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Finite-damping corrections to the conductivity of the sine-Gordon chain: General formalism and asymptotic results

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We develop a systematic method for obtaining finite-damping corrections to the Smoluchowski equation describing the Brownian motion of coupled nonlinear oscillators. The formalism is applied to the driven sine-Gordon pendulum chain to obtain the lowest-order correction to the conductivity theory by Trullinger *et al.*, and asymptotic results are presented in the limits of (i) low-torque, low-temperature, strong coupling, (ii) high-torque, all temperatures and coupling strengths, and (iii) vanishing coupling between pendula.

I. INTRODUCTION

In a recent Letter (hereafter referred to as I), Trullinger *et al.*[1] studied the steady-state dynamical behavior of a set of torsion-coupled pendula (the sine-Gordon chain) in the presence of damping, fluctuating thermal torques, and constant applied torque. It was found that for small applied torques, the average angular velocity of the pendula at low temperature is associated with the motion of thermalized[2,3] sine-Gordon (SG) solitons[4] and as the torque is increased the velocity response ("conductivity") becomes strongly nonlinear. These results have importance for several condensed-matter systems,[5] such as weakly pinned one-dimensional charge-density-wave condensates,[6] Josephson transmission lines,[7] and one-dimensional ionic conductors.[8]

In I the multiparticle Fokker-Planck[9] equation for the sine-Gordon chain was solved in the steady-state situation by reduction,[10] in the large-damping limit, to the Smoluchowski equation and the application of transfer-operator techniques[11,12] familiar from equilibrium statistical mechanics.[2,3,11-15] Although this represents the first known solution to the problem of Brownian motion of *coupled, nonlinear* oscillators in an external field, the restriction to the large-damping limit is unsatisfying since the response is, of course, strictly zero in this limit (the angular velocity is proportional to the reciprocal of the damping constant). Thus finite-damping corrections to the results obtained in I are clearly desirable, particularly if one wishes to make comparisons with actual experiments[16] or molecular-dynamics simulations.[17,18]

The influence of finite damping on the response of a *single* pendulum has been studied previously by many authors. The behavior at zero temperature has been treated by McCumber[19] and Stewart[20] who have found two branches of solutions to the deterministic equation of motion (no noise), leading to a damping-dependent threshold torque.[21] Possible branch switching[22] at finite temperature is believed to occur in molecular-dynamics calculations carried out by Kurkijärvi and Ambegaokar[23] who studied finite-damping corrections to the exact infinite-damping result[24-28] for the single pendulum. Very recently Nozières and Iche[29] have developed a stochastic formulation for the single-pendulum problem in the underdamped limit. The influence of finite damping on a recently proposed analogy[28,30] between the single-pendulum threshold behavior and a continuous phase transition has been studied by Schneider *et al.*,[18] in a molecular-dynamics simulation.

Much less attention has been paid, however, to finite-damping effects on the *coupled-pendulum* chain system, since it is more difficult to treat the many-body problem, both analytically[1] and via computer simulations.[18] Recently, Schneider *et al.*[18] have proposed the use of Wilemski's method[31] to obtain an expansion of the steady-state velocity response in inverse powers of the damping constant. We find that such an expansion is only possible in the *steady-state* situation and this assumes that the steady-state is unique and can be reached.

It has been generally recognized that in the lowest-order approximation, the Fokker-Planck (FP) equation reduces to the Smoluchowski equation in either the large-damping or long-time limits.[10,32] Although there have been numerous attempts[31,32] to reduce the Fokker-Planck equation to a diffusion equation in coordinate space in the general case, no successful reduction scheme has yet been found. One approach which is often used in these attempts is to integrate the FP equation over all velocity variables (or momenta if one uses the phase-space probability distribution function). This yields a diffusion equation for the undetermined "current" which is the first-moment function of velocity. In order to determine the velocity moment function, one may try to construct an equation for it by multiplying the FP equation by one velocity variable before integrating the entire equation over all velocities. However, one finds that the resulting equation for the *first*-velocity-moment function involves the *second*-moment function of the velocity and the equation for the second-moment function obtained by a similar procedure involves the *third*-moment function, etc. In other words, this procedure generates an infinite hierarchy of equations for the velocity moment functions. Although the Smoluchowski equation is obtainable by truncating the hierarchy at the equation for the second velocity moment (i.e., dropping the third moment in this equation) and collecting the terms of lowest order in the reciprocal of the damping constant, a similar procedure involving truncation at higher-moment equations does *not* necessarily yield a systematic expansion in powers of the reciprocal damping constant.

When the system is not in a steady state, it is impossible to obtain a reciprocal-damping-constant expansion of the velocity moments in which the expansion coefficients are independent of the damping constant but still necessarily dependent on the time variable. Steady states are attained by waiting an infinite time without altering the external conditions and the relaxation procedure *depends* on the damping mechanism in the system. This is why it is impossible to separate the time dependence of the velocity moments from their dependence on the damping constant. However, when the system *is* in the steady state, it is possible to obtain a reciprocal-damping-constant expansion for the time-independent velocity moment functions.

In Sec. II below we construct a recursion relation among the coefficients in such an expansion. In particular, we focus on an expansion for the first-velocity-moment function, i.e., the current whose lowest-order value is known as the Smoluchowski current. These results for the Brownian motion of an interacting system of particles are then applied in Sec. III to the sine-Gordon chain of torsion-coupled pendula treated in I (in the infinite-damping limit) to obtain corrections to the "current" up to third order in the reciprocal damping constant. In Sec. IV we explicitly evaluate the current (with corrections) in the asymptotic limits of high and low applied torques, respectively. Sec. V contains a summary of our results and remarks of a general nature.

II. CONFIGURATION-SPACE PROJECTION OF THE FOKKER-PLANCK EQUATION IN THE STEADY STATE

In this section we consider the Brownian motion of a system of N particles of mass M interacting with each other and with an external applied field through a potential function $U(x_1, \ldots, x_N)$ where the $\{x_i\}$ denote the configurational positions of the particles. The Langevin equations are written

$$M\ddot{x}_i = -\frac{\partial U}{\partial x_i} - \Gamma \dot{x}_i + f_i(t), \qquad (2.1)$$

where Γ is the damping constant and the random thermal noise forces $\{f_i(t)\}$ satisfy

$$\langle f_i(t)\rangle = 0 \qquad (2.2a)$$

and

$$\langle f_i(t) f_j(t')\rangle = 2\Gamma k_B T \delta_{ij}\delta(t-t'). \qquad (2.2b)$$

From Eqs. (2.1) and (2.2), a Fokker-Planck equation for the probability distribution function,

$$P(x_1, \ldots, x_N; \dot{x}_1, \ldots, \dot{x}_N; t) = P(\{x_i\}, \{\dot{x}_i\}; t),$$

in velocity-configuration space may be derived by straightforward methods.[9,33] The result is

$$\frac{\partial P}{\partial t} = \sum_i \left[-\dot{x}_i \frac{\partial P}{\partial x_i} + \frac{1}{M}\frac{\partial U}{\partial \dot{x}_i}\frac{\partial P}{\partial x_i} + \frac{\Gamma}{M}\frac{\partial}{\partial \dot{x}_i}\left(\dot{x}_i P + \frac{k_B T}{M}\frac{\partial P}{\partial \dot{x}_i}\right)\right]. \qquad (2.3)$$

We shall project this FP equation onto the configuration space (i.e., integrate over velocities) and obtain a recurrence relation for the velocity moments in the steady state. This will lead to an expansion in inverse powers of the damping constant Γ. The results will be specialized in Sec. III to the case of the sine-Gordon pendulum chain.[1]

We define a set of velocity moment functions by

$$D_n(i_1, i_2, \ldots, i_n) \equiv M^{n-1} \prod_{l=1}^{N} \int d\dot{x}_l \left(\prod_{r=1}^{n} \dot{x}_{i_r}\right) P(\{x_i\}, \{\dot{x}_i\}, t), \qquad (2.4)$$

where $n = 0, 1, 2, \ldots, \infty$. We note that whereas n can range from zero to infinity, the number of distinct i_r values must be less than or equal to N since these label the particles. We shall have occasion to use special notations for D_0 and D_1:

$$D_0 = \sigma/M, \qquad (2.5)$$

where

$$\sigma(\{x_i\}; t) \equiv \prod_{l=1}^{N} \int d\dot{x}_l P(\{x_i\}, \{\dot{x}_i\}; t) \qquad (2.6)$$

is the configuration probability distribution function, and

$$D_1(j) = J(j), \qquad (2.7)$$

where

$$J(j)=\prod_{l=1}^{N}\int d\dot{x}_l\,\dot{x}_j P(\{x_i\},\{\dot{x}_i\};t) \qquad (2.8)$$

is the jth "component" of the "current,"

$$\vec{J}(\{x_i\};t)=[J(1),J(2),\ldots,J(N)].$$

We construct an equation for the nth-order velocity moment function by multiplying both sides of Eq. (2.3) by $M^n\dot{x}_{i_1}\dot{x}_{i_2},\ldots,\dot{x}_{i_n}$ and integrating over all N velocity variables. This yields

$$\begin{aligned}M\frac{\partial}{\partial t}D_n(i_1,\ldots,i_n;t)\\ &=-\sum_{m=1}^{N}\frac{\partial}{\partial x_m}D_{n+1}(i_1,\ldots,i_n,m;t)-M\sum_{r=1}^{n}\frac{\partial U}{\partial x_{i_r}}D_{n-1}(i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_n;t)\\ &\quad -n\Gamma D_n(i_1,\ldots,i_n;t)+\Gamma Mk_BT\sum_{r=1}^{n}\sum_{\substack{s=1\\ r\neq s}}^{n}D_{n-2}(i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_{s-1},i_{s+1},\ldots,i_n;t)\delta_{i_r,i_s}.\end{aligned} \qquad (2.9)$$

In a steady-state situation $(\partial D_n/\partial t)=0$, and Eq. (2.9) can then be rewritten $(n>0)$

$$\begin{aligned}D_n(i_1,\ldots,i_n)=\frac{Mk_BT}{n}\sum_{r=1}^{n}\sum_{\substack{s=1\\ r\neq s}}^{n}D_{n-2}(i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_{s-1},i_{s+1},\ldots,i_n)\delta_{i_r,i_s}\\ -\frac{1}{n\Gamma}\left(\sum_{m=1}^{N}\frac{\partial}{\partial x_m}D_{n+1}(i_1,\ldots,i_n,m)+M\sum_{r=1}^{n}\frac{\partial U}{\partial x_{i_r}}D_{n-1}(i_1,\ldots,i_{r-1},i_{r+1},\ldots,i_n)\right).\end{aligned} \qquad (2.10)$$

Furthermore, steady-state moment functions may be expanded in inverse powers of the damping constant as

$$D_n=\sum_{l=0}^{\infty}\left(\frac{1}{\Gamma}\right)^{l}D_n^{(l)}, \qquad (2.11)$$

where the coefficients $\{D_n^{(l)}\}$ are independent of Γ.

For the sake of simplicity in the following discussion, we introduce three operators $\hat{O}_1$, $\hat{O}_2$, and $\hat{O}_3$, which are defined by the relations

$$\begin{aligned}\hat{O}_1D_{n-2}&=\hat{O}_1(i_1,\ldots,i_n)D_{n-2}(j_1,\ldots,j_{n-2})\\ &=\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\cdots\sum_{j_{n-2}=1}^{N}\left(\frac{Mk_BT}{n}\sum_{r=1}^{n}\sum_{\substack{s=1\\ r\neq s}}^{n}\delta_{i_r,i_s}\right)\\ &\quad\times\delta_{i_1,j_1}\delta_{i_2,j_2}\cdots\delta_{i_{r-1},j_{r-1}}\delta_{i_{r+1},j_r}\delta_{i_{r+2},j_{r+1}}\cdots\delta_{i_{s-1},j_{s-2}}\delta_{i_{s+1},j_{s-3}}\cdots\delta_{i_n,j_{n-2}}D_{n-2}(j_1,\ldots,j_{n-2}),\end{aligned} \qquad (2.12)$$

$$\begin{aligned}\hat{O}_2D_{n+1}&=\hat{O}_2(i_1,\ldots,i_n)D_{n+1}(j_1,\ldots,j_{n+1})\\ &=\sum_{j_1=1}^{N}\cdots\sum_{j_{n+1}=1}^{N}\left(-\frac{1}{n}\sum_{m=1}^{N}\frac{\partial}{\partial x_m}\right)\delta_{i_1,j_1}\cdots\delta_{i_n,j_n}\delta_{m,j_{m+1}}D_{n+1}(j_1,\ldots,j_{n+1}),\end{aligned} \qquad (2.13)$$

and

$$\begin{aligned}\hat{O}_3D_{n-1}&=\hat{O}_3(i_1,\ldots,i_n)D_{n-1}(j,\ldots,j_{n-1})\\ &=\sum_{j_1=1}^{N}\cdots\sum_{j_{n-1}=1}^{N}\left(-\frac{M}{n}\sum_{r=1}^{n}\frac{\partial U}{\partial x_{i_r}}\right)\delta_{i_1,j_1}\cdots\delta_{i_{r-1},j_{r-1}}\delta_{i_{r+1},j_r}\cdots\delta_{i_n,j_{n-1}}D_{n-1}(j_1,\ldots,j_{n-1}).\end{aligned} \qquad (2.14)$$

Equation (2.10) may now be written in the compact form

$$D_n=\hat{O}_1D_{n-2}+(1/\Gamma)(\hat{O}_2D_{n+1}+\hat{O}_3D_{n-1}). \qquad (2.15)$$

The recursion relation among the expansion coefficients, $D_n^{(l)}$, in Eq. (2.11) is obtained by substituting Eq. (2.11) into Eq. (2.15) to yield

$$D_n^{(l)}=\hat{O}_1(n)D_{n-2}^{(l)}+\hat{O}_2(n)D_{n+1}^{(l-1)}+\hat{O}_3(n)D_{n-1}^{(l-1)}. \qquad (2.16)$$

This equation is recognized as a two-dimensional difference equation in the variables n and l. It can be solved in principle once we specify boundary conditions. We note first that $D_0^{(l)}=0$ for odd l. This follows from the defining equations, (2.5) and (2.6), for D_0 and the fact that

$$P_{-\Gamma}(\{x_i\},\{\dot{x}_i\})=P_\Gamma(\{x_i\},\{-\dot{x}_i\})$$

for the steady-state situation, which can be easily verified by reversing the sign of Γ in Eq. (2.3) after setting the left-hand side equal to zero. We therefore define

$$D_0^{(l)} \equiv (1/M)\Gamma^l \sigma^{(l)}, \tag{2.17a}$$

so that

$$\sigma = \sigma^{(0)} + \sigma^{(2)} + \sigma^{(4)} + \ldots, \tag{2.17b}$$

with $\sigma^{(l)}$ being of order Γ^{-l}. Another boundary condition can be constructed by solving the difference equation (2.16) for $l=0$. Since $D_n^{(-l)} \equiv 0$, Eq. (2.16) with $l=0$ becomes

$$D_n^{(0)} = \hat{O}_1(n) D_{n-2}^{(0)}. \tag{2.18}$$

Since $D_{-1}^{(0)} \equiv 0$ and Eq. (2.18) connects successive odd-order moments, we have

$$D_n^{(0)} = 0 \quad (\text{odd } n). \tag{2.19}$$

For even n, a successive reduction of n may be accomplished via Eq. (2.18) and repeated application of the operator $\hat{O}_1$, eventually leading to an expression for $D_n^{(0)}$ in terms of $D_0^{(0)}$, or

$$D_n^{(0)}(i_1, \ldots, i_n) = \frac{(Mk_BT)^{n/2}}{n!}\frac{\sigma^{(0)}}{M} \times \sum_{\mathcal{P}} \delta_{i_1, i_2}\delta_{i_3, i_4} \cdots \delta_{i_{n-1}, i_n} \quad (\text{even } n), \tag{2.20}$$

where the summation over $\mathcal{P}$ implies summation over the $n!$ permutations of $(i_1, \ldots, i_n)$.

The recursion relation (2.16) together with the boundary conditions (2.17), (2.19), and (2.20), provides a systematic method for obtaining successive coefficients in the expansion of velocity moments of the FP equation in inverse powers of the damping constant (in the steady state). We see that all of these coefficients are expressible in terms of the zeroth moment $D_0 = \sigma/M$ and its derivatives with respect to the configuration coordinates. Thus, in particular, $D_1(j) = J(j)$ may be regarded as a functional of $\sigma(\{x_i\})$, and Eq. (2.9) with $n=0$ becomes

$$0 = \frac{\partial \sigma}{\partial t} = -\sum_{m=1}^{N} \frac{\partial}{\partial x_m} J\{m; \sigma\}. \tag{2.21}$$

This equation represents the projection of the Fokker-Planck equation (2.3) onto the configuration space alone in the steady state. The current $J\{m; \sigma\}$ may be expanded in inverse powers of the damping constant using the general formalism above. The first nonvanishing term occurs for $l=1$ and retention of only this term gives the Smoluchowski current

$$J_s(m) \equiv (1/\Gamma) J^{(1)}(m),$$

where $J^{(1)}(m) = D_1^{(1)}(m)$, and Eq. (2.21) then becomes the Smoluchowski diffusion equation.

In the remainder of this section we focus on obtaining the first nonvanishing correction to the Smoluchowski equation resulting from the expansion of $J(m)$ in inverse powers of Γ. The resulting modified diffusion equation is then used in Sec. III to obtain the lowest-order finite-damping correction to the "conductivity" of the sine-Gordon pendulum chain.[1]

By making use of the recurrence relation (2.16) and the relations (2.17), (2.19), and (2.20), we obtain the following results for $D_n^{(l)}$ for the values of n and l necessary to obtain the first correction to the Smoluchowski current:

$l=0$:

$$D_0^{(0)} = \frac{\sigma^{(0)}}{M}, \quad D_1^{(0)} = D_3^{(0)} = 0,$$

$$D_2^{(0)}(i,j) = Mk_BT D_0^{(0)} \delta_{ij},$$

$$D_4^{(0)}(i,j,k,l) = (Mk_BT)^2 D_0^{(0)} \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{2.22}$$

$l=1$:

$$D_0^{(1)} = D_2^{(1)} = D_4^{(1)} = 0,$$

$$D_1^{(1)}(i) = -M\left(k_BT \frac{\partial D_0^{(0)}}{\partial x_i} + \frac{\partial U}{\partial x_i} D_0^{(0)}\right) \equiv j_s(i),$$

$$D_3^{(1)}(i,j,k) = Mk_BT\left[j_s(i)\delta_{jk} + j_s(j)\delta_{ik} + j_s(k)\delta_{ij}\right], \tag{2.23}$$

$l=2$:

$$D_0^{(2)} = \frac{\Gamma^2}{M}\sigma^{(2)}, \quad D_1^{(2)} = D_3^{(2)} = 0,$$

$$D_2^{(2)}(i,j) = Mk_BT\delta_{ij} D_0^{(2)} - \frac{Mk_BT}{2}\left(\frac{\partial j_s(i)}{\partial x_j} + \frac{\partial j_s(j)}{\partial x_i} + \vec{\nabla}\cdot\vec{j}_s(i)\delta_{ij}\right) - \frac{M}{2}\left(\frac{\partial U}{\partial x_i} j_s(j) + \frac{\partial U}{\partial x_j} j_s(i)\right), \tag{2.24}$$

$l=3$:

$$D_0^{(3)} = D_2^{(3)} = 0,$$

$$D_1^{(3)}(i) = -M\frac{\partial U}{\partial x_i} D_0^{(2)} - Mk_BT\frac{\partial}{\partial x_i} D_0^{(2)} + \frac{Mk_BT}{2}\left(\nabla^2 j_s(i) + 2\frac{\partial}{\partial x_i}[\vec{\nabla}\cdot\vec{j}_s(i)]\right) + \frac{M}{2}\left[\frac{\partial U}{\partial x_i}\vec{\nabla}\cdot\vec{j}_s(i) + \left(\nabla\frac{\partial U}{\partial x_i}\right)\cdot\vec{j}_s(i) + j_s(i)\nabla^2 U + \nabla U\cdot\nabla j_s(i)\right]. \tag{2.25}$$

In these expressions we have used the definition of a reduced Smoluchowski current

$$j_s(i)=D_1^{(1)}(i)=\Gamma J_s(i)\,, \qquad (2.26)$$

and also the following definitions of the differential operators $\vec{\nabla}\cdot=\text{div}$, $\nabla=\text{grad}$, and ∇^2:

$$\vec{\nabla}\cdot\vec{j}_s=\sum_{m=1}^{N}\frac{\partial}{\partial x_m}j_s(m)\,, \qquad (2.27a)$$

$$\nabla f=\left(\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\ldots,\frac{\partial f}{\partial x_N}\right), \qquad (2.27b)$$

$$\nabla^2 f=\sum_{m=1}^{N}\frac{\partial^2 f}{\partial x_m^2}\,, \qquad (2.27c)$$

where f is a function of $x_1,x_2,\ldots,x_n$.

The current defined by Eq. (2.7) can be written according to Eqs. (2.11) and (2.26) as

$$J(i)=\frac{1}{\Gamma}j_s(i)+\frac{1}{\Gamma^3}D_1^{(3)}(i)+\cdots\,, \qquad (2.28)$$

where the neglected terms are of $O(\Gamma^{-5})$. In the steady state, Eq. (2.21) implies to this order that

$$\vec{\nabla}\cdot\vec{j}_s+\frac{1}{\Gamma^2}\vec{\nabla}\cdot\vec{D}_1^{(3)}=0\,, \qquad (2.29)$$

which in turn implies that the divergence of the reduced Smoluchowski current is of order Γ^{-2}. Therefore, we can drop the term $2\partial[\vec{\nabla}\cdot\vec{j}_s(i)]/\partial x_i$ in Eq. (2.25) up to an accuracy of order Γ^{-5}. Thus we have

$$J(i)=J_s(i)-\frac{M}{\Gamma^3}\frac{\partial U}{\partial x_i}D_0^{(2)}-\frac{Mk_BT}{\Gamma^3}\frac{\partial}{\partial x_i}D_0^{(2)}+\frac{M}{2\Gamma^2}\left(k_BT\sum_{m=1}^{N}\frac{\partial^2 J_s(i)}{\partial x_m^2}+\frac{\partial U}{\partial x_i}\sum_{m=1}^{N}\frac{\partial J_s(m)}{\partial x_m}+\sum_{m=1}^{N}\frac{\partial^2 U}{\partial x_i\partial x_m}J_s(m)+J_s(i)\sum_{m=1}^{N}\frac{\partial^2 U}{\partial x_m^2}+\sum_{m=1}^{N}\frac{\partial U}{\partial x_m}\frac{\partial J_s(i)}{\partial x_m}\right)+O(\Gamma^{-5})\,. \qquad (2.30)$$

We now define

$$\tilde{J}(i)\equiv J_s(i)-\frac{M}{\Gamma^3}\frac{\partial U}{\partial x_i}D_0^{(2)}-\frac{Mk_BT}{\Gamma^3}\frac{\partial}{\partial x_i}D_0^{(2)}\,, \qquad (2.31)$$

and note that, to the order in which we are working, J_s in large parentheses in Eq. (2.30) can be replaced by $\tilde{J}$, so that

$$J(i)=\tilde{J}(i)+\frac{M}{2\Gamma^2}k_BT\left(\sum_{m=1}^{N}\frac{\partial^2\tilde{J}(i)}{\partial x_m^2}+\frac{\partial U}{\partial x_i}\sum_{m=1}^{N}\frac{\partial\tilde{J}(m)}{\partial x_m}+\sum_{m=1}^{N}\frac{\partial^2 U}{\partial x_i\partial x_m}\tilde{J}(m)+\tilde{J}(i)\sum_{m=1}^{N}\frac{\partial^2 U}{\partial x_m^2}+\sum_{m=1}^{N}\frac{\partial U}{\partial x_m}\frac{\partial\tilde{J}(i)}{\partial x_m}\right)+O(\Gamma^{-5})\,. \qquad (2.32)$$

This result, together with Eq. (2.21), corrects Eq. (27) of Ref. 18. Although the contribution of the second term in large parentheses in Eq. (2.32) is of order Γ^{-5}, as are the terms we have dropped, we retain this term for computational convenience below. We also note for later reference that $\tilde{J}(i)$ can be written from Eqs. (2.17), (2.23), and (2.31) as

$$\tilde{J}(i)=-\frac{1}{\beta\Gamma}e^{-\beta U}\frac{\partial}{\partial x_i}(e^{\beta U}\sigma)+O(\Gamma^{-5})\,, \qquad (2.33)$$

where $\beta=(k_BT)^{-1}$.

Equation (2.32) provides a general expression for the steady-state current associated with particle i valid up through order Γ^{-3}. In Sec. III we make use of this result to calculate the lowest-order finite-damping corrections to the "conductivity" of the sine-Gordon chain in particular.

III. CONDUCTIVITY OF THE SINE-GORDON PENDULUM CHAIN

We now specialize the general results obtained above to the particular case of the sine-Gordon pendulum chain considered in Ref. 1. This system consists of $N(N\gg 1)$ simple pendula of mass m and length l whose points of support are equally spaced on a large supporting ring. Each pendulum is coupled to its nearest neighbors by a torsion spring (torsion constant κ) and is free to move only in the vertical plane containing its point of support and the center of the support ring. The motion of each pendulum can thus be described by an angular coordinate θ (measured from the vertical) and an angular velocity $\dot{\theta}\equiv d\theta/dt$.

Using the notation of Ref. 1 (referred to as I), we write the Langevin equation of motion for the ith pendulum in terms of its angular momentum $p_i\equiv I\dot{\theta}_i=ml^2\dot{\theta}_i$:

$$\dot{p}_i=K(\theta_{i-1},\theta_i,\theta_{i+1})-\eta p_i+F_i(t)\,, \qquad (3.1)$$

where

$$K(\theta_{i-1},\theta_i,\theta_{i+1})=\kappa(\theta_{i+1}+\theta_{i-1}-2\theta_i)-mgl\sin\theta_i+\tau=-\frac{\partial}{\partial\theta_i}U(\theta_1,\ldots,\theta_N)\,; \qquad (3.2)$$

g is the acceleration due to gravity and τ is a constant torque applied to each pendulum. As in I, we have $\theta_{N+1}=\theta_1$ for the ring (periodic) configuration. The potential function U includes the external torque τ and is given explicitly by

$$U(\theta_1,\ldots,\theta_N)=\sum_{i=1}^{N}\left[mgl(1-\cos\theta_i)+\tfrac{1}{2}\kappa(\theta_{i+1}-\theta_i)^2-\tau\theta_i\right]. \quad (3.3)$$

The thermal noise torques $F_i(t)$ satisfy $\langle F_i(t)\rangle=0$ and

$$\langle F_i(t)F_j(t+t')\rangle=2Ik_BT\eta\delta_{ij}\delta(t'),$$

where η is the damping constant.

The FP equation of I can be written

$$\frac{\partial P}{\partial t}=\sum_{m=1}^{N}\left[-\dot\theta_m\frac{\partial P}{\partial\theta_m}+\frac{1}{I}\frac{\partial U}{\partial\theta_m}\frac{\partial P}{\partial\dot\theta_m}+\frac{\Gamma}{I}\frac{\partial}{\partial\dot\theta_m}\left(\dot\theta_mP+\frac{k_BT}{I}\frac{\partial P}{\partial\dot\theta_m}\right)\right], \quad (3.4)$$

where $P(\{\theta_i\},\{\dot\theta_i\};t)$ is the probability distribution function and $\Gamma\equiv I\eta$. Equation (3.4) becomes identical to Eq. (2.3) with the identifications $\theta_m\leftrightarrow x_m$ and $I\leftrightarrow M$. Therefore, the results of Sec. II may be carried over directly if we replace x_m by θ_m and M by I.

As discussed in I, we seek an expression for the steady-state average angular velocity, $\overline{\omega}\equiv\langle\dot\theta_i\rangle$, of the pendula as a function of temperature and applied torque, since this is the relevant quantity in several physical contexts[6-8,24] of interest. In I the dimensionless average angular velocity

$$\Omega\equiv\overline{\omega}\eta/\omega_0^2 \quad (\omega_0\equiv g/l)$$

was calculated in the infinite-damping limit. In the remainder of this section we focus on obtaining an expression for Ω which includes finite-damping corrections through order Γ^{-2}. We shall employ the notation of Ref. 1 as much as possible in the following discussion.

In order to obtain $\overline{\omega}$, we make use of the result from Sec. II that the steady-state diffusion equation

$$\frac{\partial\sigma}{\partial t}=-\sum_{i=1}^{N}\frac{\partial J(i)}{\partial\theta_i}=0 \quad (3.5)$$

is satisfied, in next to lowest order in η^{-1}, by

$$J(i)=\tilde J(i)+\frac{1}{2I\eta^2}\left(\frac{1}{\beta}\sum_{m=1}^{N}\frac{\partial^2\tilde J(i)}{\partial\theta_m^2}+\frac{\partial U}{\partial\theta_i}\sum_{m=1}^{N}\frac{\partial\tilde J(m)}{\partial\theta_m}+\sum_{m=1}^{N}\frac{\partial^2U}{\partial\theta_i\partial\theta_m}\tilde J(m)+\tilde J(i)\sum_{m=1}^{N}\frac{\partial^2U}{\partial\theta_m^2}+\sum_{m=1}^{N}\frac{\partial U}{\partial\theta_m}\frac{\partial\tilde J(i)}{\partial\theta_m}\right), \quad (3.6)$$

where

$$\tilde J(i)=-\frac{1}{\beta I\eta}e^{-\beta U}\frac{\partial}{\partial\theta_i}\sigma e^{\beta U}, \quad (3.7)$$

and

$$\sigma(\{\theta_i\})=\int\cdots\int P(\{\theta_i\},\{\dot\theta_i\})\prod_{l=1}^{N}d\dot\theta_l. \quad (3.8)$$

We note for later convenience that βU may be expressed in dimensionless form as

$$\beta U=\frac{\gamma}{2}\sum_{i=1}^{N}\left[(1-\cos\theta_i)+\tfrac{1}{2}d^2(\theta_{i+1}-\theta_i)^2-\chi\theta_i\right], \quad (3.9)$$

where we have used the definitions[1] $\gamma\equiv2\beta mgl$, $d\equiv(\kappa/mgl)^{1/2}$, $\tau_c\equiv mgl$, and $\chi\equiv\tau/\tau_c$. The dimensionless parameter γ is the ratio of the gravitational potential barrier height to thermal energy, d is a characteristic length scale (the "width" of the soliton excitation[1-8] measured in numbers of pendula), and τ_c is the critical torque required to give a nonzero average angular velocity at $T=0$.

To find $\overline{\omega}$ we single out one of the angles (say, θ_j) and integrate Eq. (3.5) over all other angles to obtain an expression involving the single-particle distribution function,

$$\sigma(\theta_j)\equiv\int\cdots\int\sigma(\{\theta_i\})\prod_{\substack{l=1\\ l\neq j}}^{N}d\theta_l. \quad (3.10)$$

In the steady state, we have

$$\frac{\partial\sigma(\theta_j)}{\theta t}=0\equiv-\frac{\partial w}{\partial\theta_j}, \quad (3.11)$$

where w is a constant diffusion current given [from Eqs. (3.5),(3.10), and (3.11)] by

$$w=\int\cdots\int J(j)\prod_{\substack{l=1\\ l\neq j}}^{N}d\theta_l. \quad (3.12)$$

Since $\sigma(\theta_j)$ is periodic $[\sigma(\theta_j+2\pi)=\sigma(\theta_j)]$, we consider the interval $0\leq\theta_j\leq2\pi$ and normalize $\sigma(\theta_j)$ by the condition

$$\int_0^{2\pi}d\theta_j\,\sigma(\theta_j)=1. \quad (3.13)$$

With this condition, w^{-1} is the average time required for θ_j to evolve by 2π. Hence

$$\overline{\omega}=2\pi w. \quad (3.14)$$

To perform the integrations in Eq. (3.12) we employ the same techniques as used in I. Since these are described in detail in Refs. 34 and 35, we limit ourselves here to a brief sketch of the approach. First, we substitute Eq. (3.6) into Eq.

(3.12) to obtain

$$w=\int\cdots\int\left(\prod_{\substack{l=1\\l\neq j}}^{N}d\theta_l\right)\left\{\tilde{J}(j)+\frac{1}{2I\eta^2}\left[\frac{1}{\beta}\sum_{m=1}^{N}\frac{\partial^2\tilde{J}(j)}{\partial\theta_m^2}+\sum_{m=1}^{N}\frac{\partial}{\partial\theta_m}\left(\frac{\partial U}{\partial\theta_j}\tilde{J}(m)+\frac{\partial U}{\partial\theta_m}\tilde{J}(j)\right)\right]\right\}. \tag{3.15}$$

Since θ_j is restricted to the range $0\leq\theta_j<2\pi$, the current $\tilde{J}(l)$ satisfies the boundary conditions[34]

$$\lim_{|\theta_i|\to\infty}\tilde{J}(l)=\lim_{|\theta_i|\to\infty}\frac{\partial\tilde{J}(l)}{\partial\theta_m}=0\,,\quad i\neq j \tag{3.16}$$

for all l and m and for all $i\neq j$. Thus Eq. (3.15) may be simplified via integrations by parts to obtain

$$w=\int\cdots\int\left(\prod_{\substack{l=1\\l\neq j}}^{N}d\theta_l\right)\times\left[\tilde{J}(j)+\frac{1}{2I\eta^2}\left(\frac{1}{\beta}\frac{\partial^2\tilde{J}(j)}{\partial\theta_j^2}+2\frac{\partial U}{\partial\theta_j}\tilde{J}(j)\right)\right]. \tag{3.17}$$

In order to carry out the integrations in Eq. (3.17), we employ the Ansatz introduced in I for the form of the N-particle distribution function, namely,

$$\sigma(\theta_1,\ldots,\theta_N)=e^{-\beta U_0}\prod_{i=1}^{N}h(\theta_i)\,, \tag{3.18}$$

where $U_0=U(\theta_1,\ldots,\theta_N)|_{I=0}$ is the equilibrium ($\tau=0$) potential and $h(\theta)$ is a periodic single-particle function which is to be determined. Using Eqs. (3.7), (3.9), and (3.18), the current $\tilde{J}(j)$ can be written in the form

$$\tilde{J}(j)=\frac{\omega_0^2}{\eta}\sigma(\{\theta_i\})\left(\chi-\frac{\partial y(\theta_j)}{\partial\theta_j}\right), \tag{3.19}$$

where

$$y(\theta_j)\equiv\frac{2}{\gamma}\ln h(\theta_j)\,. \tag{3.20}$$

By introducing the modified function,

$$\tilde{h}(\theta_j)\equiv e^{-\beta\tau\theta_j}h(\theta_j)\,, \tag{3.21}$$

Eq. (3.18) may be written

$$\sigma(\{\theta_i\})=e^{-\beta U}\prod_{i=1}^{N}\tilde{h}(\theta_i)\,. \tag{3.22}$$

The integrations in Eq. (3.17) may now be performed with the aid of a transfer operator technique[1,34] with the result that

$$w=\frac{\omega_0^2}{\eta}\left\{\sigma(\theta)\xi(\theta)+\lambda^2\left[\frac{\sigma(\theta)}{\tilde{h}(\theta)}\frac{\partial^2}{\partial\theta^2}[\tilde{h}(\theta)\xi(\theta)]-\tilde{h}(\theta)\xi(\theta)\frac{\partial^2}{\partial\theta^2}\left(\frac{\sigma(\theta)}{\tilde{h}(\theta)}\right)\right]\right\}, \tag{3.23}$$

where

$$\xi(\theta)\equiv\chi-\frac{\partial y(\theta)}{\partial\theta}\,, \tag{3.24}$$

$$\lambda^2\equiv\omega_0^2/\gamma\eta^2\,, \tag{3.25}$$

and we have dropped the subscript j since w is independent of which j we choose and hence the label is no longer needed. In the process of obtaining Eq. (3.23) using the transfer-operator technique, we also obtain[34] an expression for the single-particle distribution function

$$\sigma(\theta)=\sum_k e^{-N(\gamma/2)\epsilon_k}|\Phi_k(\theta)|^2\,, \tag{3.26}$$

where, as in I, ϵ_k and Φ_k are the eigenvalues and associated eigenfunctions of the transfer-integral operator:

$$\int_{-\infty}^{+\infty}d\theta_i\,e^{-(\gamma/2)v(\theta_{i+1},\theta_i)}\Phi_k(\theta_i)=e^{-(\gamma/2)\epsilon_k}\Phi_k(\theta_{i+1})\,, \tag{3.27}$$

where

$$v(\theta_{i+1},\theta_i)\equiv\tfrac{1}{2}[d^2(\theta_{i+1}-\theta_i)^2-\cos\theta_i-\cos\theta_{i+1}-y(\theta_i)-y(\theta_{i+1})]\,. \tag{3.28}$$

In the thermodynamic limit ($N\to\infty$), only the ground state is important[1,34] in Eq. (3.26) and thus (normalizing so that $\epsilon_0=0$)

$$\sigma(\theta)=|\Phi_0(\theta)|^2\,. \tag{3.29}$$

At first glance, it may appear that the problem of finding w is now solved, since by Eq. (3.29), $\sigma(\theta)$ is obtainable from the ground-state solution of Eq. (3.27). However, as we shall see, the function $y(\theta)$ itself depends on $\sigma(\theta)$, and therefore $y(\theta)$ and $\sigma(\theta)$ have to be determined in a self-consistent fashion.[1] To proceed, we note from Eq. (3.23) that, to order λ^2,

$$\sigma(\theta)\xi(\theta)=\frac{\sigma(\theta)}{\tilde{h}(\theta)}\tilde{h}(\theta)\xi(\theta)\cong\frac{\eta w}{\omega_0^2}\,,$$

so that

$$\tilde{h}(\theta)\xi(\theta)=\frac{w\eta}{\omega_0^2}\frac{\tilde{h}(\theta)}{\sigma(\theta)}+O(\lambda^2)\,. \tag{3.30}$$

By substituting Eq. (3.30) into the first term in large square brackets in Eq. (3.23), carrying out the differentiation, and then reusing (3.30), we can rewrite Eq. (3.23) up through order λ^2 as

$$w=\frac{\omega_0^2}{\eta}\sigma(\theta)\xi(\theta)\left(1-2\lambda^2\frac{\partial^2}{\partial\theta^2}\ln\frac{\sigma(\theta)}{\tilde{h}(\theta)}\right). \tag{3.31}$$

Using Eq. (3.24), we may formally solve equation (3.31) for $y(\theta)$ to obtain

$$y(\theta)=\chi\left[\theta-2\pi\left(\int_0^{2\pi}\frac{d\theta'}{f(\theta')}\right)^{-1}\int_0^{\theta}\frac{d\theta''}{f(\theta'')}\right], \qquad (3.32)$$

where

$$f(\theta)\equiv\sigma(\theta)[1-\lambda^2 g(\theta)], \qquad (3.33)$$

with

$$g(\theta)\equiv 2\frac{\partial^2}{\partial\theta^2}\ln\frac{\sigma(\theta)}{h(\theta)}. \qquad (3.34)$$

To obtain $\sigma(\theta)$, Eqs. (3.27) and (3.32) have to be solved self-consistently. Once this is done, we obtain w as follows. Substituting Eqs. (3.24) and (3.34) into Eq. (3.31), we obtain

$$\chi-\frac{\partial y}{\partial\theta}=\frac{w\eta}{\omega_0^2}\frac{1}{\sigma(\theta)[1-\lambda^2 g(\theta)]} \qquad (3.35)$$

or

$$2\pi\chi=\frac{w\eta}{\omega_0^2}\int_0^{2\pi}\frac{d\theta}{\sigma(\theta)[1-\lambda^2 g(\theta)]}, \qquad (3.36)$$

where we have used the fact that $y(\theta)$ is periodic $[y(0)=y(2\pi)]$ as a consequence of the periodicity of $h(\theta)$. As in I, we define a dimensionless average angular velocity (or "current") by

$$\Omega\equiv\bar{\omega}\eta/\omega_0^2=2\pi w\eta/\omega_0^2, \qquad (3.37)$$

which, from Eq. (3.36), can be written

$$\Omega=4\pi^2\chi\bigg/\left(\int_0^{2\pi}\frac{d\theta}{\sigma(\theta)[1-\lambda^2 g(\theta)]}\right). \qquad (3.38)$$

Clearly, for the expansion to be valid, we must require

$$\lambda^2 g(\theta)\ll 1. \qquad (3.39)$$

When this is the case we may further write

$$\Omega\cong 4\pi^2\chi\left(\int_0^{2\pi}\frac{d\theta}{\sigma(\theta)}\right)^{-1}\left[1-\lambda^2\left(\int_0^{2\pi}\frac{d\theta}{\sigma(\theta)}\right)^{-1}\times\int_0^{2\pi}d\theta\frac{g(\theta)}{\sigma(\theta)}\right]. \qquad (3.40)$$

IV. ASYMPTOTIC RESULTS

In order to obtain the full dependence of Ω on χ for all applied torques, χ, it is necessary to solve Eqs. (3.27) and (3.32) numerically in self-consistent fashion. Such calculations are currently in progress and the results will be reported elsewhere.[36] The procedure employed essentially involves "bootstraping" of the numerical solution to successively higher values of χ, starting from the low-χ region where analytic approximations can be made. In this section we explore this region to obtain an analytic expression for the lowest-order finite-damping correction to the conductivity of the sine-Gordon chain. In addition, we also obtain the result that at high torques, $\Omega=\chi$, and higher-order damping corrections vanish as expected. Finally, we examine the limit in which the pendula become decoupled and hence single-pendulum results are obtained.

We first consider the low-torque limit ($\chi\ll 1$) where, to the lowest order in χ, we may ignore the χ dependence of $\sigma(\theta)$ and $g(\theta)$ in the integrands of Eq. (3.40). In this case, we may use results for $\sigma(\theta)$ and $g(\theta)$ obtained by setting

$$h(\theta)=\tilde{h}(\theta)\cong 1 \qquad (4.1)$$

and

$$y(\theta)\cong 0. \qquad (4.2)$$

In the strong coupling limit ($d\gg 1$), the Fredholm integral equation (3.27) for the ground-state eigenfunction $\Phi_0(\theta)$ can be approximated[11–15,34] by a differential equation for a related function[1]

$$\psi_0\equiv\exp\{\tfrac{1}{4}\gamma[\cos\theta+y(\theta)]\}\Phi_0(\theta):$$

$$\left(-\frac{1}{2\mu}\frac{d^2}{d\theta^2}-\cos\theta-y(\theta)\right)\psi(\theta)=\epsilon\psi(\theta), \qquad (4.3)$$

where $\mu\equiv(\frac{1}{2}\gamma d)^2$. Equation (4.3) has the form of Schrödinger's equation for a "particle" of "mass" $\mu(\hbar=1)$ in a periodic potential. The solutions have the Floquet form,

$$\psi_k(\theta)=\exp(ik\theta)u_k(\theta),$$

with $u_k(\theta+2\pi)=u_k(\theta)$, and the eigenvalues form bands in k space. We need only the lowest state, corresponding to the bottom ($k=0$) of the lowest band. In the present limit ($\chi\ll 1$), where $y(\theta)$ may be set to zero [Eq. (4.2)], Eq. (4.3) becomes the Mathieu equation.[37] The ground-state function, is then the Mathieu function,[37]

$$\psi_0(\theta)=A\,ce_0(\tfrac{1}{2}\theta,-|q|), \qquad (4.4)$$

where $|q|\equiv\gamma^2 d^2$ and A is a normalization constant. The single-particle distribution function is then given in this limit by

$$\sigma(\theta)=\frac{e^{-(\frac{1}{2}\gamma)\cos\theta}ce_0^2(\frac{1}{2}\theta,-|q|)}{\int_0^{2\pi}d\theta'\,e^{-(\frac{1}{2}\gamma)\cos\theta'}ce_0^2(\frac{1}{2}\theta',-|q|)}, \qquad (4.5)$$

and $g(\theta)$ is given by

$$g(\theta)=2\frac{\partial^2}{\partial\theta^2}\ln\sigma(\theta). \qquad (4.6)$$

At low temperatures ($|q|\gg 1$), $\sigma(\theta)$ is strongly peaked about $\theta=0$, and $\sigma^{-1}(\theta)$ is strongly peaked about $\theta=\pi$. Thus to lowest order in $|q|^{-1}$, we may approximate Eq. (3.40) by

$$\Omega\cong 4\pi^2\chi\left(\int_0^{2\pi}\frac{d\theta}{\sigma(\theta)}\right)^{-1}[1-\lambda^2 g(\pi)]. \qquad (4.7)$$

From the asymptotic properties[37] of $ce_0(\frac{1}{2}\theta, -|q|)$ for large $|q|$, we find that

$$g(\pi) \cong 2|q| = 2\gamma^2 d^2 . \tag{4.8}$$

Using Eqs. (3.25) and (4.8), Eq. (4.7) becomes

$$\Omega \cong \Omega^{(0)}(1 - 2\gamma d^2 \omega_0^2/\eta^2) , \tag{4.9}$$

where

$$\Omega^{(0)} = 4\pi^2 \chi \left(\int_0^{2\pi} \frac{d\theta}{\sigma(\theta)} \right)^{-1}$$

is the infinite-damping result which was found in Ref. 1 to have the low-temperature, low-torque, strong-coupling form,

$$\frac{\Omega^{(0)}}{\chi} = 2\pi \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{E_0}{k_B T}\right)^{3/2} \exp\left[-\left(1 - \frac{1}{8d}\right)\frac{E_0}{k_B T}\right] , \tag{4.10}$$

with

$$E_0 \equiv 8mgld. \tag{4.11}$$

In Ref. 1, the "conductivity" $\Omega^{(0)}/\chi$ was interpreted as that due to the motion of thermalized sine-Gordon solitons. From Eq. (4.9), we see that the finite-damping correction to the soliton conductivity contains a temperature-dependent factor since $\gamma = 2\beta mgl$. The condition that Eq. (4.9) be valid is that this correction be small, namely,

$$2\gamma d^2 \omega_0^2/\eta^2 \ll 1 . \tag{4.12}$$

Thus, Eq. (4.9) does *not* hold for arbitrarily low T but only for temperatures such that

$$1 \gg k_B T/E_0 \gg \tfrac{1}{2} d(\omega_0/\eta)^2 , \tag{4.13}$$

where E_0 is the soliton creation energy given by Eq. (4.11).

In the limit of high torques ($\chi \gg 1$), the gravitational $\cos\theta$ potential becomes negligible and $\sigma(\theta) \to 1/2\pi$ as $\chi \to \infty$. In addition $y(\theta) \to -\cos\theta$ so that $g(\theta) \to -\gamma\cos\theta$. Thus

$$\int_0^{2\pi} \frac{g(\theta)}{\sigma(\theta)}\, d\theta \to 0$$

and the finite-damping corrections vanish, leaving

$$\Omega/\chi \to 1 \ (\chi \to \infty) , \tag{4.14}$$

which is the expected "Ohm's-law" behavior.

We now consider the limit where the coupling between pendula vanishes, i.e., $\kappa \to 0$ (or $d \to 0$). In this limit, the transfer-operator eigenequation (3.2) is solved by

$$\Phi_0(\theta) = e^{(\gamma/4)[\cos\theta + y(\theta)]} , \tag{4.15}$$

so that

$$\sigma(\theta) = \frac{e^{(\gamma/2)[\cos\theta + y(\theta)]}}{\int_0^{2\pi} e^{(\gamma/2)[\cos\theta' + y(\theta')]}\, d\theta'} \tag{4.16}$$

From Eq. (3.22), we have

$$\frac{\sigma(\theta)}{h(\theta)} = e^{(\gamma/2)\cos\theta} , \tag{4.17}$$

and hence

$$g(\theta) = -\gamma \cos\theta . \tag{4.18}$$

Eq. (3.40) then becomes

$$\Omega = 4\pi^2\chi \left(\int_0^{2\pi} \frac{d\theta}{\sigma(\theta)} \right)^{-1} \times \left[1 + \frac{\omega_0^2}{\eta^2} \left(\int_0^{2\pi} \frac{d\theta}{\sigma(\theta)} \right)^{-1} \int_0^{2\pi} d\theta \frac{\cos\theta}{\sigma(\theta)} \right], \tag{4.19}$$

and Eq. (3.32) takes the form

$$y(\theta) = \chi \left[\theta - 2\pi \left(\int_0^{2\pi} \frac{d\theta'}{\sigma(\theta')[1 + (\omega_0^2/\eta^2)\cos\theta']} \right)^{-1} \times \int_0^{\theta} \frac{d\theta''}{\sigma(\theta'')[1 + (\omega_0^2/\eta^2)\cos\theta'']} \right] . \tag{4.20}$$

The average angular velocity Ω may be obtained from Eq. (4.19) once Eqs. (4.16) and (4.20) are solved in self-consistent fashion. Substitution of Eq. (4.16) into Eq. (4.20) yields the equation satisfied by $y(\theta)$:

$$y(\theta) = \chi \left[\theta - 2\pi \left(\int_0^{\theta} d\theta'' \frac{\exp\{-(\gamma/2)[\cos\theta'' + y(\theta'')]\}}{1 + (\omega_0^2/\eta^2)\cos\theta''} \right) \Big/ \left(\int_0^{2\pi} d\theta' \frac{\exp\{-(\gamma/2)[\cos\theta' + y(\theta')]\}}{1 + (\omega_0^2/\eta^2)\cos\theta'} \right) \right] . \tag{4.21}$$

Once $y(\theta)$ is determined from this equation, $\sigma(\theta)$ may be obtained from Eq. (4.16) and then Ω is given by Eq. (4.19).

In the low-torque limit, $y(\theta)$ can be set to zero and the single pendulum conductivity becomes

$$\frac{\Omega_{sp}}{\chi} \cong \left[I_0\left(\frac{\gamma}{2}\right) \right]^{-2} \left(1 - \frac{\omega_0^2}{\eta^2} \frac{I_1(\frac{1}{2}\gamma)}{I_0(\frac{1}{2}\gamma)} \right) , \tag{4.22}$$

where $I_n(\frac{1}{2}\gamma)$ is the modified Bessel function. In the large-damping limit ($\omega_0/\eta \to 0$), this result reduces to the infinite-damping result given by Ambegaokar and Halperin.[24]

V. SUMMARY AND DISCUSSION

We have developed a systematic method for obtaining finite-damping corrections to the infinite-damping, approximate description provided by the Smoluchowski equation for the steady-state Brownian motion of coupled particles in a nonlinear local potential with an external driving field. Explicit expressions were obtained for the specific example of the sine-Gordon pendulum chain and the leading-order corrections to the configuration-space projection of the Fokker-Planck equation were displayed. The solution of the modified Smoluchowski equation was reduced to the problem of solving a set of self-consistent equations for the steady-state single-particle distribution function $\sigma(\theta)$. The solution of these equations yields the so-called conductivity of the sine-Gordon chain, correct up through order Γ^{-3}. An approximate analytic expression was obtained in the low-field, low-temperature limit in the case of strong interpendulum torsion coupling. In addition, Ohm's law was recovered in the high-field limit, where all correction terms vanish. Finally, we presented the relevant equations leading to the conductivity for the case of decoupled pendula (the single-pendulum problem). The numerical solution of the self-consistent equations for $\sigma(\theta)$ to obtain the full nonlinear response (all χ) of the sine-Gordon chain are currently underway and the results will be reported elsewhere.

We wish to remark that it may appear to be possible (in principle) to obtain the full damping constant (Γ) dependence of the conductivity for all values of Γ by carrying out the expansion in Sec. II to all orders. However, as mentioned in the Introduction, there appears to exist (in the single-pendulum case) a critical value of Γ below which *two* possible steady states exist, one or the other being metastable (depending on χ) resulting in hysteresis effects. The existence of such a value[18] of Γ (~1.2) implies a singular point in the space of Γ values and negates the validity of the expansion technique in Sec. II for values of Γ less than this critical value. It may well be that there exists a similar value of Γ in the case of the sine-Gordon chain of coupled pendula as well, at least for finite-length chains. If such a critical (Γ_c) value exists, then clearly our expansion in inverse powers of Γ can only be valid for $\Gamma > \Gamma_c$.

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