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Spin waves in itinerant-electron spin-density-wave states

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The transverse dynamical susceptibility and the spin-wave stiffness constant of spin-density-wave (SDW) states in itinerant-electron systems are studied. The Hamiltonian is transformed by using a transformation which rotates the spin coordinate system locally, so that the new z direction lies in the direction of the local spin. For this transformed Hamiltonian, a self-consistent many-body theory developed by Fedro and Wilson and extended by Kishore is exploited to derive an exact expression for the transverse dynamical susceptibility. From the long-wavelength limit of the spin-wave spectrum, given by the poles of the transverse dynamical susceptibility, an exact formula for the spin-wave stiffness constant is obtained. Within the Hartree-Fock approximation, it reduces to the Fisher's formula for antiferromagnets.

I. INTRODUCTION

In 1960, Overhauser[1] showed the possibility for the existence of spin-density-wave (SDW) states in electron-gas model within the Hartree-Fock approximation. And later on Penn,[2] Alexander and Horwitz,[3] and Morris and Cornwell[4] showed their existence in narrow band solids, using the Hubbard model within the Hartree-Fock approximation. Experimentally, SDW states have been found in transition and rare-earth metals.[5] In spite of the experimental and theoretical indications for the existence of SDW states, very little is known[6] about their collective excitations, namely, spin waves. In ferromagnets,[5,7] it is now well established that in the long-wavelength limit, the spin-wave spectrum is given as $\omega \sim D_f q^2$, where ω is the energy of the spin wave, q is the magnitude of the spin-wave wave vector, and D_f is the spin-wave stiffness constant. Edwards and Fisher,[8] Corrias and Pasquale,[9] and Kishore,[10] by using different approaches, obtained exact microscopic formulas for D_f, from the exact expressions for the transverse dynamical susceptibility. In antiferromagnets Fisher,[11] by using an approach similar to that of Edwards and Fisher[8] for ferromagnets, obtained an exact formula for the stiffness constant D_a, given by the spin-wave spectrum $\omega \sim D_a q$. The disadvantage of this approach, as pointed out by Fisher[11] himself, is that one cannot work out an exact expression for the transverse dynamical susceptibility. In this paper, we show that, by using an approach similar to that of Kishore[10] for ferromagnets, it is possible to obtain exact expressions for the transverse dynamical susceptibility, as well as the spin-wave stiffness constant for SDW states of which antiferromagnetic state is a special case.

We describe our itinerant-electron system by the Hamiltonian

$$H=\sum_{ij\sigma} T_{ij} a_{i\sigma}^{\dagger} a_{j\sigma} + H_{e-e} \ , \tag{1}$$

where T_{ij} is the transfer integral between the lattice site $\vec{R}_i$ and $\vec{R}_j$; $a_{i\sigma}^{\dagger}$ ($a_{i\sigma}$) are the creation (annihilation) operators of the electrons at the site $\vec{R}_i$ and spin σ; and H_{e-e} is the Coulomb interaction between the electrons. In absence of spin-orbit interaction, which we assume, it is invariant to local spin rotations. For our formal calculation, we do not need the explicit form of H_{e-e}. But for actual calculation, one has to choose a particular form for it.

The coordinate transformation corresponding to local spin rotation, can be considered as follows. Let θ_i, ϕ_i be the polar and azimuthal angles of the direction of local spin at the site $\vec{R}_i$ in the laboratory spherical coordinate system, then under a transformation, which rotates the spin-coordinate system locally so that the new z direction lies in the direction of the local spin, the annihilation operator $a_{i\sigma}$ in the laboratory frame is transformed into the corresponding annihilation operator $b_{i\sigma}$, in the locally transformed frame according to

$$a_{i\sigma}=e^{-i\sigma\phi_i/2}\cos(\tfrac{1}{2}\theta_i)\, b_{i\sigma} - \sigma e^{-i\sigma\phi_i/2}\sin(\tfrac{1}{2}\theta_i)\, b_{i-\sigma} \ . \tag{2}$$

For SDW states ϕ_i is independent of the site index i and $\theta_i=\vec{Q}\cdot\vec{R}_i$, where $\vec{Q}$ is the wave vector of the SDW state. Ferromagnetic and antiferromagnetic states correspond to $\vec{Q}=0$ and $\vec{Q}=\vec{K}$ (half of the minimum reciprocal-lattice vector), respectively.

The new operators, $b_{i\sigma}$, satisfy the same commutation relations as $a_{i\sigma}$

$$[b_{i\sigma}^{\dagger}, b_{j\sigma'}]_{+}=\delta_{ij}\delta_{\sigma\sigma'} \tag{3}$$

and the Hamiltonian (1), in terms of these new operators, takes the form

$$H=\sum_{ij\sigma} T_{ij}^{+} b_{i\sigma}^{\dagger} b_{j\sigma} + i \sum_{ij\sigma} \sigma T_{ij}^{-} b_{i\sigma}^{\dagger} b_{j-\sigma} + H'_{e-e} \ , \quad (4)$$

where

$$T_{ij}^{\pm}=\frac{1}{2N}\sum_{\vec{k}} T_{\vec{k}}^{\pm} e^{i\vec{k}\cdot(\vec{R}_i-\vec{R}_j)} \ , \quad (5)$$

with

$$T_{\vec{k}}^{\pm}=\tfrac{1}{2}(T_{\vec{k}-\frac{1}{2}\vec{Q}} \pm T_{\vec{k}-\frac{1}{2}\vec{Q}}) \ . \quad (6)$$

Here, N is the total number of lattice sites and H'_{e-e} is of the same form as H'_{e-e}, except that operator a's are replaced by the operator b's.

In Sec. II, starting from the Kubo formula,[12] we derive an exact expression for the transverse dynamical susceptibility by using an approach similar to that of Kishore[10] for ferromagnets. Finally, in Sec. III, from the long-wavelength limit of the spin-wave spectrum, given by the poles of the transverse dynamical susceptibility, an exact formula for the spin-wave stiffness constant is derived. By expressing this formula in terms of two single-particle self-energies, it is found that Fisher's expression[11] for an antiferromagnet is just a special case of our general result.

II. TRANSVERSE DYNAMICAL SUSCEPTIBILITY

The calculation of the transverse dynamical susceptibility starts from the Kubo formula[12] for the generalized dynamical susceptibility.

$$\chi^{\sigma\sigma'}(\vec{q},\omega)=-g^2\mu_B^2\sum_{ij}\chi_{ij}^{\sigma\sigma'}(\omega)e^{-i\vec{q}\cdot(\vec{R}_i-\vec{R}_j)} \ , \quad (7)$$

where $\chi_{ij}^{\sigma\sigma'}(\omega)$ is the Fourier transform

$$\chi_{ij}^{\sigma\sigma'}(\omega)=\int_{-\infty}^{\infty} dt\, \chi_{ij}^{\sigma\sigma'}(t)e^{-i\omega t} \quad (8)$$

and $\chi_{ij}^{\sigma\sigma'}(t)$ is the double time-retarded Green's function[13]

$$\chi_{ij}^{\sigma\sigma'}(t)=i\Theta(t)\langle[S_i^{\sigma},S_j^{-\sigma'}(t)]_-\rangle \ , \quad (9)$$

with

$$S_i^{\sigma}\equiv S_i^x+i\sigma S_i^y\equiv b_{i\sigma}^{\dagger}b_{i-\sigma} \ . \quad (10)$$

$S_i^{\sigma}(t)$ is the Heisenberg operator, $S_i^{\sigma}=S_i^{\sigma}(t=0)$, and the angular brackets $\langle\ \rangle$ denote the ensemble average. For $\sigma'=\sigma$, Eq. (7) corresponds to the transverse dynamical susceptibility.

Now, we use a self-consistent many-body theory developed by Fedro and Wilson[14] for single-particle Green's function and extended by Kishore[10] for many-particle Green's function. A brief description of this theory is as follows. Differentation of Eq. (9) for $\chi_{ij}^{\sigma\sigma'}(t)$ with respect to time t gives the equation of motion

$$-i\frac{\partial}{\partial t}\chi_{ij}^{\sigma\sigma'}(t)=2\sigma\langle S^Z\rangle\delta_{ij}\delta_{\sigma\sigma'}\delta(t) + i\Theta(t)\langle[S_i^{\sigma},LS_j^{-\sigma'}(t)]_-\rangle \ , \quad (11)$$

where, for any arbitrary operator Θ, the Liouville operator L is defined as

$$LQ\equiv[H,Q]_- \quad (12)$$

and it is assumed that

$$\tfrac{1}{2}\sum_{\sigma}\sigma\langle b_{i\sigma}^{\dagger}b_{i\sigma}\rangle\equiv\langle S_i^Z\rangle\equiv\langle S^Z\rangle \quad (13)$$

is independent of the lattice index i. The Green's function, on the right-hand side of Eq. (11), is calculated by breaking the operator $S_j^{-\sigma'}(t)$ into two parts

$$S_j^{-\sigma'}(t)=PS_j^{-\sigma'}(t)+(1-P)S_j^{-\sigma'}(t) \ , \quad (14)$$

where the projection operator P is chosen as

$$P=\sum_{j\sigma}P_{j\sigma} \ , \quad (15)$$

with

$$P_{j\sigma}Q=\sigma S_j^{-\sigma}\langle[S_j^{\sigma},Q]_-\rangle/2\langle S^Z\rangle \ . \quad (16)$$

On substituting the identity (14) in Eq. (11) we get

$$-i\frac{\partial}{\partial t}\chi_{ij}^{\sigma\sigma'}(t)=2\sigma\langle S^Z\rangle\delta_{ij}\delta\sigma\sigma'\delta(t)+\sum_{l\sigma_1}\Omega_{il}^{\sigma\sigma_1}\chi_{lj}^{\sigma_1\sigma'}(t) + i\Theta(t)\langle[S_i^{\sigma},L(1-P)S_j^{-\sigma'}(t)]_-\rangle \ , \quad (17)$$

where

$$\Omega_{il}^{\sigma\sigma_1}=\sigma_1\langle[S_i^{\sigma},LS_l^{-\sigma_1}]_-\rangle/2\langle S^Z\rangle \ . \quad (18)$$

From the solution of the equation of motion of the operator $(1-P)\Theta(t)S_j^{-\sigma'}(t)$, it can be shown that[10,14]

$$(1-P)\Theta(t)S_j^{-\sigma'}(t)=\sum_{l\sigma_1}\int_0^{\infty}d\tau\, e^{i\tau(1-P)L}(1-P)L \times\left[\frac{\sigma_1 S_l^{-\sigma_1}}{2\langle S^Z\rangle}\right]\chi_{lj}^{\sigma_1\sigma'}(t-\tau) \ , \quad (19)$$

which, after substituting in Eq. (17), gives a closed equation for the Green's function $\chi_{ij}^{\sigma\sigma'}(t)$

$$-i\frac{\partial}{\delta t}\chi_{ij}^{\sigma\sigma'}(t)=2\sigma\langle S^Z\rangle\delta_{ij}\delta\sigma\sigma'\delta(t)+\sum_{l\sigma_1}\Omega_{il}^{\sigma\sigma_1}\chi_{lj}^{\sigma_1\sigma'}(t) + \sum_{l\sigma_1}\int_{-\infty}^{\infty}d\tau\,\gamma_{il}^{\sigma\sigma_1}(\tau)\chi_{lj}^{\sigma_1\sigma'}(t-\tau) \ , \quad (20)$$

where

$$\gamma_{il}^{\sigma\sigma_1}(\tau)=\frac{i\sigma_1\Theta(\tau)\langle[S_i^{\sigma},Le^{i\tau(1-P)L}(1-P)LS_l^{-\sigma_1}]_-\rangle}{2\langle S^Z\rangle} . \tag{21}$$

In terms of Fourier transform, Eq. (20) can be rewritten

$$\omega\chi_{ij}^{\sigma\sigma'}(\omega)=2\sigma\langle S^Z\rangle\delta_{ij}\delta\sigma\sigma'+\sum_{l\sigma_1}\Omega_{il}^{\sigma\sigma_1}\chi_{lj}^{\sigma_1\sigma'}(\omega)+\sum_{l\sigma_1}\gamma_{il}^{\sigma\sigma_1}(\omega)\chi_{lj}^{\sigma_1\sigma'}(\omega) , \tag{22}$$

where $\gamma_{il}^{\sigma\sigma_1}(\omega)$ is the Fourier transform of $\gamma_{il}^{\sigma\sigma_1}(t)$, and is defined according to Eq. (8). Equation (22) can be solved by taking its Fourier transform in momentum space, defined as

$$F_{ij}^{\sigma\sigma'}(\omega)=\frac{1}{N}\sum_{\vec{q}}F^{\sigma\sigma'}(\vec{q},\omega)e^{i\vec{q}\cdot(\vec{R}_i-\vec{R}_j)} , \tag{23}$$

where $F_{ij}^{\sigma\sigma'}(\omega)$ stands for $\gamma_{ij}^{\sigma\sigma'}(\omega)$, or $\Omega_{ij}^{\sigma\sigma'}$. The substitution of Eq. (23) in Eq. (22) gives us

$$[\omega-\Omega^{\sigma\sigma}(\vec{q})-\gamma^{\sigma\sigma}(\vec{q},\omega)]\chi^{\sigma\sigma'}(\vec{q},\omega)=-2\sigma Ng^2\mu_B^2\langle S^Z\rangle\delta\sigma\sigma'+[\Omega^{\sigma-\sigma}(\vec{q})+\gamma^{\sigma-\sigma}(\vec{q},\omega)]\chi^{-\sigma\sigma'}(\vec{q},\omega) . \tag{24}$$

The solution of Eq. (24) gives an exact expression for the transverse dynamical susceptibility as

$$\chi^{\sigma\sigma}(\vec{q},\omega)=\frac{-\sigma g^2\mu_B^2N\langle S^Z\rangle[\omega-A^{-\sigma-\sigma}(\vec{q},\omega)]}{[\omega-A^{\sigma\sigma}(\vec{q},\omega)][\omega-A^{-\sigma-\sigma}(\vec{q},\omega)]-A^{\sigma-\sigma}(\vec{q},\omega)A^{-\sigma\sigma}(\vec{q},\omega)} , \tag{25}$$

where

$$A^{\sigma\sigma'}(\vec{q},\omega)=\Omega^{\sigma\sigma'}(\vec{q})+\gamma^{\sigma\sigma'}(\vec{q},\omega) \tag{26}$$

and from Eqs. (8), (18), (21), and (23)

$$\Omega^{\sigma\sigma'}(q)\doteq\frac{\sigma'}{2N\langle S^Z\rangle}\langle[S_{\vec{q}}^{\sigma},L_0S_{-q}^{-\sigma'}]_-\rangle \tag{27}$$

and

$$\gamma^{\sigma\sigma'}(\vec{q},\omega)=\frac{-i\sigma'}{2N\langle S^Z\rangle}\int dt\, e^{-i\omega t}\Theta(t)\langle[L_0S_{\vec{q}}^{\sigma},e^{it(1-P)L}(1-P)L_0S_{\vec{q}}^{-\sigma'}]_-\rangle . \tag{28}$$

Here,

$$S_{\vec{q}}^{\sigma}=\sum_i e^{-i\vec{q}\cdot\vec{R}_i}S_i^{\sigma}=\sum_{\vec{k}}b_{\vec{k}+\vec{q}\sigma}^{\dagger}b_{k-\sigma} \tag{29}$$

and the operator L_0 is defined as

$$L_0Q\equiv[H_0,Q]_- , \tag{30}$$

with

$$H_0=H-H'_{e-e} . \tag{31}$$

In deriving Eqs. (27) and (28), we have used the fact that $[H'_{e-e},S_i^{\sigma}]_-=0$ because of its invariance under local spin rotations and for any arbitrary operators X and Y

$$\langle[LX,Y]_-\rangle=-\langle[X,LY]_-\rangle , \tag{32}$$

which follows from the cycle invariance property of the trace implied in the ensemble average.

III. SPIN-WAVE STIFFNESS CONSTANT

The spin-wave stiffness constant is obtained from the long-wavelength limit of the spin-wave spectrum which, from the poles of the transverse dynamical susceptibility (25), is given by the quadratic equation

$$\omega^2-P(\vec{q},\omega)\omega-T(\vec{q},\omega)=0 , \tag{33}$$

where

$$P(\vec{q},\omega)=\sum_{\sigma}A^{\sigma\sigma}(\vec{q},\omega) \tag{34}$$

and

$$T(\vec{q},\omega)=A^{\sigma-\sigma}(\vec{q},\omega)A^{-\sigma\sigma}(\vec{q},\omega)-A^{\sigma\sigma}(\vec{q},\omega)A^{-\sigma-\sigma}(\vec{q},\omega) . \tag{35}$$

From Eqs. (27) and (28), it can be shown that at $\vec{q}=0$

$$P(0,\omega)=T(0,\omega)=0 \tag{36}$$

and

$$A^{\sigma\sigma'}(0,\omega)=A^{-\sigma\sigma'}(0,\omega)=-A^{-\sigma-\sigma'}(0,\omega) \tag{37}$$

and therefore, in the long-wavelength limit, Taylor's expansion of $P(\vec{q},\omega)$ and $T(\vec{q},\omega)$ gives the spin-wave spectrum as

$$\omega=D_Aq , \tag{38}$$

where q is the magnitude of the spin-wave wave vector $\vec{q}$ and the spin-wave stiffness constant D_A is

$$D_A=\lim_{\substack{q\to 0,\\ \omega\to 0}}[(1/q^2)T(\vec{q},\omega)]^{1/2}\ , \tag{39}$$

which, from Eqs. (35)–(37), is given as

$$D_A=-\lim_{\substack{q\to 0,\\ \omega\to 0}}\left[\frac{1}{2}\sum_{\sigma}\sigma A^{\sigma\sigma}(\vec{q},\omega)\right]\lim_{\substack{q\to 0,\\ \omega\to 0}}\left[\frac{1}{q}\sum_{\sigma\sigma'}\sigma A^{\sigma\sigma'}(\vec{q},\omega)\right]^{1/2}\ . \tag{40}$$

It can be rewritten in a form similar to that of Fisher[11] [see Eq. (40) of his paper] for antiferromagnets as

$$D_A=(2ad)^{1/2}/2N\langle S^Z\rangle\ , \tag{41}$$

where from Eqs. (26)–(28) and (32)

$$a=\lim_{\substack{q\to 0,\\ \omega\to 0}}\left[\frac{1}{2}\sum_{\sigma}\langle[L_0S^{\sigma}_{\vec{q}},S^{-\sigma}_{-\vec{q}}]_-\rangle+\frac{1}{2}i\int dt\, e^{-i\omega t}\Theta(t)\sum_{\sigma}\langle[L_0S^{\sigma}_{\vec{q}},e^{it(1-P)L}(1-P)L_0S^{-\sigma}_{-\vec{q}}]_-\rangle\right] \tag{42}$$

and

$$d=\lim_{\substack{q\to 0,\\ \omega\to 0}}\frac{1}{2q^2}\left\{\left\langle\left[L_0\sum_{\sigma}\sigma S^{\sigma}_{\vec{q}},\ \sum_{\sigma}\sigma S^{\sigma}_{-\vec{q}}\right]_-\right\rangle+i\int dt\, e^{-i\omega t}\Theta(t)\left\langle\left[L_0\sum_{\sigma}\sigma S^{\sigma}_{\vec{q}},e^{it(1-P)L}(1-P)L_0\sum_{\sigma}\sigma S^{\sigma}_{-\vec{q}}\right]_-\right\rangle\right\}\ . \tag{43}$$

In the long-wavelength limit, the term

$$L_0\sum_{\sigma}\sigma S^{\sigma}_{\vec{q}}\equiv\left[H_0,\ \sum_{\sigma}\sigma S^{\sigma}_{\vec{q}}\right]$$

is proportional to $\vec{q}$ and the terms, multiplied by the projection operator P, are proportional to q^2. Therefore, up to the order q^2, the terms multiplied by the projection operator P do not contribute, and Eqs. (42) and (43) become

$$a=\lim_{\substack{q\to 0,\\ \omega\to 0}}\left[\frac{1}{2}\sum_{\sigma}\langle[L_0S^{\sigma}_{\vec{q}},S^{-\sigma}_{-\vec{q}}]_-\rangle+\frac{1}{2}i\int dt\, e^{-i\omega t}\Theta(t)\sum_{\sigma}\langle[L_0S^{\sigma}_{\vec{q}},e^{itL}L_0S^{-\sigma}_{-\vec{q}}]_-\rangle\right] \tag{44}$$

and

$$d=\lim_{\substack{q\to 0,\\ \omega\to 0}}\frac{1}{2q^2}\left\{\left\langle\left[L_0\sum_{\sigma}\sigma S^{\sigma}_{\vec{q}},\ \sum_{\sigma}\sigma S^{\sigma}_{-\vec{q}}\right]_-\right\rangle+i\int dt\, e^{-i\omega t}\Theta(t)\left\langle\left[L_0\sum_{\sigma}\sigma S^{\sigma}_{\vec{q}},e^{itL}L_0\sum_{\sigma}\sigma S^{\sigma}_{-\vec{q}}\right]_-\right\rangle\right\}\ . \tag{45}$$

It should be noted that the second terms on the right-hand sides of Eqs. (44) and (45) are nothing but the Fourier transform of the Greens' functions

$$i\Theta(t)\sum_{\sigma}\langle[L_0S^{\sigma}_{\vec{q}},L_0S^{-\sigma}_{-\vec{q}}(t)]_-\rangle$$

and

$$i\Theta(t)\left\langle\left[L_0\sum_{\sigma}\sigma S^{\sigma}_{\vec{q}},L_0\sum_{\sigma}\sigma S^{\sigma}_{-\vec{q}}(t)\right]_-\right\rangle\ ,$$

respectively. Thus, the quantities a and d given by Eqs. (44) and (45), respectively, are the same as that of Fisher.[11] The only difference is that he worked in two sublattice model of antiferromagnets, while we worked in terms of operators transformed under a transformation which rotates the spins locally. Also, our results are more general in the sense that they are applicable to SDW states of which antiferromagnetic state is a special case.

In Eqs. (44) and (45), the first term can be calculated exactly after substituting the commutator

$$L_0S^{\sigma}_{\vec{q}}=[H_0,S^{\sigma}_{\vec{q}}]_-=\sum_{\vec{k}}(T^{\pm}_{\vec{k}+\vec{q}}-T^{\pm}_{\vec{k}})b^{\dagger}_{\vec{k}+\vec{q}\sigma}b_{\vec{k}-\sigma}-i\sigma\sum_{\vec{k}}(T^{\mp}_{\vec{k}+\vec{q}}b^{\dagger}_{\vec{k}+\vec{q}-\sigma}b_{\vec{k}-\sigma}-T^{\mp}_{\vec{k}}b_{\vec{k}+\vec{q}\sigma}b_{\vec{k}\sigma})\ , \tag{46}$$

which one gets directly from Eqs. (4) and (31). But the second terms, in general, cannot be calculated without making some kind of approximation. The difficulty arises because of the presence of the exponential e^{itL}. We calculated these terms by dividing the Liouville operator L into two parts.

$$L=\tilde{L}_0+\tilde{L}_1 \ , \tag{47}$$

where $\tilde{L}_0$ and $\tilde{L}_1$ are assumed to correspond to the Hamiltonians

$$\tilde{H}_0=\sum_{\vec{k}\sigma}(T^{\pm}_{\vec{k}}+\Sigma^{\pm}_{\vec{k}\sigma})b^{\dagger}_{\vec{k}\sigma}b_{\vec{k}\sigma} + i\sum_{\vec{k}\sigma}\sigma(T^{\mp}_{\vec{k}}+\Sigma^{\mp}_{\vec{k}\sigma})b^{\dagger}_{\vec{k}\sigma}b_{\vec{k}-\sigma} \tag{48}$$

and

$$\tilde{H}_1=H-\tilde{H}_0 \ , \tag{49}$$

respectively. The self-energies $\Sigma^{\pm}_{\vec{k}\sigma}$ are supposed to arise from the Hamiltonian H'_{e-e}, in such a way that the effect of $\tilde{H}_1$ is a small perturbation to $\tilde{H}_0$. It is possible to diagonalize $\tilde{H}_0$ in the form

$$\tilde{H}_0=\sum_{\vec{k}}E_{\vec{k}\sigma}d^{\dagger}_{\vec{k}\sigma}d_{\vec{k}\sigma} \ , \tag{50}$$

where

$$d_{\vec{k}\sigma}=\cos(\tfrac{1}{2}\Theta_{\vec{k}})b_{\vec{k}\sigma}+i\sin(\tfrac{1}{2}\Theta_{\vec{k}})b_{\vec{k}-\sigma} \ , \tag{51}$$

$$\tan\Theta_{\vec{k}}=\frac{\sum_{\sigma}(T^{\mp}_{\vec{k}}+\Sigma^{\mp}_{\vec{k}\sigma})}{\sum_{\sigma}\sigma(T^{\pm}_{\vec{k}}+\Sigma^{\pm}_{\vec{k}\sigma})} \ , \tag{52}$$

and

$$E_{\vec{k}\sigma}=\frac{1}{2}\left\{\sum_{\sigma}(T^{\pm}_{\vec{k}}+\Sigma^{\pm}_{\vec{k}\sigma})+\sigma\left[\left[\sum_{\sigma}\sigma(T^{\pm}_{\vec{k}}+\Sigma^{\pm}_{\vec{k}\sigma})\right]^2+\left[\sum_{\sigma}(T^{\mp}_{\vec{k}}+\Sigma^{\mp}_{\vec{k}\sigma})\right]^2\right]^{1/2}\right\} . \tag{53}$$

For the exponential e^{itL}, we use the operator identity

$$e^{itL}\equiv e^{it\tilde{L}_0}+i\int_0^t d\tau\, e^{i\tau\tilde{L}_0}\tilde{L}_1e^{i\tau L} \tag{54}$$

and substitute Eq. (46) for $L_0S^{\sigma}_{q}$ in Eqs. (44) and (45). Then, after converting the operators $b_{\vec{k}}$'s into $d_{\vec{k}}$'s from Eq. (51), using the equation

$$\exp(it\tilde{L}_0)d^{\dagger}_{\vec{k}_1\sigma_1}d_{\vec{k}_2\sigma_2}=\exp[it(E_{\vec{k}_1\sigma_1}-E_{\vec{k}_2\sigma_2})d^{\dagger}_{\vec{k}_1\sigma_1}d_{\vec{k}_2\sigma_2} \tag{55}$$

obtained from Eq. (50), and by performing the time integrals, Eqs. (44) and (45) become

$$a=\sum_{\vec{k}}T^{\mp}_{\vec{k}}\sin\theta_{\vec{k}}\left[1-T^{\mp}_{\vec{k}}\sin^2\theta_{\vec{k}}\Big/\left(\frac{1}{2}\sum_{\sigma}(T^{\mp}_{\vec{k}}+\Sigma^{\mp}_{\vec{k}\sigma})\right)\right]\sum_{\sigma}\sigma\langle d^{\dagger}_{\vec{k}\sigma}d_{\vec{k}\sigma}\rangle$$
$$-\lim_{\substack{q\to 0,\\ \omega\to 0}}\left\{\frac{1}{2}\int_{-\infty}^{\infty}dt\, e^{-i\omega t}\Theta(t)\sum_{\sigma}\left\langle\left[L_0S^{\sigma}_{\vec{q}},\int_{\sigma}^{t}d\tau\, e^{i\tau\tilde{L}_0}\tilde{L}_1e^{i\tau L}L_0S^{-\sigma}_{-\vec{q}}\right]_-\right\rangle\right\} \tag{56}$$

and

$$d=\frac{1}{2q^2}\sum_{\vec{k}}(\vec{q}\cdot\vec{\nabla}_{\vec{k}})T^{\pm}_{\vec{k}}\sum_{\sigma}\langle d^{\dagger}_{\vec{k}\sigma}d_{\vec{k}\sigma}\rangle+\frac{1}{2q^2}\sum_{\vec{k}}\left[(\vec{q}\cdot\vec{\nabla}_{\vec{k}})^2T^{-}_{k}\sin\theta_{\vec{k}}-\frac{2(\vec{q}\cdot\vec{\nabla}_{\vec{k}}T^{\pm}_{\vec{k}})^2\cos^3\nabla_{\vec{k}}}{\Sigma^{\pm}_{\vec{k}_+}-\Sigma^{\pm}_{\vec{k}_-}}\right]\sum_{\sigma}\sigma\langle d^{\dagger}_{\vec{k}\sigma}d_{\vec{k}\sigma}\rangle$$
$$-\lim_{\substack{q\to 0,\\ \omega\to 0}}\left\{\frac{1}{2q^2}\int dt\, e^{-i\omega t}\Theta(t)\left\langle\left[L_0\sum_{\sigma}\sigma S^{\sigma}_{\vec{q}},\int_0^t d\tau\, e^{i\tau\tilde{L}_0}\tilde{L}_1e^{i\tau L}L_0\sum_{\sigma}\sigma S^{\sigma}_{-\vec{q}}\right]_-\right\rangle\right\} . \tag{57}$$

Equations (56) and (57) are the exact expressions for the quantities a and d of the spin-wave stiffness constant D_A for itinerant-electron-spin-density-wave states. They hold good for all one-band itinerant-electron Hamiltonians provided that electron-electron interaction part H_{e-e} is invariant to the local spin rotations. Usually the Hamiltonian $\tilde{H}_0$ can be considered a good approximation to the total Hamiltonian H, and hence the contributions from the third terms on the right-hand side of Eqs. (56) and (57) are quite small and can be neglected for all practical purposes. Thus, a good estimate of spin-wave stiffness constant depends on the reliability of the self-energies $\Sigma^{\pm}_{\vec{k}\sigma}$. It is easy to see that within the Hartree-Fock approximations ($\Sigma^{\pm}_{\vec{k}\sigma}=In_{-\sigma}$, $\Sigma^{\mp}_{\vec{k}\sigma}=0$ for the Hubbard model), the expressions for a and d are identical to that of Fisher[11] for antiferromagnets.

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