Scaling theory of the Potts-model multicritical point

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A theory of the scaling behavior near the $q_c = 4$ state multicritical point of the two-dimensional Potts lattice gas model is developed. Proceeding from the assumption that a dilution field becomes marginal at the multicritical point while the thermal and ordering fields are relevant, a set of differential renormalization group (RG) equations for these fields are constructed. Keeping terms through second order we find that these equations are characterized by five universal parameters which we evaluate using exact as well as conjectured results. Based upon these RG equations, we investigate the physical properties of the two-dimensional Potts lattice gas for q near q_c. For the pure Potts model with $q = q_c$ we find logarithmic temperature corrections to the specific heat and the spontaneous magnetization. At T_c we find $\ln(r)$ corrections to the power law behavior of the spin-spin correlation function. For the dilute Potts model with $q = q_c$ we find that the latent heat, the discontinuity in the magnetization, and the discontinuity in the coexisting densities vanish with an essential singularity as T approaches the multicritical point from the first-order side. Results for $q > q_c$ and $q < q_c$ are also given.

I. INTRODUCTION

The two-dimensional Potts model,[1] originally introduced as a formal generalization of the Ising model, has in recent years become the focus of increasing interest to both theorists and experimentalists. From a theoretical point of view, a rich variety of models can be obtained from various limits of the Potts model[2] and conversely limits of yet other models reduce to particular cases of the Potts model.[3] Experimentally[4] a variety of submonolayer adsorbed gas phase transitions, which provide realizations[5] of various Potts models, have been studied. In these experimental systems the coverage can vary leading to the study of Potts lattice gases in which a dilution field enters in a natural way. Recently, Nienhuis *et al.*[6] proposed a renormalization procedure in which disordered spin cells of a pure Potts model were mapped onto vacancies and found that the dilution field played an essential role. In particular it provided a natural way of understanding the abrupt change in the pure two-dimensional Potts model from a second-order to a first-order transition when the number of Potts states q exceeds a critical value $q_c = 4$. In the extended parameter space of dilute Potts Hamiltonians, the behavior near q_c is seen to arise from a smooth line of fixed points which change at q_c from critical to tricritical.

We have been interested in the behavior of this system near q_c both because we believe it provides a general picture of what happens in a class of similar multicritical points and because it offers a rich variety of experimental tests. In a previous publication,[7] two of us proposed a set of differential renormalization group (RG) equations for the temperature and the dilution fields. The form of these equations were originally deduced on the basis of examining the flows observed in an approximate Migdal-Kadanoff renormalization procedure. They led naturally to a thermal eigenvalue having the parabolic form of the extended den Nijs[8] conjecture near q_c and to a latent heat essential singularity in $(q - q_c)^{1/2}$ of the form found by Baxter.[9] Fitting parameters in these equations to the extended den Nijs conjecture and Baxter's result, it was argued that these equations provided an accurate representation of the renormalization group in the vicinity of the multicritical point. Here we explore further the behavior of this multicritical point by adding an ordering field to break the Potts symmetry. Carrying out nonlinear scaling field transformations[10] we identify the structure and universal parameters which enter the differential renormalization group relations through second order. While the universal parameters entering the thermal and dilution field equations can be determined from the den Nijs conjecture and Baxter's latent heat result, the parameters which enter the ordering field equation can only be estimated on the basis of presently known results. We discuss this and then proceed to explore the experimental consequences of our renormalization group, examining a variety of thermal and magnetic properties of the pure and dilute Potts model with q near q_c.

The physical variable space of the dilute Potts models depends upon the number of Potts states q, the temperature, the ordering field, and on the concentration. As discussed in Sec. II, the renormalization group equations can be expressed in terms of universal parameters when scaling fields are used. These scaling fields are analytically related to the physical variables and for convenience we will call φ, h, and ψ the thermal, ordering and dilution fields, respectively.

The fixed point structure for q near q_c found by Nienhuis *et al.*,[6] is shown in the q-ψ-φ space in Fig. 1. Here the pure Potts model is approached for increasingly negative values of ψ while positive values correspond to a dilute system. The critical surface corresponds to the plane $\varphi=0$. On this critical surface there is a smooth curve consisting of a (solid) line of critical points and a (dashed) line of tricritical points which join smoothly at q_c. Along the critical line, the eigenvalue associated with the dilution field is negative and hence irrelevant while along the tricritical line it is positive and relevant. When these lines meet at q_c, this eigenvalue goes through zero and becomes marginal. As discussed by Wegner,[10] this can lead to logarithmic corrections as well as essential singularities.

In Sec. II, a scaling field analysis is carried out to determine the structure of the renormalization group equations expanded about the multicritical point. Here the thermal and ordering scaling fields φ and h are relevant while the dilution field ψ is marginal. Making the fundamental assumption of analyticity, we find through second order that under an infinitesimal scale change dl

$$\frac{d\psi(l)}{dl}=a\,[\psi^2(l)+\epsilon]\ , \tag{1.1}$$

$$\frac{d\varphi(l)}{dl}=[y_T+b\,\psi(l)]\varphi(l)\ , \tag{1.2}$$

$$\frac{dh(l)}{dl}=[y_H+c\,\psi(l)]h(l)\ . \tag{1.3}$$

Here $\epsilon=q-q_c$, and a, b, c, y_T, and y_H are universal parameters. The line of fixed points shown in Fig. 1 corresponds to $\varphi(l)=h(l)=0$ and $\psi=\pm(q_c-q)^{1/2}$. Equations (1.1) and (1.2) were introduced in Ref. 7 and their analytic solutions are given there.

Setting $h=0$ and using Eqs. (1.1) and (1.2), we have constructed some renormalization flows in the φ-ψ plane corresponding to $q<q_c$, $q=q_c$, and $q>q_c$. These are plotted as Figs. 2(a), 2(b), and 2(c),

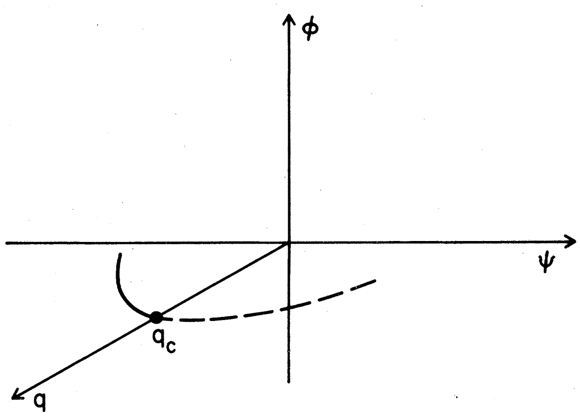

FIG. 1. Fixed point structure in the q-ψ-φ space. A continuous line of critical points (solid curve) meets a continuous line of tricritical points at a multicritical point ($q=q_c$, $\psi=\varphi=0$).

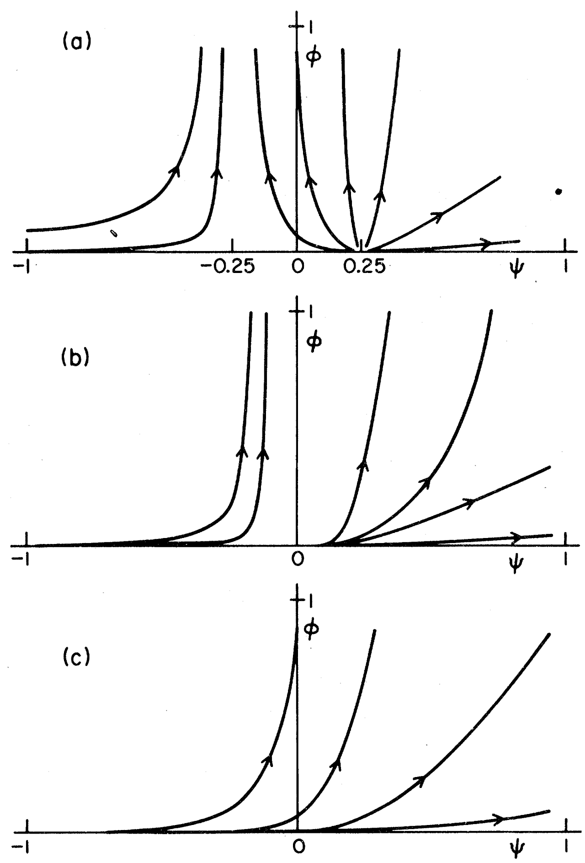

FIG. 2. Renormalization flow lines in the φ-ψ plane for $h=0$. The top figure (a) corresponds to $q_c>q$ ($q_c-q=\frac{1}{16}$) and shows the effect of the critical and tricritical points on the flow. In the center figure (b), $q=q_c$ and the critical and tricritical points have coalesced leading to flows having an essential singularity at the multicritical point $\psi=\varphi=0$. The bottom figure (c) is for $q>q_c$.

respectively. Note that they are symmetric with respect to ϕ going to $-\phi$. For $q < q_c$, Fig. 2(a) shows the critical-tricritical crossover behavior. As q approaches q_c, the critical and tricritical points coalesce leading to the flow shown in Fig. 2(b). This flow accounts for the essential singularity.

Following our derivation of the form of the renormalization group equation, we turn in Sec. III to a discussion of their physical consequences. In order to make quantitative predictions it is necessary to determine the values of the universal parameters. Following a brief review of how y_T, b, and a were previously evaluated from the den Nijs conjecture and from Baxter's result for the latent heat, we discuss two possible choices for y_H and c. Here we lack the exact results necessary to determine these parameters uniquely.

Using the basic renormalization group equations (1.1)–(1.3) we investigate the thermal, magnetic, and number density properties of both pure and dilute two-dimensional Potts models with q near 4. For the physically interesting case of $q=4$ we find in the pure Potts model $\ln[(T-T_c)/T_c]$ corrections to the usual power law behavior of the specific heat and the spontaneous magnetization as well as $\ln r$ corrections to the $r^{-\eta}$ power law falloff of the order parameter correlation function at $T=T_c$. For the dilute $q=4$ state Potts model we find that the latent heat, the discontinuity in magnetization, and the discontinuity in density $\Delta\rho$ across the first-order transition vanish with an essential singularity as the temperature approaches the multicritical point. The schematic illustration Fig. 3(a) of the phase diagram for a $q=4$ Potts lattice gas shows the cusp associated with this essential singularity in $\Delta\rho$. This is in contrast to the theoretically expected[11] behavior of the three states Potts lattice gas illustrated in Fig. 3(b). Unfortunately, temperature variations and finite substrate size effects can alter the $q=3$ phase diagram of Fig. 3(b) producing a cusplike behavior.[11] However, with the increasing number of systems being explored, as well as the introduction of x-ray and neutron scattering techniques, we hope that it will prove possible to observe the type of phenomena we predict. In a more formal context the results we have obtained for the logarithmic corrections can be used to guide Pade approximations of series expansions. This same type of analysis has also proved fruitful in understanding the related bifurcation of the Ashkin-Teller model.[12]

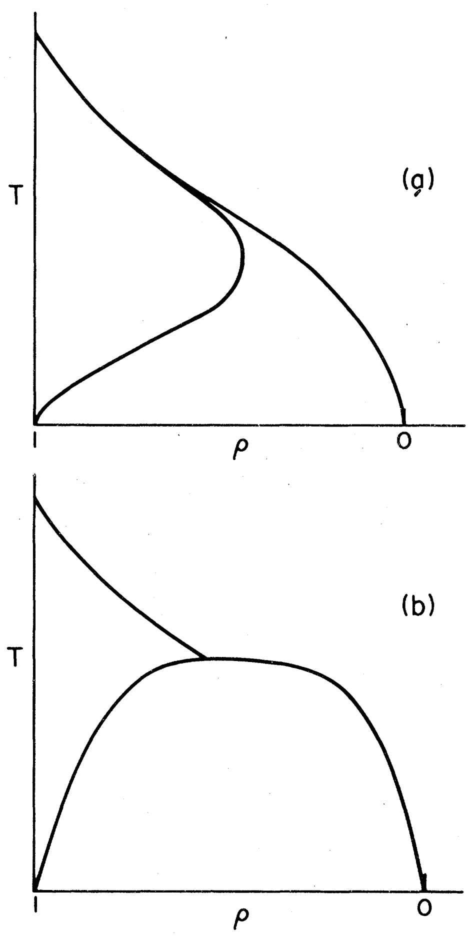

FIG. 3. (a) Phase diagram for a $q=4$ Potts lattice gas schematically indicating the cusp which arises from the essential singularity in $\Delta\rho$ at the multicritical point. (b) Schematic phase diagram for the $q=3$ state Potts lattice gas.

II. SCALING THEORY NEAR THE MULTICRITICAL POINT

In this section the form of the renormalization group equations given in Eqs. (1.1)–(1.3) is derived. The basic idea is simple. We expand the RG equations about the fixed point at $q=q_c$. Then, using the fact that two of the fields are relevant while one is marginal, we select new nonlinear combinations of fields so as to eliminate as many nonlinear terms in these equations as possible. Just as for the Ising model, there could be an additional relevant field in the symmetry breaking sector, but this will not alter our results. The reader who wishes to skip this derivation of Eqs. (1.1)–(1.3) can go directly to Sec. III where the scaling predictions based on these three equations are discussed.

In differential form the RG equations for the fields x_i are

$$\frac{dx_i}{dl}=f_i(\vec{x},\epsilon) \ , \qquad (2.1)$$

where the f_i are nonlinear functions of their arguments $\vec{x}$ and $\epsilon=q-q_c$. The multicritical fixed point lies at $\vec{x}=0$, $\epsilon=0$, i.e., $f_i(\vec{0},0)=0$. The fundamental assumption is that $f_i(\vec{x},\epsilon)$ is analytic in a neighborhood of the fixed point, so we may expand it in a

Taylor series in $\vec{x}$ and ϵ. To obtain the dominant singularities for our problem we need consider only terms up to second order, and we shall confine our main discussion to this case, only briefly mentioning the effect of third-order terms at the end of this section. Most of the following general discussion parallels that of Wegner,[10] with the addition of the ϵ dependence.

To second order we have

$$f_i(\vec{x},\epsilon)=c_i\epsilon+\sum_j \bar{T}_{ij}x_j+\sum_{jk}\bar{T}_{ijk}x_jx_k+\sum_j \bar{R}_{ij}x_j\epsilon+d_i\epsilon^2+\cdots \quad (2.2)$$

Let ψ_{ir} and ϕ_{ri} be the right and left eigenvectors of $\bar{T}_{ij}$ corresponding to the eigenvalue y_r. Introducing variables

$$u_r=\sum_i \phi_{ri}x_i \ , \quad (2.3)$$

or

$$x_i=\sum_r \psi_{ir}u_r \ , \quad (2.4)$$

we obtain the RG equations for the u_r

$$\frac{du_r}{dl}=y_ru_r+c_r'\epsilon+\sum_{st}T'_{rst}u_su_t+\sum_s R'_{rs}u_s\epsilon+d_r'\epsilon^2+\cdots \ , \quad (2.5)$$

where

$$c_r'=\sum_i \phi_{ri}c_i \ , \quad (2.6)$$

$$d_r'=\sum_i \phi_{ri}d_i \ , \quad (2.7)$$

$$T'_{rst}=\sum_{ijk}\phi_{ri}\bar{T}_{ijk}\psi_{js}\psi_{kt} \ , \quad (2.8)$$

$$R'_{rs}=\sum_{ij}\phi_{ri}\bar{T}_{ij}\psi_{js} \ . \quad (2.9)$$

We can simplify Eq. (2.5) further by making a nonlinear transformation to the scaling fields ψ_r, which, to second order, is of the form

$$\psi_r=u_r+\sum_{st}a_{rst}\,u_su_t+\cdots \ . \quad (2.10)$$

The coefficients a_{rst} are determined by the requirement that in the RG equations for ψ_r as many nonlinear terms as possible are eliminated. These equations then are

$$\frac{d\psi_r}{dl}=y_r\psi_r+c_r'\epsilon+\sum_{st}T_{rst}\psi_s\psi_t+\sum_s R_{rs}\psi_s\epsilon+d_r'\epsilon^2+\cdots \ , \quad (2.11)$$

where

$$T_{rst}=\begin{cases}T'_{rst}, & \text{if } y_r=y_s+y_t \ ,\\ 0, & \text{otherwise} \ ,\end{cases} \quad (2.12)$$

and

$$a_{rst}=T'_{rst}/(y_r-y_s-y_t) \ , \ \text{if } y_r\neq y_s+y_t \ , \quad (2.13)$$

and is undetermined otherwise (and can be set equal to zero to the order at which we are working). Also

$$R_{rs}=R'_{rs}+2\sum_t a_{rst}c_t' \ . \quad (2.14)$$

When $y_r=y_s+y_t$, the coefficients T_{rst} are universal in nature, like the eigenvalues y_r, and the simplified RG equations (2.11) can be used to determine universal scaling laws for the physical fields.

We now specialize to the dilute q-state Potts model in the presence of an external ordering field. We assume that there are two relevant scaling fields ψ_2 and ψ_3, and a marginal field ψ_1, so that $y_2>0$, $y_3>0$, and $y_1=0$. In the absence of an ordering field, ψ_1 and ψ_2 are analytically related to the reduced temperature t and the chemical potential μ which controls the concentration. The field ψ_3 has the symmetry-breaking properties of the ordering field h, and is proportional to h for small fields.

From our general analysis, the nonvanishing second-order coefficients in Eq. (2.11) are T_{111}, $T_{212}=T_{221}$, and $T_{313}=T_{331}$, giving the RG equations

$$\frac{d\psi_1}{dl}=T_{111}\psi_1^2+(R_{11}\psi_1+R_{12}\psi_2+R_{13}\psi_3+c_1')\epsilon+d_1'\epsilon^2+\cdots \ , \quad (2.15)$$

$$\frac{d\psi_2}{dl}=(y_2+T_{221}\psi_1)_2+(R_{21}\psi_1+R_{22}\psi_2+R_{23}\psi_3+c_2')\epsilon+d_2'\epsilon^2+\cdots \ , \quad (2.16)$$

$$\frac{d\psi_3}{dl}=(y_3+T_{331}\psi_1)\psi_3+(R_{31}\psi_1+R_{32}\psi_2+R_{33}\psi_3+c_3')\epsilon+d_3'\epsilon^2+\cdots \ . \quad (2.17)$$

Rescaling and shifting the origin by the further transformations

$$\psi_1(c_1'/T_{111})^{1/2}\psi \ , \quad (2.18)$$

$$\psi_2=\phi-c_2'\epsilon/y_2 \ , \quad (2.19)$$

$$\psi_3=h-c_3'\epsilon/y_3 \ , \quad (2.20)$$

we obtain

$$\frac{d\psi}{dl}=a(\psi^2+\epsilon)+(\tilde{R}_{11}\psi+\tilde{R}_{12}\phi+\tilde{R}_{13}h)\epsilon+\tilde{d}_1\epsilon^2+\cdots \ , \quad (2.21)$$

$$\frac{d\phi}{dl}=(y_T+b\psi)\phi+(\tilde{R}_{21}\psi+\tilde{R}_{22}\phi+\tilde{R}_{23}h)\epsilon+\tilde{d}_2\epsilon^2+\cdots \ , \quad (2.22)$$

$$\frac{dh}{dl}=(y_H+c\psi)h+(\tilde{R}_{31}\psi+\tilde{R}_{32}\phi+\tilde{R}_{33}h)\epsilon+\tilde{d}_3\epsilon^2+\cdots \ , \quad (2.23)$$

with $a = (c_1' T_{111})^{1/2}$, $b = T_{212}(c_1'/T_{111})^{1/2}$, $c = T_{313}(c_1'/T_{111})^{1/2}$, and with $\tilde{R}_{ij}$ and $\tilde{d}_i$ linearly related to R_{ij} and d_i, respectively. From Eq. (2.21) we now see that, at the fixed points, ϵ is $O(\psi^2)$, and thus we should drop all terms involving $\tilde{R}_{ij}$ and $\tilde{d}_i$, consistent with our original assumption of dropping cubic terms in the fields.

We have thus derived Eqs. (1.1)–(1.3) quoted in the Introduction. Note that we have assumed $c_1' T_{111} > 0$. If this were not the case, we should have identified ϵ as $q_c - q$ instead. However, this would lead to $q > q_c$ being the interesting region of critical and tricritical behavior. We know that this is not the case. If we keep the higher order terms, Eqs. (2.21)–(2.23) yield two fixed points at

$$\psi_{\pm} = \pm(-\epsilon)^{1/2} + (\tilde{R}_{11}/2a)\epsilon + \cdots \quad , \tag{2.24}$$

$$\phi = h = 0 \quad , \tag{2.25}$$

and we interpret the upper (lower) signs as giving the tricritical (critical) fixed points for $\epsilon < 0$. The corresponding eigenvalues are

$$y_T(q)_{\pm} = y_T \pm b(-\epsilon)^{1/2} + [\tilde{R}_{22} - (b/2a)\tilde{R}_{11}]\epsilon + \cdots \quad , \tag{2.26}$$

$$y_H(q)_{\pm} = y_H \pm c(-\epsilon)^{1/2} + [\tilde{R}_{33} - (c/2a)\tilde{R}_{11}]\epsilon + \cdots \quad , \tag{2.27}$$

$$y'(q)_{\pm} = \pm 2a(-\epsilon)^{1/2} + \cdots \quad . \tag{2.28}$$

The last eigenvalues are the first nonleading ones in the thermal sector. Note that while our Eq. (2.21) gives no $O(\epsilon)$ correction to Eq. (2.28), these can arise through possible $O(\psi^3)$ terms on the right-hand side of Eq. (2.21). Such terms can also give additional contributions to the $O(\epsilon)$ coefficients in the equations above.

III. SCALING PREDICTIONS

We now examine the consequences of Eqs. (1.1)–(1.3) for the pure and dilute Potts models. First we evaluate the universal constants (y_T, y_H, a, b, c) by comparison with available information. For $\epsilon = q - 4 < 0$, the q dependence of the thermal and magnetic eigenvalues is given by Eqs. (2.26) and (2.27), respectively. Equation (2.26) agrees with the expansion for $q \lesssim q_c$ of the conjecture of den Nijs,[8] connecting the thermal exponent of the q-state Potts model with that of the eight-vertex model, if we take $y_T = \frac{3}{2}$ and $b = 3/(4\pi)$. Next we turn to the problem of the determination of the constants y_H and c in the expansion of the leading magnetic eigenvalue $y_H(q)$, Eq. (2.27). As noted previously, the plus and minus signs in Eqs. (2.26) through (2.28) give the tricritical and critical fixed points, respectively. To avoid confusion we will in this section explicitly label the magnetic eigenvalues using sub t for tricritical and sub c for critical. From the Ising model one knows that the critical index $y_H(2)_c = \frac{15}{8}$. The recent result of Baxter[13] that $\delta = 14$ for the hard hexagon model implies that $y_H(3)_c = \frac{28}{15}$. This model would appear to be in the same universality class as the three-state Potts mode. This is corroborated by the series work of Kogut, Pearson, and Shigemitsu[14] for the three-state Potts model which strongly suggests that $\beta = \frac{1}{9}$ and $\nu = \frac{5}{6}$ for $q = 3$, which implies that $\delta = 14$ by scaling. In addition to these exact pieces of information, there is the conjecture of Barber and Baxter[15] that $\delta = 15$ for the eight-vertex model. Relations[16] between the eight-vertex model, the Ashkin-Teller model, and the four-state Potts model then suggest that $\delta = 15$ for $q = 4$.

The problem is to make use of this information to obtain the expansion coefficients y_H and c. As a first try, we ignored the linear term in Eq. (2.27) and simply used a two-parameter fit to the Ising $y_H(2)_c = \frac{15}{8}$ and Baxter $y_H(3)_c = \frac{28}{15}$ values obtaining $y_H = 1.847$ and $c = -0.020$. The result for $y_H(q)$ is plotted as the curve labeled (I) in Fig. 4. A similar two-parameter fit for the thermal eigenvalue $y_T(q)$ leads to a 12% error in y_T, compared with the exact value of $\frac{3}{2}$ and even larger errors in the tricritical value $y_T(2)_+$. However, it does reproduce the general trend of the q dependence implied by the continua-

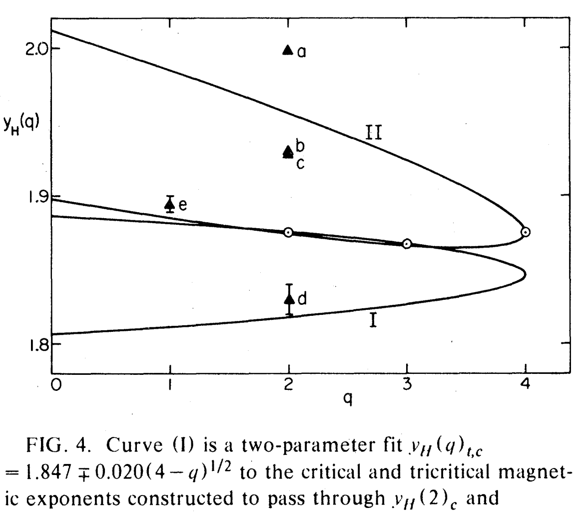

FIG. 4. Curve (I) is a two-parameter fit $y_H(q)_{t,c} = 1.847 \mp 0.020(4-q)^{1/2}$ to the critical and tricritical magnetic exponents constructed to pass through $y_H(2)_c$ and $y_H(3)_c$, while curve (II) is a three-parameter fit $y_H(q)_{t,c} = 1.875 \pm 0.0285(4-q)^{1/2} + 0.020(4-q)$ which also passes through the Baxter-Barber conjecture of $y_H(4) = \frac{15}{8}$. The data points a (Ref. 19), b (Ref. 20), c (Ref. 21), and d (Ref. 22) correspond to various calculations of the tricritical exponent $y_H(2)_t$ while e (Ref. 17) is a percolation value for $y_H(1)_c$.

tion of den Nijs's conjecture, with the tricritical value $y_T(q)_+$ lying above the critical value $y_T(q)_-$ as it should.

Now, our fit (I) for the magnetic exponent $y_H(q)$ fails to pass through the Baxter-Barber conjecture $y_H(4)=\frac{15}{8}$. In order to take this into account, we will keep the linear (q_c-q) term in Eq. (2.27), obtaining the three-parameter fit

$$y_H(q)_{t,c}=1.875\pm 0.0285(4-q)^{1/2}+0.020(4-q) \quad . \tag{3.1}$$

This gives $y_H=\frac{15}{8}$, $c=0.0285$ and is shown as the curve labeled (II) in Fig. 4.

Also shown in Fig. 4 are approximate results for the exponents of the percolation problem[17] $y_H(1)_c$ and for the tricritical point $y_H(2)_t$ of the Blume, Emery, and Griffiths model.[18] The approximate estimate of δ for the percolation problem gives a value of $y_H(1)_c$ (point e in Fig. 4) which is in fair agreement with both fits. It is along the tricritical branch that the fits (I) and (II) significantly depart from one another. Unfortunately the current estimates of the tricritical value $y_H(2)_t$ are inconsistent with each other and thus do not rule out either fit. The results of a d-3 expansion[19] (point a of Fig. 4) as well as real space renormalization group calculations[20,21] (points b and c of Fig. 4) favor fit (II), and are consistent with the Barber-Baxter conjecture that $y_H(4)=\frac{15}{8}$. On the other hand, the Monte Carlo estimate[22] $\delta=10.8\pm 0.7$ (point d of Fig. 4) favors fit (I), and would imply that $\delta\approx 12$ for the four-state Potts model, in disagreement with the Barber-Baxter conjecture.

After obtaining these results, we have learned that a curve similar to fit (II) has been recently obtained by Nienhuis *et al.*[23] In addition, Nienhuis *et al.*[23] and Pearson[24] have independently conjectured that

$$y_H=\frac{15+8x+x^2}{8+4x} \quad . \tag{3.2}$$

Here $x=2/\pi\cos^{-1}(\frac{1}{2}\sqrt{q})$, with $-1\leq x\leq 0$ corresponding to the critical phase and $0\leq x\leq 1$ to the tricritical phase. Expanding this near $q=4$, one obtains a curve similar to fit (II) with $y_H=\frac{15}{8}$ and $c=1/16\pi$. We will use these parameters in the same spirit as those obtained from the den Nijs conjecture.[8] As we will see, they give very simple exponents in the results which we will obtain.

There is one more piece of exact information available which can be used to fix the parameter a. This is Baxter's result for the latent heat[9] in the pure model for $q>4$. This was previously discussed in Ref. 7 and will be reviewed below. Following this, we proceed on the basis of our equations to derive a number of scaling laws valid at, or close to, $q=4$. The existence of the marginal field at $q=4$ leads to several interesting consequences, notably logarithmic corrections at $q=4$, and essential singularities in some quantities. Our results are summarized at the end of this section. Since the derivations tend to be rather similar, we describe in detail only a few of them, and comment on those which are physically most relevant.

The singular part of the free energy per site, f_s, satisfies the scaling relation

$$f_s(\psi(0),\phi(0),h(0))=e^{-dl}f_s(\psi(l),\phi(l),h(l)) \quad . \tag{3.3}$$

Here the dimensionality $d=2$. We now imagine, beginning with $h(0)=0$, close to the critical surface $[\phi(0)$ small$]$ in the pure model $[\psi(0)$ negative and $O(-1)]$. For $q>4$, if $\phi(0)=0$ initially, the RG flows go into the discontinuity fixed point. However, we choose to terminate the renormalization when $\psi(l)$ is positive and $O(1)$. At this point, cubic and higher order terms in the RG equations become important. However, $\psi(l)=O(+1)$ represents a strongly dilute system where we expect the latent heat to be $O(1)$ also. We therefore incorporate this as an assumption. Within a specific RG scheme, one can iterate all the way out to the vicinity of the discontinuity fixed point and calculate the latent heat. However, the iterations from $\psi(l)=O(+1)$ to the fixed point will give only an $O(1)$ renormalization of the latent heat. The scaling behavior which we wish to extract will be dominated by the flow in the vicinity of the multicritical fixed point. Similar arguments and assumptions must be made for all our other results, and we shall not repeat them. Ultimately, one can always appeal to a specific RG scheme.

Since ϕ is a temperaturelike field, the latent heat L is proportional to the discontinuity in the derivative $\partial f_s/\partial\phi(0)$ across the critical surface. By Eqs. (3.3), (1.1), and (1.2) this is

$$\begin{aligned} L(\psi(0)) &= \frac{\partial}{\partial\phi(0)}f_s(\phi(0),\psi(0),0)\Big|_{\phi(0)=0-}^{0+} \\ &= e^{-dl}\frac{\partial\phi(l)}{\partial\phi(0)}\frac{\partial}{\partial\phi(l)}f_s(\phi(l),\psi(l),0)\Big|_{\phi(l)=0-}^{0+} \quad . \end{aligned} \tag{3.4}$$

As long as $\delta l \ll 1$, we can approximate from Eq. (1.2)

$$\frac{\partial\phi(l+\delta l)}{\partial\phi(l)}=1+\delta l[y_T+b\psi(l)]+O(\delta l^2) \quad . \tag{3.5}$$

By iterating this infinitesimal transformation, Eq. (3.4) can be written

$$L(\psi(0))=e^{-dl}\exp\left(\int_0^l[y_T+b\psi(l')]dl'\right)L(\psi(l)) \quad . \tag{3.6}$$

We have argued that if we choose l such that

$\psi(l)=O(+1)$, then the last factor in Eq. (3.6) is $O(1)$. The exponent in Eq. (3.6) is easily evaluated by writing it as

$$\int_{\psi(0)}^{\psi(l)} \frac{(y_T-d+b\psi)d\psi}{d\psi/dl}=\int_{\psi(0)}^{\psi(l)} \frac{(y_T-d+b\psi)d\psi}{a(\psi^2+\epsilon)} \quad . \tag{3.7}$$

The integral is elementary and yields,[7] for $\psi(0)<0$,

$$L(\psi(0)) \propto |\psi(0)|^{-b/a} \exp[-(d-y_T)/a|\psi(0)|] \times \exp[-(d-y_T)\pi/a\sqrt{\epsilon}][1+O(\epsilon)] \quad . \tag{3.8}$$

Baxter's exact result[9] implies that $L \propto \exp -\pi^2/2\sqrt{\epsilon}$, so since $d=2$ and $y_T=\frac{3}{2}$ we conclude that $a=1/\pi$. For the pure model $|\psi(0)|=O(1)$, but we have explicitly kept the $\psi(0)$ dependence to show that, as $\psi(0)\to 0$, the coefficient of the essential singularity vanishes. For $\psi(0)>0$ we can repeat the calculation to show that the latent heat is smooth on passing through $q=4$. A similar calculation to the above, with $h(l)$ replacing $\phi(l)$, leads to a similar essential singularity in the jump in the spontaneous magnetization at $T=T_c$ but with y_T and b replaced by y_H and c, respectively, in Eq. (3.8).

We now give an example of a calculation for $q=4$. In Ref. 7, the temperature dependence of f_s in zero external field was evaluated. This gives rise to a logarithmic correction to the specific heat. We shall calculate the temperature dependence of the zero-field magnetization M as $T\to T_c^-$. An almost identical argument to that leading to Eq. (3.6) gives

$$M(\phi(0),\psi(0)) = e^{-dl}\exp\left[\int_0^l [y_H+c\psi(l')]dl'\right]M(\phi(l),\psi(l)) \quad . \tag{3.9}$$

If we now renormalize out to $\phi(l)=-1$, the last factor in Eq. (3.9) will be $O(1)$. At $q=4$,

$$\psi(l)=-\frac{1}{al-\psi(0)^{-1}} \quad , \tag{3.10}$$

so that

$$\ln\left[\frac{\phi(l)}{\phi(0)}\right]=y_T l-\frac{b}{a}\ln\left[\frac{al-\psi(0)^{-1}}{-\psi(0)^{-1}}\right] \quad , \tag{3.11}$$

where, of course $\psi(0)<0$. Solving Eq. (3.11) for l with $\phi(l)=-1$

$$l \cong -\frac{1}{y_T}\ln[\phi(0)] + \frac{b}{ay_T}\ln\left[\frac{-(a/y_T)\ln|\phi(0)|+[-\psi(0)]^{-1}}{[-\psi(0)]^{-1}}\right] \quad . \tag{3.12}$$

Then, integrating Eq. (3.10)

$$\int_0^l \psi(l')\,dl' = -\frac{1}{a}\ln\left[\frac{al-\psi(0)^{-1}}{-\psi(0)^{-1}}\right] \quad , \tag{3.13}$$

substituting in Eq. (3.9), and noting that $\phi(0)$ is analytically related to the reduced temperature t, we obtain

$$M(t,\psi(0)) \propto (-t)^{\beta}[1+(a/y_T)|\psi(0)|(-\ln|t|)]^{-(b\beta+c)/a} \quad . \tag{3.14}$$

Here, as usual,

$$\beta=(d-y_H)/y_T \quad . \tag{3.15}$$

Note that, in principle, it is possible for $\psi(0)$ to vanish, in which case there are no logarithmic corrections. This appears to be the case for the free energy in the Baxter-Wu model,[25] which is in the same universality class as the four-state Potts model.

For $q \leqslant 4$, the dilute model exhibits tricritical behavior, in the sense that in the (T,μ) plane there is a line of second-order transitions which become first order at a tricritical point. We can use our equations to find the singular behavior of the discontinuities of various quantities across the first-order line, as we approach the tricritical point, for $q \lesssim 4$.

As an example, we calculate the latent heat, by applying Eq. (3.6), beginning with $\psi(0) \gtrsim \epsilon' \equiv (4-q)^{1/2}$, and integrating out to $\psi(l)=O(+1)$. A similar integral to Eq. (3.7) leads to

$$L(\psi(0)) \propto \left[\frac{\psi(0)+\epsilon'}{\psi(0)-\epsilon'}\right]^{-(d-y_T)/2a\epsilon'} \times \{[\psi(0)+\epsilon'][\psi(0)-\epsilon']\}^{-b/2a} \quad . \tag{3.16}$$

We emphasize that our results are given to lowest order in $(|4-q|)^{1/2}$. Both the exponents and the coefficients have corrections which are linear in $(4-q)$ which we have not calculated. This discontinuity ΔM in the magnetization has the same form as Eq. (3.16) with the parameters (y_T,b) replaced by (y_H,c).

If we fix q and let $\psi(0)\to(4-q)^{1/2}$, the quantity $[\psi(0)-(4-q)^{1/2}]$ is a measure of the distance down the first-order line from the tricritical point, which is proportional to $T_c(\mu_{tr})-T_c(\mu)$. Thus we find

$$L \propto [T_c(\mu_{tr})-T_c(\mu)]^{\omega_1} \quad , \tag{3.17}$$

where

$$\omega_1 \cong \frac{(d-y_T)}{2a(4-q)^{1/2}} \quad . \tag{3.18}$$

On the other hand, if we first let $q\to 4$ in Eq. (3.16), we obtain,

$$L(\psi(0)) \propto \psi(0)^{-b/a}\exp[-(d-y_T)/a\psi(0)] \quad , \tag{3.19}$$

where now $\psi(0)$ is proportional to $T_c(\mu_{tr})-T_c(\mu)$. Since the chemical potential μ is also coupled to ϕ, the discontinuity $\Delta\rho$ in the concentration of vacancies behaves in a similar way. This has interesting consequences for the phase diagram of the dilute four-state Potts model. In the (T,ρ) plane, the region of phase separation should disappear as $T_c \rightarrow T_c(\mu_{tr})$ with a sharp cusp [see Fig. 3(a) in Sec. I], which should exhibit an essential singularity rather than a finite exponent ω_1. This is in striking contrast with the $q=2$ and 3 cases where $\omega_1<1$, [see Fig. 3(b)].

IV. SUMMARY OF RESULTS

Our results will be expressed in terms of the five universal constants (y_T, y_H, a, b, c). Then, following these general expressions which show the particular combinations of constants which enter, we will give the result which is found when we take the values for these parameters obtained from the exact and conjectured results: $y_T=\frac{3}{2}$, $y_H=\frac{15}{8}$, $a=1/\pi$, $b=3/4\pi$, and $c=1/16\pi$.

A. $q>4$

(i) The latent heat, L, at $T=T_c$ in the pure model varies as

$$L \propto \exp[-(2-y_T)\pi/a(q-4)^{1/2}] = \exp-[\pi^2/2(q-4)^{1/2}] \quad . \quad (3.20)$$

A similar result holds for the discontinuity $\Delta\rho$ in the number of vacancies, for the nearly pure model.

(ii) The discontinuity in the zero-field magnetization ΔM, at $T=T_c$, in the pure model varies as

$$\Delta M \propto \exp[-(2-y_H)\pi/a(q-4)^{1/2}] = \exp-[\pi^2/8(q-4)^{1/2}] \quad . \quad (3.21)$$

B. $q=4$

(i) The singular part of the free energy f_s, in zero field, versus reduced temperature t, in the pure model has, in addition to the usual power law dependence, a logarithmic correction

$$f_s \propto |t|^{2/y_T}(-\ln|t|)^{-2b/ay_T} = \frac{|t|^{4/3}}{(-\ln|t|)} \quad . \quad (3.22)$$

(ii) The zero-field magnetization M, for $t<0$, in the pure model also has a logarithmic correction

$$M \propto (-t)^{\beta}(-\ln|t|)^{-b\beta/a-c/a} = \frac{|t|^{1/12}}{(-\ln|t|)^{1/8}} \quad , \quad (3.23)$$

with $\beta=(2-y_H)/y_T$.

(iii) The magnetization M versus external field H at $T=T_c$ for the pure model varies as

$$M \propto |H|^{1/\delta}(-\ln|H|)^{-2c/ay_H} = \left(\frac{H}{\ln H}\right)^{1/15} \quad , \quad (3.24)$$

with $\delta = y_H(2-y_H)^{-1}$.

(iv) The spin-spin correlation function at the critical point for the pure model has a logarithmic correction to the usual power law decay

$$G(r) \propto r^{-\eta}(\ln r)^{-2c/a} = r^{-1/4}(\ln r)^{-1/8} \quad , \quad (3.25)$$

where $\eta = 2(2-y_H)$. This should be contrasted with the $r^{-1/4}(\ln r)^{1/8}$ behavior of the correlation function of the two-dimensional xy model found by Kosterlitz.[26]

(v) For the dilute model, the latent heat L, and the discontinuity $\Delta\rho$ in concentration, versus distance

$$\bar{t} = [T_c(\mu_{tr})-T_c(\mu)]/T_c(\mu_{tr})$$

along the first-order line exhibits an essential singularity

$$L \propto \Delta\rho \propto e^{-(2-y_T)/aA\bar{t}} \quad . \quad (3.26)$$

Here A is nonuniversal.

(vi) Similarly, for the dilute model, the discontinuity in magnetization ΔM vs $\bar{t}$ is

$$\Delta M \propto e^{-(2-y_H)/aA\bar{t}} \quad , \quad (3.27)$$

with A the same as in Eq. (3.26).

C. $q<4$

(i) For the dilute model, the latent heat L and the discontinuity $\Delta\rho$ versus distance along the first-order line $\bar{t}$ have the power law form

$$L \propto \Delta\rho \propto \bar{t}^{-\omega_1} \quad . \quad (3.28)$$

Expanding about $q_c=4$, the leading behavior of ω_1 is given by

$$\omega_1 = \frac{(2-y_T)}{2a(4-q)^{1/2}} = \frac{\pi}{4(4-q)^{1/2}} \quad . \quad (3.29)$$

Here, as previously discussed, there are corrections of order $(4-q)$.

(ii) Similarly, for the dilute model, the discontinuity in magnetization ΔM vs $\bar{t}$ varies as

$$\Delta M \propto \bar{t}^{-\omega_2} \quad (3.30)$$

and to leading order

$$\omega_2 = \frac{(2-y_H)}{2a(4-q)^{1/2}} = \frac{\pi}{16}\frac{1}{(4-q)^{1/2}} \quad . \quad (3.31)$$

V. CONCLUSIONS

Here we have studied the structure of the logarithmic and essential singularities which occur when a line of critical fixed points merges with a line of tricritical fixed points. Logarithmic singularities are also known to occur at an upper critical dimension. Some examples are the critical point in four dimensions,[27] the tricritical point in three dimensions,[28] and the critical behavior of a three-dimensional uniaxial ferromagnet with strong dipolar coupling.[29] In these cases, however, there is a crossing of a line of nontrivial fixed points with a line of Gaussian fixed points. Below the upper critical dimension the nontrivial fixed points are stable and the Gaussian fixed points unstable, while above the critical dimensionality they exchange roles. In the case we have considered, the lines form two branches of a parabola, merging at a critical value of a parameter q_c.

In developing this theory we have used an approach which proceeds in three stages: (i) based on a knowledge of the important physical variables and their marginal or relevant character, the form of the RG equations for a set of nonlinear scaling fields are constructed; (ii) the coefficients in these equations are fit to exact, conjectured or experimental results; (iii) these RG equations are used to describe the local scaling behavior near the multicritical point and supplemented by physical conditions outside this region.

For the particular problem treated here, we believe that the parameters obtained from Baxter's latent heat result,[9] den Nijs's thermal exponent conjecture,[8] and the recent Nienhuis *et al.*[23] and Pearson[24] conjecture for the magnetic exponent are in fact exact. Thus this approach offers a way of obtaining additional exact results which may be very difficult to obtain by direct calculation. These results also bear on other two-dimensional systems such as the Askin-Teller[12] model and the planar *xy* model.[26] They should also be useful in providing information necessary in designing Pade approximate procedures and in organizing Monte Carlo calculations. Finally, we hope that these results will give further impetus to the experimental search for a submonolayer adsorbed gas system which can provide a physical realization of the four-state Potts model.

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