Nonlinear effects in two-dimensional superfluid ^{4}He

A. C. Biswas and C. S. Warke
Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India
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A nonlinear finite-amplitude density-wave-propagation equation in two-dimensional superfluid ^{4}He is derived. Though this equation is similar to that investigated by Huberman, it differs in the coefficient of the nonlinear term of the Korteweg–de Vries equation used by him. The conditions are studied for the propagation of solitons in thin superfluid films at low temperatures. Possible experiments are discussed to test the predictions of the theory.

A great deal of understanding of the properties of two-dimensional ^{4}He films has been attained during the past decade.[1] The most striking among these is the third-sound phenomenon orignally proposed by Atkins.[2] It has established the superfluid nature of two-dimensional helium films. On the theoretical front, Landau's theory of quantum hydrodynamics was generalized to the regime of two-dimensional superfluids.[3] The linearized equation of motion correctly predicted the propagation of third sound and its temperature dependence in such helium films.[3] The recent spectacular observation[4] is the universal jump in superfluid density at the transition temperature predicted by two-dimensional theories[5] of superfluid ^{4}He films. There are indications of the finite-amplitude nonlinear effects taking place in low-temperature superfluid thin films.[1,6] At larger superfluid velocity amplitudes the shock-wave effects are observed in thick saturated films. While at low temperature, in thin films, an undistorted pulse propagation is observed. Recently, a theoretical formalism for the possible nonlinear excitations in monolayer superfluid ^{4}He films has been developed by Huberman.[6] It was found that the nonlinear effects can lead to existence of gapless solitons made up of superfluid condensate. The conditions for the propagation of a solitary wave in such films were derived. These results are based on a conjectured nonlinear superfluid density equation. The aim of this paper is to present a systematic theoretical derivation of the nonlinear superfluid-fluctuation-density equation starting from the generalized version of Landau theory of two-dimensional quantum hydrodynamics. We find that the nonlinear term is different from that used in Ref. 6. It changes the conditions required for the propagation of a solitary wave in monolayer superfluid ^{4}He films.

The observed T^3 dependence of the square of the sound velocity at low temperature T suggested the two-dimensional spectrum of Landau elementary excitations.[3] Assuming the existence of superfluid condensate wave function (complex order parameter) $\psi(x,t)$, the following phenomenological equation of motion for the monolayer superfluid motion was proposed by Rutledge *et al.*[3]:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi-\frac{A\psi}{(a+|\psi|^2)^3}-\mu\psi-B\psi\vec{\nabla}^2|\psi|^2 \quad (1)$$

where m is the mass of the helium atom, A and a ($A\simeq 14$ K and a $\simeq 1.2$ atomic layers) are constants of van der Waals interaction, μ and B are the chemical-potential and the surface-tension constants, respectively. The vector $\vec{x}$ is a two-dimensional vector in the monolayer superfluid helium film. The superfluid surface density is $\rho(\vec{x},t)=|\psi(\vec{x},t)|^2$. As we are interested in finding the propagation of density waves on the two-dimensional surface, we may assume a general form of $\psi(\vec{x},t)=\psi(\vec{n}\cdot\vec{x},t)$, where $\vec{n}$ is a constant unit vector in this plane, along which the solutions will propagate. For the sake of simplicity in notation, we denote $x=\vec{n}\cdot\vec{x}$. With the chosen form of $\psi(\vec{x},t)=\rho(x,t)\exp[i\phi(x,t)]$, where ρ and ϕ are real functions, one easily obtains the equations of motion for ρ and ϕ from Eq. (1),

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0 \quad (2)$$

and

$$\frac{\hbar}{m}\frac{\partial\phi}{\partial t}=\frac{\hbar^2}{2m^2}\left[\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial x^2}\sqrt{\rho}-\left(\frac{\partial\phi}{\partial x}\right)^2\right]+\frac{A}{m(a+\rho)^3}+\frac{\mu}{m}+\frac{B}{m}\frac{\partial^2\rho}{\partial x^2} \quad (3)$$

We have defined superfluid velocity $v=(\hbar/m)\times(\partial\phi/\partial x)$. By differentiating Eq. (3) with respect to x, it can be put in a more convenient form

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial x^2}\sqrt{\rho}\right)-\frac{3A}{m}\frac{\partial\rho}{\partial x}\frac{1}{(a+\rho)^4}+\frac{B}{m}\frac{\partial^3\rho}{\partial x^3} \quad (4)$$

In searching a solution of Eqs. (2) and (4) characterizing the density-wave propagation, it is convenient to make the following coordinate transformation[7] $X = x + C_3 t$ and $t \to t$ so that

$$\rho(x,t) = \rho(X - C_3 t, t) \equiv R(X,t)$$

and

$$v(x,t) = v(X - C_3 t, t) \equiv V(X,t) \ . \tag{5}$$

The constant C_3 will be found later. Equations (2) and (4) then take the forms

$$\frac{\partial R}{\partial t} + C_3 \frac{\partial R}{\partial X} + \frac{\partial}{\partial X}(RV) = 0 \tag{6}$$

and

$$\frac{\partial V}{\partial t} + C_3 \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial X} + \frac{\hbar^2}{2m^2}\frac{\partial}{\partial X}\left[\frac{1}{4R^2}\left(\frac{\partial R}{\partial X}\right)^2 - \frac{1}{2R}\frac{\partial^2 R}{\partial X^2}\right] - \frac{B}{m}\frac{\partial^3 R}{\partial X^3} + \frac{3A}{m}\frac{\partial R}{\partial X}\frac{1}{(a+R)^4} = 0 \ . \tag{7}$$

The solution we are looking for corresponds to a uniform liquid-helium density at rest; on the average superimposed on it is a characteristic collective density oscillation mode propagating through the two-dimensional liquid. Let us make a scaling transformation[7]

$$X \to \epsilon^{1/2} X, \quad t \to \epsilon^{3/2} t \ . \tag{8}$$

The long-wavelength approximation of Eqs. (6) and (7) then can be carried out by expanding R and V in powers of the small parameter ϵ consistent with the above restrictions. This leads to

$$R = \rho_0 + \epsilon \rho_1(X,t) + \epsilon^2 \rho_2(X,t) + \cdots$$

and

$$V = \epsilon v_1(X,t) + \epsilon^2 v_2(X,t) + \cdots \ . \tag{9}$$

In Eq. (9), ρ_0 is the average uniform-surface superfluid density. Using Eqs. (8) and (9), the expansion in powers of $\epsilon^{1/2}$ of Eqs. (6) and (7) is carried out. The following equations are then obtained by equating to zero the coefficients of the two lowest-order terms $\epsilon^{3/2}$ and $\epsilon^{5/2}$:

$$C_3 \frac{\partial \rho_1}{\partial X} + \frac{\partial}{\partial X}(\rho_0 v_1) = 0 \ , \tag{10}$$

$$C_3 \frac{\partial v_1}{\partial X} + \frac{3A}{m}\frac{\partial \rho_1}{\partial X}\frac{1}{(a+\rho_0)^4} = 0 \ , \tag{11}$$

$$\frac{\partial \rho_1}{\partial t} + C_3 \frac{\partial \rho_2}{\partial X} + \frac{\partial}{\partial X}(\rho_0 v_2 + \rho_1 v_1) = 0 \ , \tag{12}$$

$$\frac{\partial v_1}{\partial t} + C_3 \frac{\partial v_2}{\partial X} + v_1 \frac{\partial v_1}{\partial X} + \frac{3A}{m(a+\rho_0)^4}\frac{\partial \rho_2}{\partial X} - \frac{\hbar^2}{4m^2\rho_0}\frac{\partial^3 \rho_1}{\partial X^3} - \frac{12A}{m(a+\rho_0)^5}\rho_1 \frac{\partial \rho_1}{\partial X} - \frac{B}{m}\frac{\partial^3 \rho_1}{\partial X^3} = 0 \ . \tag{13}$$

It is straightforward to solve Eqs. (10) and (11) with the boundary conditions that both ρ_1 and v_1 go to zero as X tends to infinity. It gives

$$v_1 = -\frac{C_3 \rho_1}{\rho_0}, \quad v_1 = -\frac{3A}{mC_3}\frac{\rho_1}{(a+\rho_0)^4} \ . \tag{14}$$

From this one obtains

$$C_3^2 = 3A\rho_0 / m(a+\rho_0)^4 \ . \tag{15}$$

There are two solutions corresponding to C_3 being positive or negative, which we denote by $C_3 = \pm C_0$ ($C_0 > 0$). The coordinate transformation in Eq. (5) is correspondingly indicated by R^+ and R^-, respectively. The choice of R^- corresponds to the coordinate frame used in Ref. 6. This expression of the adiabatic velocity is the same as that obtained in Ref. 6 for the ordinary dispersionless third-sound mode. It is interesting to note that the unwanted terms ρ_2 and v_2 in Eqs. (12) and (13) can be eliminated by multiplying Eq. (13) with ρ_0 and subtracting it from the products of C_3 and Eq. (12). The velocity v_1 in the resulting equation can be expressed in ρ_1 using Eq. (14). The final expression for the equation of motion of the density fluctuation ρ_1 comes out to be

$$\frac{\partial \rho_1}{\partial t} + \frac{C_3(\rho_0 - 3a)}{2\rho_0(a+\rho_0)}\rho_1 \frac{\partial \rho_1}{\partial X} + \left(\frac{\hbar^2 + 4mB\rho_0}{8m^2 C_3}\right)\frac{\partial^3 \rho_1}{\partial X^3} = 0 \ . \tag{16}$$

This completes the derivation of the required nonlinear differential equation describing the superfluid density-fluctuation motion propagating along $\vec{n}$ in a monolayer helium film. It is the Korteweg–de Vries[8] (KdV) equation as was taken by Huberman, with a difference that the coefficient of nonlinear term $\rho_1 \partial\rho_1/\partial X$ is equal to $\gamma = C_3(\rho_0 - 3a)/[2\rho_0(a+\rho_0)]$ instead of C_3. If one drops the nonlinear term, the solution of the linearized equation is

$$\rho_1 = R_1 \sin(kX + i\omega t) + R_2 \cos(kX + \omega t) \ ,$$

where $\omega = (\hbar^2 + 4mB\rho_0)k^3/8m^2 C_3$. When expressed back in terms of the original coordinates, it has a form

$$\rho_1(x,t) = R_1 \sin(k\vec{n}\cdot\vec{x} + C_3kt + \omega t) + R_2\cos(k\vec{n}\cdot\vec{x} + C_3kt + \omega t) \ .$$

The frequency of the linearized sound-wave motion propagating along $\vec{n}$ turns out to be $-C_3k - \omega = -C_3(k + k^3/k_0^2)$, with $k_0^2 = 8m^2C_3^2/(\hbar^2 + 4Bm\rho_0)$, which has a value $k_0 \simeq 0.5$ Å^{-1}. This gives the positive dispersion relation implying that in two-dimensional superfluids the phase velocity v for small k is $-C_3(1 + k^2/k_0^2)$ as was found in Ref. 6, since $C_3 = -C_0$ in the coordinate system used there. In our approach the third-sound wave is propagating along $k\vec{n}$ (k can be taken positive without loss of generality).

The solitary wave solution[9] of Eq. (16) is

$$\rho_1 = A_0 \operatorname{sech}^2[(X - Ct)/\Delta] = A_0\operatorname{sech}^2[(\vec{n}\cdot\vec{x} + C_3t - Ct)/\Delta] \ . \quad (17)$$

The width Δ and the velocity C depend on the amplitude A_0 in the following way (for $k_0 > 0$):

$$\Delta = 2k_0^{-1}(C_3/C)^{1/2}, \quad \frac{C}{C_3} = \frac{A_0(\rho_0 - 3a)}{6\rho_0(a + \rho_0)} \ . \quad (18)$$

The solitary wave in Eq. (17) is propagating along $\vec{n}$ with a phase velocity

$$v = -C_3 + C = -C_3\left(1 - \frac{A_0(\rho_0 - 3a)}{6\rho_0(a + \rho_0)}\right) \ . \quad (19)$$

We require that the width Δ be real, which from Eq. (18) leads to the fact that C has the same sign as that of C_3. This further implies that C/C_3 is always positive which in turn from Eq. (18) restricts the amplitude A_0 such that

$$A_0(\rho_0 - 3a) > 0 \ . \quad (20)$$

The change of sign of C_3 does not alter any conclusions derived here, except that it merely changes the direction of propagation of the waves. Because of the appearance of a factor $(\rho_0 - 3a)$ in the nonlinear term in Eq. (16), the shape of the solitary wave and the conditions for its propagation differ from those discussed in Ref. 6. It can be seen from Eqs. (19) and (20) that the solitary wave phase velocity is always smaller than that of the ordinary third sound for all values of film density ρ_0. The solitary waves always lag behind the third-sound waves. The amplitude of the solitary wave is smaller for a faster moving soliton as can be seen from Eq. (19).

The nonlinear term in Eq. (16) is a small perturbation for values of average density $\rho_0 \simeq 3a$, therefore the solutions are the third-sound waves discussed above. It is seen from Eq. (20) that the amplitude $A_0 > 0$ for the values of mean density of the superfluid $\rho_0 > 3a$. It describes the propagation of a local compression of the superfluid density relative to its average density ρ_0. While for $\rho_0 < 3a$ Eq. (20) implies a negative amplitude $A_0 < 0$. These solutions describe the propagation of local rarefaction of the superfluid density relative to ρ_0. Following Berezin and Karpman,[10] we define a dimensionless parameter σ in terms of the amplitude A_0 and width Δ (now arbitrary) of the initial perturbation of the form given in Eq. (17) and the constants appearing in Eq. (16)

$$\sigma = k_0\Delta(\gamma A_0/C_3)^{1/2} > 0 \ . \quad (21)$$

When $\sigma << (12)^{1/2}$, the perturbation can be considered almost linear. The long-wavelength part of its spectral expansion in powers of k then propagates with velocity C_3 as the third-sound waves. However, for $\sigma > (12)^{1/2}$, the perturbation transforms asymptotically into a weakly nonlinear wave packet of a number of solitons, which have widths and amplitudes as that given in Eq. (18). The propagation of an arbitrary initial shape perturbation can be studied by the well-known inverse-scattering method as applied to study the solutions of KdV equation.[10]

From the knowledge of the van der Waals energy parameters $a \simeq 1.2$ atomic layers, we predict the following effect in the two-dimensional superfluid helium at low temperatures (below 0.4 K). For the superfluid density $\rho_0 > 3.6$ atomic layers, the solitary waves can be created by applying a cooling pulse to a localized region of the film resulting in a local compression of the superfluid density. On the contrary, for $\rho_0 < 3.6$ atomic layers the application of a localized heat pulse, resulting in the depression of the superfluid density produces the solitons describing the propagation of this local rarefied superfluid density. In the neighborhood of $\rho_0 \simeq 3.6$ atomic layers, it is not possible to produce the solitons and only the third-sound waves can be propagated with the initial heating or cooling pulse. As the third-sound propagation has been studied[3] in the surface density range $0.16 < \rho_0 < 5.25$ atomic layers, it can be possible to test the above theoretical predictions in two-dimensional superfluids. In the case of thicker films, the additional dispersion may change its sign[6] in Eq. (16) thereby complicating the observation as it has the opposite effect. The propagation of solitons can be detected by their well-known properties.

[1]K. R. Atkins and I. Rudnick, Prog. Low Temp. Phys. 6, 37 (1970); K. L. Telshow, R. K. Galkiewiez, and R. B. Hallock, Phys. Rev. B 14, 4883 (1976).
[2]K. R. Atkins, Phys. Rev. 113, 962 (1959).
[3]J. E. Rutledge, W. L. McMillan, J. M. Mochel, and T. E. Washburn, Phys. Rev. B 18, 2155 (1978).
[4]I. Rudnick, Phys. Rev. Lett. 40, 1454 (1978).
[5]D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. 39, 1201 (1977).
[6]B. A. Huberman, Phys. Rev. Lett. 41, 1389 (1978).
[7]H. Washimi and T. Taniuti, Phys. Rev. Lett. 17, 996 (1966).
[8]D. J. Korteweg and G. de Vries, Philos. Mag. 39, 422 (1895).
[9]B. B. Kadomtsev and V. I. Karpman, Usp. Fiz. Nauk 103, 193 (1971) [Sov. Phys. Usp. 14, 40 (1971)].
[10]Yu A. Berezin and V. I. Karpman, Zh. Eksp. Fiz. 51, 1557 (1966) [Sov. Phys. JETP 24, 1049 (1967)].