Flux-lattice melting in thin-film superconductors

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(Received 3 April 1980)

Melting of flux lattices in thin-film type-II superconductors is analyzed, and the phase diagram as a function of magnetic field is discussed. Several experimental consequences are explored, in particular the flux-flow resistance of inhomogeneous thin films.

Beginning with Abrikosov's calculations on type-II superconductors in a magnetic field,[1] it has become clear that in a certain range of magnetic fields and temperatures, there exists a triangluar flux lattice consisting of regions of superconductor interspersed with a regular array of vortex lines, each containing one quantum of flux

$$\Phi_0 = hc/2e \quad . \tag{1.1}$$

Several authors[2,3] have shown that triangular vortex lattices are stable and have a well-defined shear modulus. They have been observed experimentally, both directly[4] and through a novel interference experiment performed by Fiory[5,6] from which the shear modulus of vortex lattices in thin films can be obtained.

In most of the previous work on flux lattices, the effects of thermal motion of the vortices have been ignored. In this paper we investigate the effects of finite temperature fluctuations on the properties of flux lattices and flux flow in thin films. In particular we show that above a melting temperature, T_M, vortices will no longer form a lattice but will rather form a fluid with considerably different properties. Doniach and Huberman[7] have recently obtained some results on flux-lattice melting independently, several discrepancies between their results and those of this author will be discussed.

In this paper, we emphasize some of the experimental consequences of flux-lattice melting, in particular the effects of vortex pinning. We first review the properties of vortices in films and recent results for films in zero magnetic field.[8–10] In Sec. II Kosterlitz-Thouless[11,12] melting of vortex lattices and the resulting phase diagram as a function of temperature and magnetic field are discussed. In Sec. III we discuss the effects of pinning, and in Sec. IV we analyze the resistive properties of thin films in a magnetic field. Finally in Sec. V we consider measurements of the shear modulus near the vortex-lattice melting temperature. The Appendix deals with more accurate estimation of the melting temperature, T_M.

I. VORTICES

In a two-dimensional film geometry, it is not possible to exclude flux in a magnetic field perpendicular to the film.[3] Thus the lower critical field, H_{c1}, vanishes, and in the absence of pinning there will be a finite resistance due to flux flow in any nonzero field. [Doniach and Huberman[7,8] define a nonzero $H_{c1}(T)$ which depends on the sample diameter D and vanishes as $D \to \infty$. At $T = 0$ in a *finite* film there will be flux exclusion for small enough fields; however at any nonzero temperature there will sometimes be vortices in the film due to thermal fluctuations in any field. The average vortex density will thus change smoothly as a function of field for $T > 0$ and there will be *no* phase transition and no H_{c1}, in contrast to statements and the figures in Refs. 7 and 8.]

Several authors[8–10] have recently pointed out that in *zero-field,* on the other hand, there can be a superconducting-normal transition analogous to the Kosterlitz-Thouless[13] transition in thin ^{4}He films. They predict that below a temperature, T_c, there are no free vortices and the resistance vanishes. Above T_c thermally excited vortices appear and the resistance becomes finite.

While strictly speaking, this transition will occur only in the limit of infinitely thin films, in practice the resistance will be immeasurably small below the temperature, T_c, implicitly determined by

$$T_c = \Phi_0^2/16\pi^2\Lambda(T_c) \quad , \tag{1.2}$$

where $\Lambda(T) \equiv 2\lambda_B^2(T)/d$ is an effective two-dimensional penetration depth (typically on the order of 1 cm at T_c) with $\lambda_B(T)$ the bulk penetration depth and d the film thickness.

In thin films where $d << \lambda_B$, the super current density and vector potential are uniform through the thickness of the films and the vortices interact at separation $r >> \xi$ [where $\xi(T)$ is the temperature-dependent Ginzburg-Landau coherence length] with a potential (for vortices of the same sign)[14]

$$V(r) = \frac{\Phi_0^2}{8\pi\Lambda(T)}\left[H_0\left(\frac{r}{\Lambda(T)}\right) - Y_0\left(\frac{r}{\Lambda(T)}\right)\right] \quad , \tag{1.3}$$

where H_0 is a Struve function and Y_0 a Neumann function. At very long distances $r \gg \Lambda$,

$$V(r) \simeq \frac{\Phi_0^2}{4\pi^2}\frac{1}{r} \tag{1.4}$$

and at intermediate distances,

$$\xi(T) \ll r \ll \Lambda(T) \ ,$$
$$V(r) \simeq -\frac{\Phi_0^2}{4\pi^2\Lambda}\ln\left|\frac{r}{\xi}\right| \ . \tag{1.5}$$

We note that in this intermediate regime, the vortex-vortex interactions are proportional to their length, d. Thus, as the film gets thinner, the interactions get weaker. It is this logarithmic interaction out to distances of the order of Λ (which diverges as $d \to 0$) which is responsible for the Kosterlitz-Thouless transition in zero magnetic field.

At high magnetic fields, $B \leqslant H_{c2}(T)$, where H_{c2} is the *bulk* upper critical field, the approximation that the vortices are far apart compared to $\xi(T)$ breaks down, the short distance form of the interaction $(r \sim \xi)$ becomes important and it no longer has a simple form.

II. FLUX-LATTICE MELTING AND PHASE DIAGRAM

In this paper we consider a film in a nonzero perpendicular field B with an areal denstiy of free vortices of one sign given by

$$n = B/\Phi_0 \ . \tag{2.1}$$

At low temperatures these vortices will form a triangular lattice with lattice spacing, a_0, given by

$$(\tfrac{3}{4})^{1/2}a_0^2 = 1/n \ . \tag{2.2}$$

Kosterlitz and Thouless[11,12] have noted that any two-dimensional lattice becomes unstable to the formation of dislocations above a temperature, T_M, given in terms of the two-dimensional Lamé coefficients, $\mu(T)$ and $\lambda(T)$,[15] by

$$\frac{\mu(T_M)[\mu(T_M)+\lambda(T_M)]a_0^2}{T_M[2\mu(T_M)+\lambda(T_M)]} = 4\pi \ . \tag{2.3}$$

They predict that the lattice will melt via a second-order transition at T_M and the shear modulus $\mu(T)$ (which is nonzero for $T \leqslant T_M$) will drop discontinuously to zero.

In the case of a vortex lattice, λ is infinite at $T=0$ due to the long-range vortex-vortex interactions (i.e., the system is incompressible). At finite temperature it will be renormalized to a large but finite value by the presence of defects. However, if we ignore this small effect, the criteria for melting [Eq. (2.3)] can be approximated by

$$\mu(T_M)a_0^2/T_M = 4\pi \ . \tag{2.4}$$

To determine the melting curve it is thus necessary to calculate the shear modulus of the vortex lattice.

Several authors have calculated μ in various regimes which we now consider[2,16]: (1) intermediate fields $a_0 \ll \Lambda$ but $B \ll H_{c2}(T)$, where H_{c2} is the *bulk* upper critical field, (2) high fields $a_0 \ll \Lambda$ and $B \lesssim H_{c2}(T)$, and (3) extremely low fields $a_0 \gg \Lambda$.

All the previous calculations[2,16] are valid only in the limit that $T \ll T_M$. At finite temperatures, $T \lesssim T_M$, the shear modulus will be renormalized by nonlinear lattice vibrations[17] and defects, notably dislocation pairs.[11,12,18] We can estimate the melting temperature for the vortex lattice, however, by use of the calculated "bare" shear modulus μ_0 ignoring these effects, and a correction of the order of unity which takes into account these renormalizations.

We consider first the most interesting case.

1. *Intermediate fields*

In the intermediate-field regime,

$$\frac{1}{\xi^2(T)} \gg \frac{B}{\Phi_0} \gg \frac{1}{\Lambda^2(T)} \tag{2.5}$$

it is found that the bare shear modulus (ignoring thermal fluctuations of the flux lattice) is[2]

$$\mu_0(T) = \frac{\Phi_0^2}{4\pi^2\Lambda(T)}\frac{1}{8}\frac{B}{\Phi_0} \tag{2.6}$$

and T_M is hence *independent of magnetic field* and is implicitly determined by

$$T_M = \frac{1}{2\pi\sqrt{3}}\frac{1}{8}\frac{\Phi_0^2}{4\pi^2\Lambda(T_M)}A_1 \ , \tag{2.7}$$

where A_1 is a constant of the order of unity (for particles interacting with a logarithmic potential) which arises from renormalization of the elastic constants at finite temperatures from their "bare" valves $\mu_0(T)$ and $\lambda = \infty$ due to nonlinear lattice vibrations[17] and defects[11,12,18].

$$A_1 = \frac{\mu(T_M)[\mu(T_M)+\lambda(T_M)]}{[2\mu(T_M)+\lambda(T_M)]\mu_0(T_M)} \ . \tag{2.8}$$

In the Appendix we discuss estimation of the renormalization constant A_1 and conclude it probably lies in the range

$$0.4 \leqslant A_1 \leqslant 0.75 \ . \tag{2.9}$$

The effects of renormalization of the shear modulus have not been discussed by Doniach and Huberman[7]; they assume $A_1 = 1$.

As Beasley *et al.*[10] have noted, it is useful to express the transition temperatures in terms of experimentally measurable quantities. In the dirty limit[19]

$$\Lambda(T)=3.56\frac{\Phi_0^2}{4\pi^2}\frac{R_n}{R_c}\frac{1}{T_{c0}}f^{-1}\left(\frac{T}{T_{c0}}\right)\;, \tag{2.10}$$

where R_n is the normal-state sheet resistance, $R_c=\hbar/e^2=4.12\ \mathrm{k}\Omega/\Box$, T_{c0} is the *bulk* BCS transition temperature and

$$f\left(\frac{T}{T_{c0}}\right)=\frac{\Delta(T)}{\Delta(0)}\tanh\frac{\Delta(T)}{2T}$$
$$\simeq 2.66\left(\frac{T_{c0}-T}{T_{c0}}\right)\ \text{ as } T\to T_{c0}\ . \tag{2.11}$$

As R_n increases (i.e., the film gets thinner), the melting temperature T_M, decreases continuously from T_{c0}; for small R_n in the experimentally important intermediate-field regime,

$$\frac{T_M}{T_{c0}}\simeq\left(1+\frac{3.8}{A_1}\frac{R_n}{R_c}\right)^{-1}\ . \tag{2.12}$$

This will typically be valid for fields down to small fractions of a milligauss. We can compare this with the resistive "transition" temperature in zero field (marked by an × in the figure[10])

$$\frac{T_c}{T_{c0}}\simeq\frac{1}{1+0.173R_n/R_c}\ . \tag{2.13}$$

(We note that the coefficient 0.173 above should probably be somewhat larger due to renormalization effects.[9])

In the limit of extremely high R_n, on the other hand, Λ will be temperature independent near T_M and T_c and we find that

$$\frac{T_M}{T_{c0}}\simeq 0.10A_1\frac{R_c}{R_n} \tag{2.14}$$

and

$$T_c(B=0)=(4\pi\sqrt{3}/A_1)T_M\ . \tag{2.15}$$

We now turn to the high magnetic field regime.

2. *High fields*

In the high-field limit,

$$B\leqslant H_{c2}(T)\ , \tag{2.16}$$

the melting temperature is determined implicitly by

$$T_M=A_2\frac{1}{2\pi\sqrt{3}}0.353\,\frac{1}{4}\left(\frac{H_{c2}(T_M)-B}{H_{c2}(T_M)}\right)^2\frac{\Phi_0^2}{4\pi^2\Lambda(T_M)}\ , \tag{2.17}$$

where A_2 is again a renormalization coefficient of the order of unity [analogous to A_1, see Eq. (2.8)] which can in this case depend weakly on B. From Eq. (2.17) we see that at low temperatures and high fields the melting curve has the form

$$T_M\propto[H_{c2}(0)-B]^2 \tag{2.18}$$

(which we note is incorrectly sketched in the figure in Ref. 7).

It should be emphasized that in thin films there is *no* phase transition at the bulk $H_{c2}(T)$; there is just a gradual crossover to normal metallic behavior. Its place is taken by the flux-lattice melting transition. As can be seen from the above discussion, as the film gets thicker the melting line $T_M(B)$ will approach the bulk $H_{c2}(T)$.

3. *Very low fields*

In the very low-field limit,

$$B<<\Phi_0/\Lambda^2(T)$$

we find that

$$T_M=A_3\frac{1}{4\pi}(0.245)\left(\frac{B}{\Phi_0}\right)^{1/2}\frac{2}{\sqrt{3}}\frac{\Phi_0^2}{4\pi^2\Lambda(T_m)}\ , \tag{2.19}$$

where A_3 is again a constant of the order of unity, appropriate to particles interacting with a $1/r$ potential. For this case A_3 can be estimated by analogy with experiments, numerical and analytical results on the two-dimensional electron crystal[17,20–22]:

$$A_1\simeq 0.6\ . \tag{2.21}$$

This regime is almost inaccessible experimentally and for the rest of this paper we will restrict ourselves to fields much larger than Φ_0/Λ^2.

A sketch of the vortex-lattice melting curve as a function of B is shown as a solid line in the figure. The very low-field region is not shown—it would, in any case, not be visible on this scale.

A. Hexatic phase

In addition to the vortex-lattice melting transition in a finite magnetic field, there will be another transition at higher temperatures.

Halperin and Nelson[12] have shown that if any two-dimensional solid melts with a second-order transition via dislocation unbinding, it will not do so directly into the isotropic liquid, but rather it will melt to an anisotropic or "hexatic" liquid-crystal-like phase. This phase will be characterized by the presence of free dislocations (and hence exponential decay of positional order) but only *bound* pairs of disclinations. Halperin and Nelson[12] define a bond-orientation or-

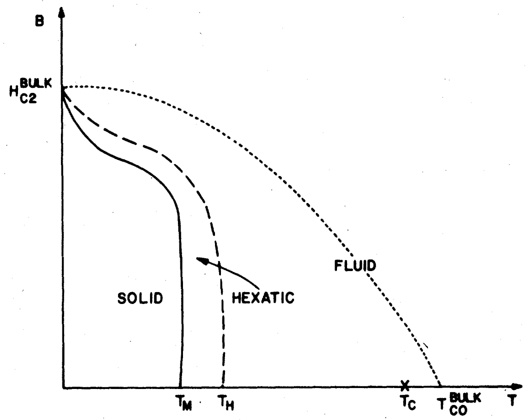

FIG. 1. Phase diagram of a thin-film superconductor showing vortex solid, hexatic, and fluid phases. The solid line is the flux-lattice melting curve (T_M) and the dashed line the hexatic-isotropic fluid phase boundary (T_H). The resisitive transition temperature, T_c, in zero field, is marked by an ×. For comparsion, the *bulk* upper critical field $H_c^2(T)$ is shown as a dotted line, and the *bulk* transition temperature, T_{c0}, is also marked.

der parameter $\psi(r)=\exp[6i\theta(r)]$ where θ is the "bond-angle" between an atom (or in our case a vortex) and its neighbor. This order parameter will have long-range order in the solid phase and power-law decays in the hexatic phase. At a temperature T_H the disclinations will unbind and for $T > T_H$ there will be an isotropic liquid with exponential decay of all correlation functions.

In our case, the phase boundary $T_H(B)$ will lie above $T_M(B)$ and roughly parallel it (see Fig. 1). Though its exact location is hard to predict theoretically, its position can be crudely estimated if the dislocation core energy, E_{DC}, is known (see Appendix). Unfortunately, as in other experimental systems, there is no known probe which couples directly to the hexatic order parameter ψ and hence the hexatic-isotropic transition will be very hard to observe experimentally.

III. PINNING

So far we have considered only homogeneous films. However, in a real film there are always (at least) small scale inhomogeneities which can act to pin vortices by interactions with their normal cores, and the large current densities near these cores.

In this section we will be concerned with pinning of vortices due to small scale inhomogeneities. The resulting pinning potential, $U_p(\vec{r})$ felt by a vortex can be characterized by the correlation function between pinning potentials at different places[6]

$$\Gamma_U(\vec{r}-\vec{r}\,')=\langle U_p(\vec{r})U_p(\vec{r}\,')\rangle \ , \tag{3.1}$$

where the angle brackets indicate averaging over the distribution of pinning potentials. If a film consists of randomly deposited grains of size b_{gr} with only extremely short-ranged ($\sim b_{gr}$) correlations in their positions, (as has been asserted for many different thin-film systems but may in fact not be correct for *any* of them), then $\Gamma_U(r)$ will be zero for $r \gtrsim b_{gr}$. (Note: It will also have this form for any inhomogeneity with only short-range correlations.) For *simplicity* we will consider this to be the case of the purposes of this paper. However, there is at least preliminary evidence that there are in fact long-range correlations in the pinning potential.[23] The effects of correlations will be dicussed in a future paper.

If, as we will assume henceforth in this paper, the correlations in the pinning potential are very short ranged, most of the contribution to the pinning will be from the interaction of normal vortex cores with the small scale inhomogeneities. The maximum core pinning potential, U_p, possible can be obtained by punching a hole in the film[24] of radius ξ which results in

$$U_p^{\max}\simeq(H_c^2/8\pi)(\pi\xi^2)d=\frac{\Phi_0^2}{32\pi^2\Lambda(T)}=\frac{\Lambda(T_c)}{\Lambda(T)}\frac{T_c}{2} \ . \tag{3.2}$$

In the intermediate-field regime (which we will assume henceforth) this yields

$$U_p^{\max}(T_M)\simeq(2\pi\sqrt{3}/A_2)T_M \ . \tag{3.3}$$

Often, however, the pinning potential will be considerably smaller than $U_p^{\max}$. Granular aluminum films (with which several previous experiments on flux flow have been carried out[5,24,25]) typically have inhomogeneities on a scale, b_{gr}, considerably less than the coherence length. In this case typical values $\tilde{U}_p$, of the pinning potential will be

$$\tilde{U}_p\simeq U_p^{\max}(b_{gr}^3/\pi\xi^2 d)^{1/2}\simeq(T_{c0}-T)^{-3/2} \tag{3.4}$$

and the typical pinning force on a vortex will be

$$\tilde{U}_p/\xi\sim(T_{c0}-T)^{-2} \ . \tag{3.5}$$

If $R_n \lesssim 100\ \Omega$ so that T_M is relatively close to T_{c0} and $\xi(T_M)$ is sufficiently large compared to b_{gr} (or the film is relatively thick), $U_p(T_M)$ can be of the order of T_M.

At the other extreme, however, for films in which the pinning has correlations on scales much larger than ξ, there are large contributions to the pinning (and nonlinear effects due to the possible presence of other vortices nearby) due to the dependence on film thickness of the energy of the large currents around the core.

In this limit, the pinning potential can be considerably larger than the $U_p^{\max}$ [Eq. (3.2)] arising from the cores. Consideration of this limit may be necessary to explain flux-flow resistance data[23,25] (see Sec. IV). While it might be expected that even pinning potentials with $\tilde{U}_p$ on the order of the flux-lattice melting temperature (probably the *weakest* pinning which can be hoped for in granular films) would drastically affect the vortex phase diagram, in fact due to the presence of a large number of pinning "sites" per vortex in a granular film with only short-range correlations; the effects are rather small in the intermediate-field regime for $B << H_{c2}$. A detailed calculations (planned to be published elesewhere) shows that the change in melting temperature in an uncorrelated granular film is

$$\frac{\delta T_M}{T_M} \propto \frac{b_{\rm gr}^2}{a_0^2} \quad . \tag{3.6}$$

Thus only for $B \lesssim H_{c2}$ where the intervortex spacing is only a few times $b_{\rm gr}$ will the flux-lattice melting curve be appreciably altered by pinning.

The effects of pinning on the *dynamic* properties of the vortex system are much larger, however. These we discuss in Sec. IV.

IV. FLUX-FLOW RESISTANCE

In the absence of pinning forces, the vortices present in a perpendicular magnetic field will move transverse to an applied current, causing dissipation.[26] There will thus be a voltage caused by this flux flow at all temperatures and applied currents. In a real film, as discussed in Sec. III, there will always be inhomogeneities which tend to pin vortices. In this section we consider the effects of pinning on the flux-flow resistance in various regions of the vortex phase diagram.

1. *Very low temperatures* $T << T_M$

At temperatures well below the vortex-lattice melting temperature, T_M, thermal motion of the vortices can be ignored and we are in the familiar regime analogous to that in thicker films where there is a static flux lattice at all temperatures for $B < H_{c2}(T)$. This limit has been considered theoretically by several authors[6] and has been well studied experimentally.[5]

It is found that below some critical areal current density, J_p, the pinning forces dominate the Lorentz force of the applied current on the vortices and there is no flux flow. For transport areal current densities, J_T, larger than J_p, the flux lattice will move with an average velocity roughly proportional to $J_T - J_p$, and there will be a voltage drop across the sample: *IV* data roughly fit an expression of the form[6]

$$J_T = R_f^{-1} E + J_p \quad , \tag{4.1}$$

where E is the electric field and R_f is the flux-flow resistance per square given approximately by[26]

$$R_f = [B/H_{c2}(T)] R_n \quad , \tag{4.2}$$

with R_n the normal-state resistance. Schmid and Hauger[6] find that the critical current density is

$$J_p \propto (1/\mu) \tilde{U}_p^2 \quad , \tag{4.3}$$

with μ the shear modulus of the vortex lattice and $\tilde{U}_p$ a typical pinning potential (see Sec. III). (Actually, the critical current, J_p, is related to the correlation function Γ_U, in a complicated way, not just to $\tilde{U}_p$.[6])

2. *High temperatures* $T >> T_M$

At temperatures well above the vortex-lattice melting, a very different picture is necessary. [It should be noted that the relevant inequality is really

$$\frac{T}{\Lambda(T)} >> \frac{T_M}{\Lambda(T_M)} \quad , \tag{4.4}$$

not $T >> T_M$, however, for simplicity we will refer to this regime as high temperature.] As long as $T < T_c$, thermally excited free vortices will be present only in very small numbers (see discussion at the end of this section), and we can consider the system as just a one-component plasma of the free vortices present due to the appled magnetic field.

Even for $T \lesssim T_c$, the potential energy of a typical pair of vortices will be a factor of about 4 times the temperature and will hence dominate the kinetic energy. However, since the temperature is well above melting, we will assume that each vortex feels only a random force from the others: i.e., that the vortex motion is uncorrelated. To calculate the flux-flow properties of the vortex fluid, it is then sufficient to consider the motion of individual vortices in the presence of random noise (simulating temperture). With a transport areal current density $\vec{J}_T$, and a pinning potential $U_p(\vec{r})$ the equation of motion for a vortex becomes

$$\Gamma \vec{v} = (\Phi_0/c)\hat{z} \times \vec{J}_T - \vec{\nabla} U_p(\vec{r}) + \vec{\zeta}(t) \quad , \tag{4.5}$$

where $\hat{z}$ is a unit vector perpendicular to the film, Γ is a friction coefficient given in the Bardeen-Stephen[26] approximation by

$$\Gamma = \frac{\pi\hbar}{2\xi^2} \frac{R_c}{R_n} \quad . \tag{4.6}$$

ζ is a Gaussian white-noise source satisfying

$$\langle \zeta^\alpha(t) \zeta^\beta(t') \rangle = 2\Gamma T \delta^{\alpha\beta} \delta(t-t') \quad , \tag{4.7}$$

and $\vec{v}$ and $\vec{r}$ are the vortex velocity and position, respectively.

Due to the Lorentz force from the transport current, $\vec{F}_L$ [the first term in Eq. (4.5)], the vortices will move transverse to the transport current with a drift velocity $\vec{V}_D$, causing dissipation. It is useful to define a vortex mobility

$$\mu_v = V_D/F_L \quad . \tag{4.8}$$

In terms of this the flux-flow resistance will be

$$\frac{R}{R_c} = \frac{\pi e B}{c}\mu_v \tag{4.9}$$

and we thus need merely to estimate the form of the vortex mobility as a function of temperature. For weak pinning, $\tilde{U}_p(T)/T << 1$, vortices are thermally depinned and the mobility is given by

$$\mu_v = (1/\Gamma)[1 - \tfrac{1}{2}\Gamma_U(r=0)/T^2] \quad . \tag{4.10}$$

While for strong pinning, $\tilde{U}_p/T >> 1$, the mobility will be exponentially activated:

$$\mu_v \sim (1/\Gamma)e^{-\tilde{U}_p(T)/T} \quad . \tag{4.11}$$

[The exact form of the mobility in this limit will depend on details of the pinning potential and on $\Gamma_U(r)$.] At all temperatures in the vortex fluid phase, there will thus be some nonzero resistance due to flux flow, in contradistinction to the low-temperature critical current behavior discussed above, where there is no resistance for small J_T in the solid phase at $T=0$. However, as in the vortex solid phase, a large enough transport current in the fluid phase will dominate the pinning forces and in this limit the differential resistance will again be given by R_f. This crossover from the pinning dominated to the Lorentz-force-dominated regime, will occur at currents

$$J_T \sim (\nabla \tilde{U}_p)c/\phi_0 \quad , \tag{4.12}$$

where $\nabla \tilde{U}_p$ is a typical value of the gradient of U_p, i.e., the pinning force.

Flux-flow measurements on granular aluminum films carried out by Horn and Parks[25] show a regime in which the temperature dependence of the resistance is of the form [Eq. (4.11)]. However, in order to explain these data, it would be necessary to invoke a $\tilde{U}_p$ considerably larger than possible from just core pinning. This suggests that in fact inhomogeneities in granular aluminum films have considerable long-ranged correlations, in agreement with some recent preliminary results mentioned earlier.[23]

3. *Intermediate temperatures $T \sim T_M$*

Between the two limits discussed above, the flux-flow behavior is much more complicated, and the details, along with discussion of the ac conductivity, will be left for future investigation. Several points, though, should be made here.

As discussed earlier, in the absence of thermal fluctuations a vortex lattice will not move in the presence of a small current. However, at any nonzero temperature the lattice will be thermally depinned and can drift under the influence of an arbitrarily small J_T, causing a nonzero resistance. In addition, there will be thermally activated defects in the lattice (such as vacancies and interstitials) which will move and contribute to the resistance. Both of these processes should lead to resistances which vanish as $\exp(-1/T)$ as $T \rightarrow 0$. The first (and probably the second) will depend in a complicated way on the pinning-pinning correlation function $\Gamma_U(G)$ at reciprocal-lattice vectors, G, and there will be several regimes with quantitatively varying behavior—depending on the nature and strength of the pinning. There will be no measurable singularity in the flux-flow resistance at the melting temperature since the fluid for $T > T_M$ will be very solidlike out to length scales of the vortex-vortex positional correlation length, $\xi_v(T)$, which diverges extremely rapidly as $T \rightarrow T_M$[12,18]

$$\xi_v(T) \sim \exp\{-b[T_M/(T-T_M)]\}^{\bar{\nu}} \quad , \tag{4.13}$$

where $\bar{\nu} \simeq 0.37$ and b is a numerical constant which will be independent of magnetic field in the intermediate-field regime.

As the temperature is increased from T_M, the flux-flow resistance will cross over smoothly from the complicated latticelike behavior to the simple independent vortex regime far above T_M. This crossover will probably be sharpest if the pinning is very weak, i.e., if $\tilde{U}_p(T_M)/T_M \lesssim 1$.

In this weak pinning limit, the pinning will not significantly alter the flux-flow resistance in the fluid phase until the vortex fluid becomes strongly correlated. There should be a relatively sharp crossover from unpinned individual vortex behavior to pinned latticelike behavior at a temperature, T, above T_M where

$$n\xi_v^2(T)\tilde{U}_p(T)/T \sim 1 \quad . \tag{4.14}$$

At this temperature the resistance should start to decrease much more rapidly as T is decreased. The crossover will be the closest to T_M and the sharpest for films in which the pinning is weakest. For this reason it may be desirable experimentally to use (relatively) thick, high resistivity films (where the relative pinning energy may be smaller) rather than extremely thin films of lower resistivity material.

When $U_p << U_p^{\max}$ another flux-flow experiment is possible which may see a stronger crossover between the high- and low-temperature regimes. Hebard, Fiory, and Somekh[24] have observed very large critical currents (on the order of the theoretical Ginzburg-Landau critical current) in an aluminum film perforated with a regular array of holes to provide strong pinning centers (in contrast to the high density of pinning "centers" considered thus far). If a similar experiment is performed in a high resistance film with the separation between holes, S_h, large compared to the vortex-vortex spacing and the holes as small as possible (to avoid occupation of a single hole by many vortices) there will be a large change in the flux-flow resistance somewhere above T_M. For $T >> T_M$, most of the vortices will not be in holes and hence will be only weakly pinned and the flux-flow resistance will be only somewhat reduced from its value in the absence of the holes. However, when the vortex-vortex positional correlation length, ξ_v, becomes of the order of S_h, all the vortices will be strongly pinned up to large transport currents. Since the vortex density can be varied via the magnetic field, this experiment could provide a very rough estimate of ξ_v as a function of temperature. While this experiment clearly will not give a quantitative measurement of the vortex-vortex correlations, it may be possible to observe the effects of a rapidly diverging ξ_v.

In order to actually measure properties of the vortex system near melting, it is necessary to use a probe sensitive to very long wavelength properties. Most experiments (like the flux-flow resistance and the "hole" experiment discussed above) will observe only a gradual crossover from vortex liquidlike to vortex solidlike behavior.

In all the above discussion, we have assumed that the vortex-lattice melts via a second-order transition through dislocation unbinding. It is possible, however, that the Kosterlitz-Thouless melting theory does not apply for this system. In this case the melting would probably occur via a *first-order* transition at a lower temperature than predicted above and there would be a corresponding first-order jump in the flux-fow resistance and other properties of the system. While a small first-order jump can of course not be excluded experimentally, a large first-order jump should be readily observable.

In Sec. V we will discuss an experiment which, at least in principle, can be sensitive to very long-wavelength properties of the vortex system.

Before proceeding, however, we briefly digress to discuss a possible complication due to thermally excited vortices.

A. Effects of thermally excited vortices

So far we have considered only the vortices of one sign which are present due to the applied magnetic field. At finite temperatures, however, there will in addition be thermally excited vortices, with roughly an equal number of each sign. (It is the unbinding of pairs of these thermally excited vortices which gives rise to the resistive transition, T_C, in zero magnetic field.[8-10]) In nonzero applied field the distinction between these thermally excited vortices and the free vortices present due to the applied field is ambiguous. However, for $T << T_C$, the thermally excited vortices will be present primarily in tightly bound pairs and the distinction can be relatively clear. These pairs will be polarized by the free vortices and will give rise to an effective temperature-dependent dielectric constant, ϵ_B, which will modify the interactions between free vortices.[9] At low temperatures, this effect will be exponentially small,

$$\epsilon_B - 1 \sim e^{-2E_C/T} , \tag{4.15}$$

where E_C is the core energy of a vortex

$$E_C \simeq T_C . \tag{4.16}$$

Since $T_M << T_C$, $\epsilon_B(T_M) << 1$ and the effects of these bound vortex pairs can thus be ignored for consideration of melting and the vortex-lattice phase.

At temperatures well above T_M, we have already ignored the correlations and the details of the interactions between the free vortices in estimating the resistivity (Sec. IV). This approximation will be strengthened by the screening effects of the polarized bound pairs and the linear resistance will only be altered by replacing Γ in Eq. (4.5) by a new Γ_{eff} which includes the effects of bound pairs.

In addition, for $T \lesssim T_C$ there will be a nonlinear resistance due to the breakup of bound pairs by a finite transport current J_T (Ref. 9) which will decrease rapidly at T decreases from T_C. This will be difficult to distinguish experimentally in finite magnetic fields from the nonlinear resistance due to pinning forces discussed in Sec. IV. For $T \lesssim T_C$, the qualitative behavior of the IV curves will thus not be altered.

Near and above T_C, however, there will be free vortices thermally excited which give rise to the resistance in zero magnetic field.[9] These free vortices will contribute a term roughly independent of B to the resistance and the distinction between "thermally excited" and "applied" vortices will break down. The results of this paper, therefore, are valid in nonzero fields for temperatures up to around T_C and will break down qualitatively only when the number of thermally excited free vortices becomes on the order of B/Φ_0.

V. MEASUREMENTS OF THE SHEAR MODULUS

If a vortex lattice is driven through a random pinning potential at a velocity, V_D, by a transport current, $J_T > J_p$, dissipative fluctuations of the vortex lattice will be excited.[5,6] By coupling to these fluctuations with a small rf, current parallel to J_T, of frequency, f, Fiory[5] was able to extract the shear modulus of the lattice from current-voltage measurements by using the theoretical analysis of Schmid and Hauger (SH).[6] Steps were observed in the dc current, J_T, at dc electric fields

$$E_{m1} = B a_0 f / mc \tag{5.1}$$

(where m is an integer) corresponding to motion of the lattice one lattice spacing, a_0, in m periods of the rf current. These steps, which would be sharp for a perfectly rigid lattice, have a width

$$\delta x_m = \Delta E_{m1} / E_{m1} \tag{5.2}$$

arising from fluctuations of the lattice as it moves. For high frequencies and drift velocities,[6]

$$\delta x_m \approx \frac{1}{m^3} \frac{f}{\mu} \frac{\Phi_0 B}{2\pi\sqrt{3} c^2 R_f} \ , \tag{5.3}$$

where R_f is the flux-flow resistance. At low frequencies or velocities, on the other hand, there are large fluctuations in the local drift velocity and the width is[6]

$$\delta x_m \propto m / f \ . \tag{5.4}$$

Because of this $1/f$ dependence of δx_m at low frequencies, it is necessary to carry out the experiments at frequencies large enough so that the $1/f$ contribution to the width can be ignored and the shear modulus can be extracted by Eq. (5.3). This implies that there is a minimum value of δx_m as a function of frequency which can be used to calculate μ. As noted by SH,[6] if δx_m is extremely small, it will be sensitive only to the very long-wavelength transverse modes of the vortex lattice and hence to the desired long-wavelength shear modulus. However, this is *not* true in general. Analysis of SH's calculations shows that δx_1 (the $m=1$ step is the primary one observed experimentally) is determined mostly by modes of wavelengths

$$L \sim a_0 / \delta x_1 \ . \tag{5.5}$$

Thus as δx_1 increases, this experiment becomes sensitive to shorter wavelengths. We find that near T_m, at frequencies large enough so that Eq. (5.3) applies, δx_1 will typically be of the order of 0.1 or larger and hence this experiment will, unfortunately, *not* be sensitive to very long-wavelength properties of the flux lattice. In fact Fiory's experiments[5] on films with $R_n \simeq 10\ \Omega/\Box$ observe steps with widths satisfying Eq. (5.3) for $B \lesssim H_{c2}$ in a region of the phase diagram in which, from the extracted shear modulus, one expects the vortex lattice to have melted by the Kosterlitz-Thouless[11] stability criterion [Eq. (2.3)]. However, it is found that in this region the experimentally observed δx_1 is $\gtrsim 0.2$ and hence the experiment is only sensitive to a short-wavelength shear modulus which will be present for $T > T_m$ as long as the correlation length $\xi_v(T)$ [given by Eq. (4.17)] is larger than a few lattice spacings. We note that for $T << T_M$, on the other hand, δx_1 becomes very small; the experiment becomes sensitive to long wavelengths and the results agree well with the calculated values of the bare shear modulus, $\mu_0(T)$.[2,5] For $T \sim \frac{1}{2} T_M$, however, deviations will be expected from the bare shear modulus due to nonlinear lattice vibrations[17] (see discussion in Appendix). While quantitative comparisons with experiment have not been made, Fiory's[5] data for $\mu(T)$ tend to lie somewhat below $\mu_0(T)$ for temperatures in this range.

While it may prove difficult to measure the long-wavelength flux-lattice shear modulus near T_M (only the long-wavelength shear modulus includes all the effects of dislocation pairs) by the interference experiment discussed above, the shear modulus may be measureable more directly. For example, if one part of the lattice is strongly pinned (say by holes) and another part is moved by a transport current, it may be possible to measure the shear restoring force of the lattice. This and other possible experiments, we leave for future consideration.

ACKNOWLEDGMENTS

The author wishes to thank W. F. Brinkman for many fruitful and inspirational discussions and for comments on a preliminary version of the manuscript. I have benefited greatly from interactions with experimental colleagues, especially A. T. Fiory and A. F. Hebard. In addition useful discussions with M. R. Beasley, B. I. Halperin, P. C. Hohenberg, and T. M. Rice are gratefully acknowledged.

APPENDIX

In this appendix, we discuss the possibility of estimating the effects of nonlinear phonon interactions[17] and thermally excited dislocation pairs[11,12,18,21] on the flux-lattice melting temperture in the intermediate magnetic field regime. To achieve this it is necessary only to calculate the effects of these nonlinearities on the long-wavelength Lamé coefficients $\mu(T)$ and $\lambda(T)$ and hence the melting temperature renormalization factor A_1 defined by Eq. (2.8).

The renormalization of the shear modulus due to nonlinear phonon interactions for a lattice of particles interacting with a potential

$$V(r) = -q^2(T) \ln(r/\epsilon) \tag{A1}$$

[for the case of superconducting vortices, $q^2 = \Phi_0^2 / 4\pi^2\Lambda$; in these units $T_M = A_1/(16\pi\sqrt{3}) q^2(T_M)$] can be systematically calculated in perturbation theory about $T=0$

$$\frac{\mu_{\rm ph}}{\mu_0}-1=+C_1\frac{T}{q^2}+C_2\left(\frac{T}{q^2}\right)^2 , \quad (A2)$$

where $C_1 \approx -28$, μ_0 is the bare shear modulus (ignoring fluctuations) and the calculation of C_2 involves numerically evaluating 12 Feynman diagrams. The other Lamé coefficient, λ, is not renormalized by phonon-phonon interactions for the case of logarithmically interacting particles.

To estimate the effects of dislocation pairs, it is necessary to know the core energy of a dislocation in the lattice under consideration, $E_{\rm DC} \propto q^2$. At low temperatures the number of dislocation pairs in the lattice will be[11,12]

$$N_{\rm pairs} \propto e^{-2E_{\rm DC}/T} \quad (A3)$$

and the renormalization of the shear modulus [and $\lambda(T)$] due to these pairs will also be exponentially small. As the temperature is increased, the effects of dislocation pairs on the long-wavelength Lamé coefficients can be calculated by using $E_{\rm DC}$ and $\mu_{\rm ph}(T)$, and integrating the renormalization-group equations[17,21] of Halperin and Nelson.[21]

The resulting shear modulus, $\mu(T)$ will generally be *less* than $\mu_{\rm ph}(T)$ and will have a singularity at the Kosterlitz-Thouless melting temperature.[12,18]

$$\mu(T)=\mu(T_M)+D_1(T_M-T)^{\bar\nu} , \quad (A4)$$

where $\bar\nu$ is defined after Eq. (4.13) and D_1 can be estimated given $E_{\rm DC}$. Note that λ will also be renormalized to a large but finite value.

For particles interacting with $1/r$ potentials (e.g., electrons or superconducting vortices in the extremely low magnetic field regime) $E_{\rm DC}$ has been calculated by Fisher, Halperin, and Morf,[22] and the first term in $\mu_{\rm ph}(T)$ by this author.[17] Morf has found, using a linearly termperature dependent $\mu_{\rm ph}(T)$ (i.e., assuming $C_2=0$, etc.) and the calculated $E_{\rm DC}$, that by integrating the renormalization-group equations, one finds a temperature-dependent shear modulus in excellent agreement with his molecular dynamics calculations,[21] and a melting temperature which agrees remarkably well with the experimentally observed phase transition in electrons on Helium.[20] This calculation yields an estimate for the renormalization factor for particles interacting with a $1/r$ potential

$$A_3 \simeq 0.6 .$$

If $E_{\rm DC}$ were known, a similar calculation would be possible for logarithmically interacting particles, which would give (one hopes) a reasonable estimate for A_1 and hence T_M. In absence of a specific numerical determination of $E_{\rm DC}$ for this case, a few comments on the probable range of A_1 are useful. In reduced units (i.e., those in which the bare melting temperature is the same for $1/r$ and $\ln r$ potentials) the linear term C_1T in $\mu_{\rm ph}(T)$ is roughly the same for $1/r$ and $\ln r$ potentials.

However there is some reason to expect that the quadratic term C_2T^2 which appears to be small (from Morf's results) for $1/r$ potentials may be considerably larger for $\ln r$ potentials.[17] If we assume for simplicity, however, that the terms in $\mu_{\rm ph}$ higher order than C_1 are negligible (or negative) for $T \lesssim T_M$ then we can set a rough upper bound for A_1, valid in the limit that $E_{\rm DC}$ is extremely large, by taking

$$\mu(T)=\mu_0(T)(1+C_1T/q^2) \quad (A5)$$

this yields

$$A_1 \leq 0.75 . \quad (A6)$$

At the other extreme, if $E_{\rm DC}$ is small ($\sim$ few times T_M or less) then there will be a large number of dislocation pairs present for $T \lesssim T_M$. The Kosterlitz-Thouless[11] theory assumes that the dislocation density is small and that nonlinear interactions between dislocations are unimportant. If the dislocation pair density is large, the theory may break down and melting may be driven by another mechanism (e.g., grain boundaries; see discussion in Ref. 22) or become first order. It is reasonable to doubt the applicability of the theory if the renormalization due to dislocation pairs are much bigger than a factor of 2. Hence if the flux-lattice melting *is* a Kosterlitz-Thouless transition, we can guess that

$$0.4 < A_1 < 0.75 . \quad (A7)$$

A numerical evaluation of $E_{\rm DC}$ and C_2 is necessary, however, to make a real estimate of A_2 and test the applicability of the theory.

We note, finally, that while the full effects of the renormalization of μ due to dislocation pairs will only occur at very long wavelengths,[12] the effects of nonlinear phonons will be measurable even at wavelengths of a few lattice spacings.

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