

Singularities near the bifurcation point of the Ashkin-Teller model

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Singularities in the $d = 2$ Ashkin-Teller model are analyzed by writing down flow equations based upon the Gaussian model in the presence of suitable fields. A scaling function for the coherence length, ξ , is set up in terms of coordinates δ and t which describe deviations from the four-state Potts point. At that point, ξ depends on the reduced temperature as $\xi^{-1} \sim |t|^{2/3} \times |\ln t|^{-1/2}$.

The Ashkin-Teller (AT) model is known to have¹ a bifurcation of the form shown in Fig. 1(a) in which the line of continuously varying critical points which occurs in region I breaks up into two Ising critical lines. In this plot $t = 2c_{AT} + b_{AT} - 1$ measures deviations from the self-dual line while $x_{AT} = 3 - 2b_{AT}/c_{AT}$ measures positions on that line, with $x_{AT} = 1$ being the four state Potts point. Figure 1(b) shows how the AT model might be considered to be a slice in a diagram for a Kosterlitz-Thouless multicritical point² of a generalized Gaussian model (GM).³ The specific mapping⁴ is one in which v (the fugacity of the Kosterlitz-Thouless model) describes an $n = \pm 4$ spin-wave excitation, t is a field for an $n = \pm 2$ excitation, and x describes the GM coupling via $x = -2 + 8/(2\pi K)$. The Hamiltonian density for the model is then $\frac{1}{2}K(\nabla\phi)^2 + t \cos 2\phi + v \cos 4\phi$. In this representation, $\cos\phi$ describes the product of the two spins in the AT model, $\sigma^{(1)}\sigma^{(2)}$. The lowest-order vortex excitations are left out because they represent a breaking of the symmetry between the spins $\sigma^{(1)}$ and $\sigma^{(2)}$. Higher-order vortex excitations are neglected because they are irrelevant in the interesting range $x \approx 0$.

Near the multicritical point, at $x = v = t = 0$, one can write approximate renormalization-group (RG) flow equations⁵

$$\frac{dx}{dl} = -v^2, \tag{1a}$$

$$\frac{dv}{dl} = -xv, \tag{1b}$$

$$\frac{dt}{dl} = \left(\frac{3}{2} - \frac{1}{4}x - \frac{1}{2}v\right)t. \tag{1c}$$

The derivation of these equations is given in the Appendix. The first two of these equations provide the standard description of flows near the Kosterlitz-Thouless phase transition.⁶ These equations may be expected to be universal features which reappear in the many different examples⁷ of bifurcation of critical

lines in two-dimensional systems. Thus, for example, these equations should describe equally well the AT model's bifurcation, the planar model's infinite-order critical point, the q -state Potts model⁸ near $q = 4$, the $\beta^2 \rightarrow 8\pi$ limit of the sine-Gordon theory, etc. However, in each case these flow equations must be supplemented by additional relations like the last member of the set (1), which describe the particular field which occurs in the individual problem of

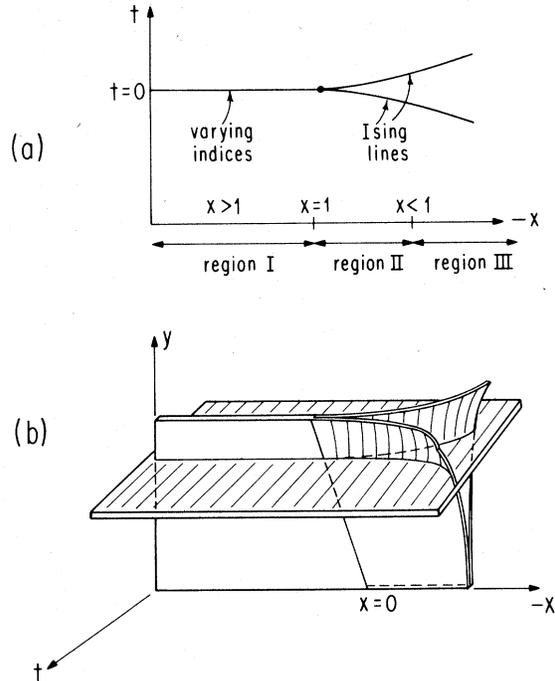


FIG. 1. Phase diagrams. In (a), we have the standard AT model phase diagram. The horizontal line at $t = 0$ is self-dual; the two branches to the right of the bifurcation have an Ising character. In (b) this phase diagram is shown inserted as the $v = 1$ plane in the xvt space of the generalized GM.

interest. For the AT case, the particular field is t , the reduced temperature which describes the deviation from self-duality. Reference 5 describes the calculation of the critical index for this quantity and this implies a flow equation of the form listed in Eqs. (1). Equations (1) are not accurate for the real AT model, which is roughly the $y \rightarrow 1$ limit, but they do describe in a semiquantitative fashion what happens to flows near the origin. Later on, we shall have to supplement these equations by statements about behavior for large x , y , and t .

The standard RG flow methods determine a scaling form for the coherence length, ξ , starting from Eqs. (1). The result is

$$\begin{aligned} \xi^{-1} &= \frac{t^{2/3}}{y^{1/6}(x+y)^{1/3}} M(\delta, \tau) \\ &= (\exp - U) M'(\delta, \tau) . \end{aligned} \quad (2)$$

Here δ and τ are the two invariants which do not change in the course of the flows of Eqs. (1). They are, $\delta = x^2 - y^2$, which is an invariant measure of the distance from the multicritical point measured parallel to the line of continuously varying exponents and

$$\tau = ty^{-1/4}(|x+y|)^{-1/2} \exp(-3U/2) ,$$

which is an invariant version of the orthogonal coordinate which measures $T - T_c$ near this line. Here U takes three different forms depending upon the region of the diagram

$$\begin{aligned} &\frac{1}{\sqrt{\delta}} \sinh^{-1} \frac{\sqrt{\delta}}{y} , \quad \text{region I} , \\ &\frac{1}{\sqrt{-\delta}} \left[\sin^{-1} \frac{\sqrt{-\delta}}{y} - \pi \right] , \quad \text{region II} , \\ &-\frac{1}{\sqrt{\delta}} \sinh^{-1} \frac{\sqrt{\delta}}{y} , \quad \text{region III} . \end{aligned} \quad (3)$$

These regions (see Fig. 18 of Ref. 3), respectively, are described by $\delta > 0$, $x > 0$ (region I); $\delta < 0$ (region II); $\delta > 0$, $x < 0$ (region III). To make these regions fit together smoothly, choose the branch of $\sin^{-1} \sqrt{-\delta}/y$ to give π as $\delta \rightarrow 0^-$, $x < 0$ and consequently to give zero as $\delta \rightarrow 0^-$, $x > 0$.

So far the calculation is very similar to a parallel analysis carried out by Nauenberg and Scalapino,⁸ who were interested in the analogous bifurcation of the q -state Potts model. The two problems differ in the structure of fixed points far from $x = y = t = 0$, a difference which must be put in "by hand."

By looking at the structure of the phase diagram in Fig. 1, one concludes that the only singularities in ξ must occur at $\tau = 0$ in region I and $\tau = \pm \tau_c(\delta)$ in regions II and III. In region III, $\tau_c(\delta)$ is of order unity and is analytic in δ . But, since there are no singularities in passing from region II to region III, M must

have the same form in both regions, and therefore the same condition for criticality holds. As $\delta \rightarrow 0$, toward the left-hand end of region II, the Ising critical temperatures are therefore $\pm t_c(\delta)$, with⁹

$$\begin{aligned} t_c(\delta) &= \tau_c(\delta) (4y)^{-1/4} |\delta|^{1/2} \exp[-(3\pi)/(2\sqrt{-\delta})] \\ &\sim (1 - x_{\text{AT}})^{1/2} \exp[-\frac{3}{2} B / (1 - x_{\text{AT}})^{1/2}] . \end{aligned} \quad (4)$$

In the second line of Eq. (4), we have expressed our result in terms of AT model parameters, i.e., $y \sim 1$, $\delta = -(\pi^2/B^2)(1 - x_{\text{AT}})$, where B is a constant to be determined. By the same logic, one concludes that for $y \sim 1$ in region III $\xi^{-1}(\delta, \tau)$ is expandable in a power series in δ and τ^2 in which all the coefficients are of order unity. Hence, so is $M'(\delta, \tau)$. Near the left-hand end of region II, if $t = 0$ and $\delta \rightarrow 0^-$, we find $\xi^{-1} = M'(0, 0) e^{\pi/\sqrt{-\delta}}$. Consequently, in the AT model, if $t = 0$ and $x_{\text{AT}} \rightarrow 1^-$, $\xi \sim \exp[B / (1 - x_{\text{AT}})^{1/2}]$.

To find B , notice that the critical index ν is directly given by Eq. (1) as $\nu^{-1} = \frac{3}{2} - \sqrt{\delta}/4$, when $y = 0$.

Therefore, in region I, in AT model language $\nu^{-1} = \frac{3}{2} + \pi(x_{\text{AT}} - 1)^{1/2}/4B$ as $x_{\text{AT}} \rightarrow 1^+$. From Ref. 4, we then conclude $B = \sqrt{2}\pi^2/4$.

Finally, focus upon region I in the limit $\delta \rightarrow 0$. For fixed t and y this limit must exist and give a scaling function of the form

$$\xi^{-1} = \frac{t^{2/3}}{y^{1/2}} M_0(\tau); \quad \tau = \frac{t}{y^{3/4}} e^{-3/2y} . \quad (5)$$

But for fixed t , as $y \rightarrow 0$, ξ should be finite and non-singular in y . Thence, for small τ

$$M_0(\tau) \sim (\ln \tau + \frac{3}{4} \ln \ln \tau + \dots)^{1/2} .$$

Hence, for $\delta = 0$ and small t , the singular part of the free energy is

$$f_{\text{sing}} \sim \xi^{-2} \sim \frac{t^{4/3}}{[-\ln t + (\frac{3}{4}) \ln(-\ln t) + \dots]} . \quad (6)$$

Equation (6) is an AT model result which holds at $x_{\text{AT}} = 1$. Notice the extra factor which appears in Eq. (6) beyond the $t^{4/3}$, which is expected from naive scaling. These extra terms perhaps explain why series¹⁰ and real-space renormalization-group¹¹ calculations do not give numerical evidence for the $\alpha = \frac{2}{3}$ expected¹² in the four-state Potts model.

Equation (6) was first derived in the context of Potts model studies.⁸ Since the AT model and the q -state Potts model exactly reduce to one another when, respectively, $x_{\text{AT}} = 1$ and $q = 4$, it is gratifying to notice that the flow equations exactly reduce to those used in Ref. 8 when the appropriate limits are taken.

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APPENDIX: DERIVATION OF FLOW EQUATIONS

To derive Eqs. (1), reach back to Ref. 3, Eqs. (5.17). In the notation of the earlier work, t is proportional to v_2 , v to v_4 , while the meaning of K remains unchanged. However, the earlier paper also contained a vorticity field, v_0 , which we do not need in the AT model calculation but which will be useful in our derivation. After suitable scale changes, the flow equations read

$$\begin{aligned} \frac{dx}{dt} &= -v^2 + v_0^2 + At^2, \\ \frac{dy}{dt} &= -xy + Bt^2, \\ \frac{dt}{dt} &= \left(\frac{3}{2} - \frac{1}{4}x\right)t + Ct^2, \\ \frac{dv_0}{dt} &= xv_0. \end{aligned} \quad (\text{A1})$$

In writing this result, one assumes that the equations in these variables are indeed analytic at the multicritical point $x - v = t = v_0 = 0$. These equations now include on the right-hand side all the quadratic terms which are permitted by the symmetry of the problem. We do not need A and B for our analysis. The value of C is crucial however.

In the real AT model, the vorticity v_0 is zero. To see this one refers back to the mapping of Ref. 4 and

notices that in the AT model v_0 actually refers to a half-integral vorticity field coupled in part to a disorder variable, and hence is not present in the usual Hamiltonian for this model.

Nonetheless this field is useful because it permits one to use the techniques of Ref. 5 to describe the structure of the multicritical point. Entering this point there are several fixed lines, one at very small v , v_0 , and t but fixed x having linearized flow equations

$$\begin{aligned} \frac{dy}{dt} &= -xy, \\ \frac{dv_0}{dt} &= xv_0, \\ \frac{dt}{dt} &= \left(\frac{3}{2} - \frac{1}{4}x\right)t. \end{aligned} \quad (\text{A2})$$

Another fixed line occurs at fixed values $v + v_0$ and small values of $v - v_0$, x , and t . For this fixed line, the linearized flows read

$$\begin{aligned} \frac{d}{dt}(x + v - v_0) &= -(v + v_0)(x + v - v_0), \\ \frac{d}{dt}(x - v + v_0) &= (v + v_0)(x - v + v_0), \\ \frac{dt}{dt} &= \left[\frac{3}{2} + \frac{1}{2}C(v + v_0)\right]t. \end{aligned} \quad (\text{A3})$$

According to the work of Ref. 5, the fixed lines described by Eqs. (A2) and (A3) are essentially identical, or rather can be mapped into one another under the transforms

$$\begin{aligned} v &\rightarrow x + v - v_0, \\ v_0 &\rightarrow x - v + v_0, \\ t &\rightarrow t, \\ x &\rightarrow v + v_0. \end{aligned} \quad (\text{A4})$$

To make this mapping hold, the constant C must be equal to -0.5 . To complete the derivation of Eqs. (1), set $v_0 = 0$ and neglect terms of order t^2 in Eqs. (A1).

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