

where $\gamma = 1 + \frac{1}{6}\epsilon + O(\epsilon^2)$ and the ordered-phase free energy is now

$$\mathfrak{F}_s(r,u,v) = -\frac{|t|^{2-\alpha}}{48u}\left(\frac{R^{1/3}-|t|^{\alpha}}{1+|v|/3u-(|v|/3u)R^{1/3}|t|^{-\alpha}}\right) - \frac{|t|^{2-\alpha}}{16u}\frac{R^{1/3}}{[1+|v|/3u-(|v|/3u)R^{1/3}|t|^{-\alpha}]}\left(\frac{1}{1+|v|/3u^*-(4|v|/3u)R^{1/3}|t|^{-\alpha}}\right) , \tag{6.32}$$

with $\alpha = \frac{1}{6}\epsilon + O(\epsilon^2)$. As mentioned earlier these results are not valid for arbitrarily small values of $|t|$. This is because if $|t|$ becomes too small (for given u and v) the condition $|v(l^*)| < u(l^*)$ will be violated. Using arguments similar to those given in the disordered phase we obtain the condition that $|t|$ must be greater than $t^-_{\min}$ where

$$t^-_{\min} = \tfrac{1}{2}(4|v|/3u^*)^{1/\alpha} = \tfrac{1}{2}t^+_{\min} . \tag{6.33}$$

In arriving at this conclusion we have again assumed for simplicity that $u = u^*$ and $|v| \ll u^*$. These conditions on the values of t for which the ordered- and disordered-phase solutions remain valid are displayed schematically in Fig. 7. The shaded region in the figure indicates those values of t, v (we have taken $u = u^*$ for simplicity) where the system is mapped to the line of instability shown in Fig. 2 before reaching the prescribed value of $t(l^*)$ ($+1$ for disordered phase and $-\frac{1}{2}$ for ordered phase). The solutions obtained thus far are of course not valid in this shaded region. One can see from these coexistence curve results that if $|v| \ll u\,(=u^*)$ then as t is raised towards $-t^-_{\min}$ from below (but with $t \ll t^-_{\min}$) the system behaves as though it were approaching a critical point with the rigid-system exponents. For instance, it is seen from Eq. (6.30) that M_0 decreases as $|t|^{\beta}$ where β is the critical exponent describing the rigid system's coexistence curve behavior. Only when $|t|$ becomes very small (of order $t^-_{\min}$) do the effects of finite compressibility become apparent.

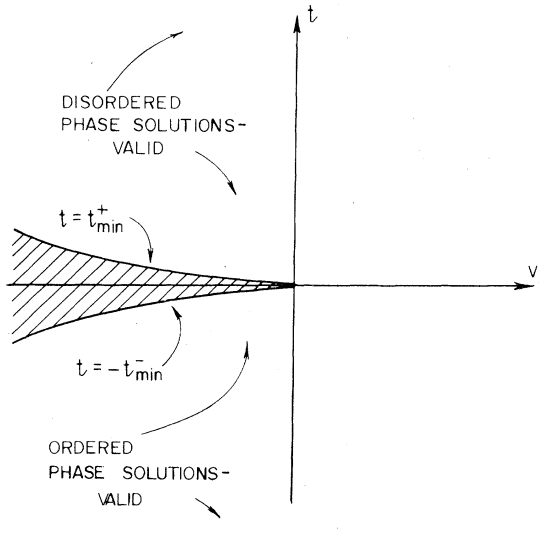

FIG. 7. Shaded region indicates (for $u = u^*$) values of t, v for which disordered- and ordered-phase expressions for thermodynamic functions are invalid.

The equation of state can be calculated (for $t \ll -t^-_{\min}$) if we retain a small nonzero external field h. For simplicity we take $u = u^*$. In this case Eq. (6.23) reduces to

$$\frac{g_t e^{\lambda_t l^*} + 4\epsilon M^2 g_v e^{\lambda_v l^*} e^{(2-\epsilon)l^*}}{(1+12K_4 g_v e^{\lambda_v l^*})} + 12u^* M^2 e^{(2-\epsilon)l^*} = 1 . \tag{6.34}$$

In the above if we let

$$e^{l^*} = M^{-(\delta+1)/d}\Phi(x,y) , \tag{6.35}$$

where $x \equiv g_t/M^{1/\beta}$, $y = |g_v|/M^{\alpha/\nu\lambda_M}$, $\delta = 3+\epsilon+O(\epsilon^2)$, and $\lambda_M = 1 - \frac{1}{2}\epsilon + O(\epsilon^2)$, we obtain an equation for $\Phi(x,y)$

$$\frac{x\Phi(x,y)^{2-\epsilon/3} - 4\epsilon y\Phi(x,y)^{2-2\epsilon/3}}{1-12K_4 y\Phi(x,y)^{-\epsilon/3}} + 12u^*\Phi(x,y)^{2-\epsilon} = 1 . \tag{6.36}$$

This equation can now be iterated for the scaling

function Φ to obtain

$$\Phi(x,y)=[(x-4\epsilon y)/(1-12K_4y)+12u^*]^{-1/2} \quad . \tag{6.37}$$

Using this result in Eq. (6.19) leads to an equation of state in scaling form, namely,

$$\frac{h}{M^\delta}=\frac{x-4\epsilon y}{1-12K_4y}+4u^* + \tfrac{1}{6}\epsilon g(x,y)\ln\left[\frac{x-4\epsilon y}{1-12K_4y}+12u^*\right] , \tag{6.38}$$

where

$$g(x,y)=\frac{1}{1-12K_4y}\left[x+12u^*-12\epsilon y - \frac{12K_4y(x-4\epsilon y)}{1-12K_4y}\right] . \tag{6.39}$$

In the limit when $y\to 0$ this of course reduces to the rigid-system result[9] as it should. Furthermore, if $|v|\ll u^*$ then $g_t\to t$ and $|g_v|\to|v|/\epsilon$ allowing us to replace x and y above by $t/M^{1/\beta}$ and $|v|/\epsilon M^{\alpha/\nu\lambda_M}$, respectively. The free energy and susceptibility can also be calculated readily using Eqs. (6.35) and (6.37).

C. First-order transition

We now consider values of t inside the shaded region of Fig. 7. Specifically, we will let the values of t be very close to but less than $t=t^+_{\min}$ given by Eq. (6.7). The essential difficulty with such values of t is that repeated RG transformations map such a system to the line of instability before $t(l^*)$ is of $O(1)$. In order to profit then from such a mapping procedure it must be established that thermodynamic functions can be calculated near the line of instability. To this end we first consider a system represented by point B of Fig. 2. If we assume that r for this system is less than zero then in anticipation of a spontaneous ordering in the system we can rewrite Eq. (4.1) (with $n=1$) replacing S_0 by M. We then obtain

$$\begin{aligned}\mathcal{H}_{\rm eff}=&[\tfrac{1}{2}rM^2+(u+v)M^4-hM]L^d+\tfrac{1}{2}L^{-d}\sum_{\vec{k}\neq\vec{0}}(r+12uM^2+4vM^2+k^2)S_{\vec{k}}S_{-\vec{k}}\\ &+4uML^{-2d}\sum_{\vec{k}_1\vec{k}_2\vec{k}_3\neq\vec{0}}S_{\vec{k}_1}S_{\vec{k}_2}S_{\vec{k}_3}\delta_{\vec{k}_1+\vec{k}_2+\vec{k}_3,\vec{0}}\\ &+uL^{-3d}\sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4\neq\vec{0}}S_{\vec{k}_1}S_{\vec{k}_2}S_{\vec{k}_3}S_{\vec{k}_4}\delta_{\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4,\vec{0}}+vL^{-3d}\left(\sum_{\vec{k}\neq 0}S_{\vec{k}}S_{-\vec{k}}\right)^2 .\end{aligned} \tag{6.40}$$

Since we will eventually make use of results in $d=4-\epsilon$ we can reasonably regard u and v in Eq. (6.40) as small parameters. Thus, the thermodynamic properties of this system can be determined by expanding in powers of u and v. A way of carrying out such an expansion would be to replace r in the Gaussian part of Eq. (6.40) with r' where r' can be determined self-consistently by solving a Dyson equation[28] for the two-spin correlation function. Then we can neglect all but the first two terms in (6.40). The validity of these approximations will ultimately be established through a satisfaction of the Ginzburg criterion.[29] Figure 8 shows graphically the equation to be solved while analytically it is given by

$$r'+k^2=r+k^2+(12u+4v)K_d\int_0^1\frac{p^{d-1}\,dp}{r'+p^2} \quad . \tag{6.41}$$

This equation can be iterated for r' and after one such iteration it becomes

$$r'\approx r+\frac{\lambda}{d-2}-\lambda\left[r+\frac{\lambda}{d-2}\right]^{(d-2)/2}\int_0^\infty\frac{x^{d-3}\,dx}{1+x^2} + \lambda\left[r+\frac{\lambda}{d-2}\right]\int_1^\infty\frac{p^{d-3}\,dp}{r+\lambda/(d-2)+p^2} , \tag{6.42}$$

where we have generalized the definition of λ given after Eq. (5.1d) to d dimensions, i.e., $\lambda\equiv 12K_du+4K_dv$. The right-hand side of Eq. (6.42) can be approximated by $r+\lambda/(d-2)$ as long as

$$\left[r+\frac{\lambda}{d-2}\right]^{(4-d)/2}\gg\lambda\int_0^\infty\frac{x^{d-3}\,dx}{1+x^2} \quad . \tag{6.43}$$

Since $\lambda\ll 1$ it follows that for $2<d<4$ the replacement $r'=r+\lambda/(d-2)$ is justified. We now rewrite $\mathcal{H}_{\rm eff}$ as

$$\mathcal{H}_{\rm eff}=[\tfrac{1}{2}rM^2+(u+v)M^4-hM]L^d + \tfrac{1}{2}L^{-d}\sum_{\vec{k}\neq\vec{0}}(r'+12uM^2+4vM^2+k^2)S_{\vec{k}}S_{-\vec{k}} , \tag{6.44}$$

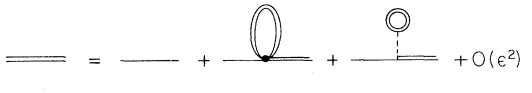

FIG. 8. Dyson equation (6.41) satisfied by the partially renormalized $\langle S_kS_{-k}\rangle$.

where $r' \equiv r + \lambda/(d-2)$. Since we are interested in a system near the line of instability we will replace $u+v$ by δ where we take $\delta > 0$ and arbitrarily small. After this replacement in Eq. (6.44) we calculate the free energy of the system according to the prescription

$$\mathcal{F}(r,u,v,h) = -L^{-d} \ln(\mathrm{Tr}_{S_{\vec{k}}} e^{-\mathcal{H}_{\mathrm{eff}}[S_{\vec{k}}]}) \quad . \tag{6.45}$$

This then results in

$$\mathcal{F}(r,u,v,h) = \tfrac{1}{2} r M^2 + \delta M^4 - hM + \tfrac{1}{2} K_d \int_0^1 k^{d-1}\, dk\, \ln(r' + 12\delta M^2 + 8|v| M^2 + k^2) \quad . \tag{6.46}$$

If we now expand the logarithm in Eq. (6.46) and perform a Legendre transformation with respect to the conjugate variables h, M we obtain

$$\mathcal{F}(r,u,v,M) = \tfrac{1}{2} u_2 M^2 + u_4 M^4 + u_6 M^6 + \cdots \quad , \tag{6.47}$$

where

$$u_2 = r + 8|v| K_d \int_0^1 \frac{k^{d-1}\, dk}{r' + k^2} \quad ,$$

$$u_4 = \delta - 16 |v|^2 K_d \int_0^1 \frac{k^{d-1}\, dk}{(r' + k^2)^2} \quad , \tag{6.48}$$

$$u_6 = \tfrac{4}{3} |v|^3 64 K_d \int_0^1 \frac{k^{d-1}\, dk}{(r' + k^2)^3} \quad ,$$

which near $d = 4$ become approximately $u_2 \approx r + 4K_4|v|$, $u_4 \approx \delta - 16|v|^2 K_4 |\ln(r')|$, and $u_6 \approx \frac{4}{9}(64) K_4 (|v|/r')^3$. The important thing to note here is that for δ arbitrarily small we can have $u_2 > 0$, $u_4 < 0$, and $u_6 > 0$. The appearance of the corresponding $\mathcal{F}$ vs M curve for such a system is shown in Fig. 9 where the equilibrium value of M corresponding to an absolute minimum in $\mathcal{F}$ could be at some nonzero value (depending on the relative sizes of u_2, u_4, and u_6). Presumably then, for suitably chosen r, u, v, the equlibrium value of M (given by $\partial \mathcal{F}/\partial M = 0$) would occur at some finite value, say M_0. As we increase r keeping u and v fixed we expect a point at which the free energy at the finite minimum, $\mathcal{F}(M_0)$, would no longer be less than its value at $M = 0$, $\mathcal{F}(M=0)$. For these values of the fields a first-order transition would occur. We can determine M_0 by setting $\partial \mathcal{F}/\partial M = 0$, however, we cannot use the low-order expansion given by Eq. (6.47) for this purpose since M_0 may be large. Instead, expecting a large M_0, we perform the integral in Eq. (6.46) treating the logarithm as a constant. This results in

$$\mathcal{F}(r,u,v,M) = \tfrac{1}{2} r M^2 + \delta M^4 + \frac{K_d}{2(d-2)} \ln(8|v| M^2) \quad . \tag{6.49}$$

We thus obtain an equation for M_0, namely,

$$r M_0 + 4\delta M_0^3 + \frac{K_d}{(d-2)} \frac{1}{M_0} = 0 \quad . \tag{6.50}$$

The last term in Eq. (6.50) can be ignored for large M_0 to give approximately

$$M_0 = \pm (-r/4\delta)^{1/2} \quad , \tag{6.51}$$

which is indeed large for arbitrarily small δ. In search of the value of r at which the first-order transition occurs, say $r = r_c$, we set $\mathcal{F}(M=0)$ equal to $\mathcal{F}(M = (|r|/4\delta)^{1/2})$ to obtain

$$\frac{r_c^2}{16\delta} = \frac{K_d}{2(d-2)} \ln\left(\frac{2v r_c}{\delta}\right) - \tfrac{1}{2} K_d \int_0^1 k^{d-1}\, dk\, \ln(r' + k^2) \quad . \tag{6.52}$$

For very small δ the second term in Eq. (6.52) can be ignored when compared with the first one to obtain

$$r_c^2 = \frac{16\delta K_d}{(d-2)} \ln\left(\frac{2v r_c}{\delta}\right) \quad , \tag{6.53}$$

which after iteration for r_c gives

$$r_c = -\left(\frac{4\delta K_d}{d-2}\right)^{1/2} \ln\left(\frac{32 K_d |v|^2}{(d-2)\delta}\right) \quad , \tag{6.54}$$

which of course corresponds to an $r_c' = r_c + \lambda/(d-2) > 0$. Using this value for r_c we can also calculate the jump in the order parameter at the

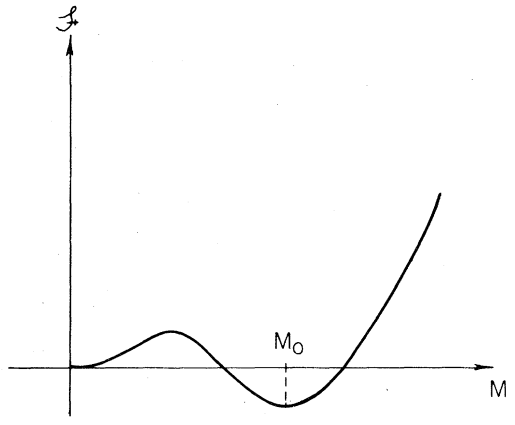

FIG. 9. Typical appearance of $\mathcal{F}$ vs M curve when $u_2 > 0$, $u_4 < 0$, and $u_6 > 0$.

transition, namely,

$$\Delta M = M_0(r = r_c) = \left(\frac{K_d}{4(d-2)\delta}\right)^{1/2} \ln^{1/4}\left(\frac{32K_d v^2}{(d-2)\delta}\right) . \quad (6.55)$$

Now we check the self-consistency of the calculation by verifying the Ginzburg criterion which requires that $\Delta \equiv u_4/(u_{2\,\mathrm{eff}})^2 << 1$. This is certainly satisfied since Δ is given approximately by $\Delta \approx \lambda\delta/2rv$ which is much less than unity and the procedures used in accounting for the effects of interactions are justifed. The methods used in this analysis are similar to those employed by Rudnick[12] in his detailed analysis of the first-order transition induced in magnetic systems with extreme cubic anisotropy. The goal of this calculation has thus been established: we have shown that near point B of Fig. 2 the system's thermodynamic functions can be determined in a self-consistent manner. With the solution at point B at hand we can now determine the system's free energy at point A by using the appropriate mapping formulas. Specifically, if we perform a Legendre transformation with respect to the variables h, M, Eq. (6.17) becomes[30]

$$\mathfrak{F}(r,u,v,M) = \mathfrak{F}_D + e^{-dl^*}\{\tfrac{1}{2}t(l^*)M^2(l^*) + [u(l^*)+v(l^*)]M^4(l^*) + \tfrac{1}{8}K_4T(l^*)^2\ln[T(l^*)]\} . \quad (6.56)$$

The condition determining l^* is now $v(l^*) = -u(l^*)$. With this choice Eq. (6.56) reduces to

$$\mathfrak{F}(r,u,v,M) = \mathfrak{F}_D + e^{-dl^*}\{\tfrac{1}{2}t(l^*)M^2(l^*) + \tfrac{1}{8}K_4[t(l^*)8u(l^*)M^2(l^*)]^2 \times \ln[t(l^*)+8u(l^*)M^2(l^*)]\} . \quad (6.57)$$

It is first interesting to note that for $t = 0$ the minimum of this expression occurs at a finite value of M while $M = 0$ is a nonanalytic local maximum. Presumably then for some value of $t = t_c > 0$ a first-order transition occurs. We can calculate this temperature by setting the ordered- and disordered-phase free energies equal to one another. When this is done we obtain

$$t_c(l^*) = 2K_4u(l^*)e^{-1} \quad (6.58)$$

and for the jump in the order parameter at this temperature

$$\Delta M(l^*) = e^{-1/2}/[8u(l^*)]^{1/2} . \quad (6.59)$$

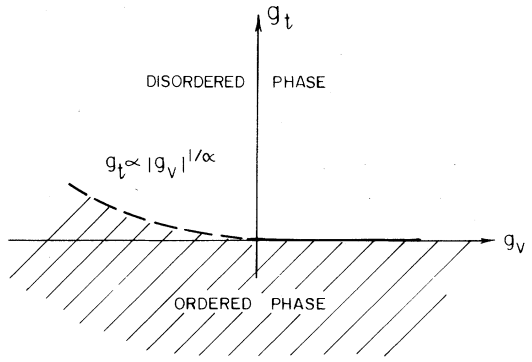

FIG. 10. Line $g_t = 0$, $g_v > 0$ is a line of second-order transitions while the broken line given by $g_t \propto |g_v|^{1/\alpha}$ which ends at a tricritical point is a line of first-order transitions. The physically accessible region corresponds to $g_v < 0$.

These two results can now be expressed in terms of the bare thermodynamic fields by utilizing the value l_2 given in Eq. (6.6). In using Eq. (6.6) we are again making approximations $u = u^*$ and $|v| << u^*$. We then obtain for the transition temperature

$$t_c = \tfrac{3}{2}K_4u^*e^{-1}(4|v|/3u^*)^{1/\alpha} \quad (6.60)$$

and for ΔM

$$\Delta M = \frac{e^{-1/2}}{(8u^*)^{1/2}}\left(\frac{4|v|}{3u^*}\right)^{\nu\lambda_M/\alpha} . \quad (6.61)$$

If we reexpress Eq. (6.60) in terms of the nonlinear scaling fields we obtain the result

$$g_{t_c} \propto |g_v|^{1/\alpha} \quad (6.62)$$

as predicted by the scaling hypothesis.[31] With α small in $d = 3$ we would then expect the phase diagram to appear as shown in Fig. 10. It is also clear that our use of recursion relation solutions for $l \approx l_2$ is valid since the solutions to the set (6.11) only break down for values of l greater than $l_{\max}$ given by Eq. (6.2) and clearly $l_2 < l_{\max}$.

VII. DISCUSSION

We were able to calculate the free energy of the compressible magnet for values of t in the ranges $t < -t^-_{\min}$, $t > t^+_{\min}$, and also at t just less than $t^+_{\min}$. We could not however calculate thermodynamic properties in the other portions of the shaded region of Fig. 8. Nevertheless if $|v|$ is sufficiently small then this region of temperature will be exponentially small since $\alpha \sim 0.1$ in $d = 3$.

Although the jump in the order parameter given by Eq. (6.61) is very small (for small $|v|$) such first-order behavior has been suspected. For example experimental data[16] on β brass (which probably under-

goes an order-disorder transition) is better fitted when the existence of a small but finite jump in the order parameter is assumed even though the jump cannot actually be observed.

ACKNOWLEDGMENTS

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