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Droplet models, renormalization group, and essential singularities at first-order phase transitions

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We propose (in $d = 2$) a modified form of the droplet model studied by Langer and Fisher. We solve this model exactly and with the renormalization group. Both methods obtain an essential singularity of the form proposed by Klein, Wallace, and Zia; however, the renormalization group misses an additional singularity identical to that of the standard droplet model. We discuss the implications of this result for Ising models in $d = 2$. Higher dimensions are also discussed.

I. INTRODUCTION

One of the most interesting and challenging problems in the study of condensed matter is to develop a theory of the metastable state. The problem is important from the materials point of view since many important substances are metastable. It is also important from the point of view of fundamental statistical mechanics as no underlying formalism equivalent to the ensemble theory of equilibrium statistical mechanics is available for the study of metastability.

One idea for treating metastability inspired by mean-field theories, is to assume that the functions of interest (e.g., equations of state or thermodynamic potentials) can be analytically continued through the transition point (Fig. 1) and that this analytic continuation describes the metastable state. This assumption of analytic continuation has been questioned by various workers in the field.[1–6] Although most agree that analytic continuation *through* the point of the first-order transition is impossible (i.e., there exists a very weak singularity at the transition point (in addition to the step function jump in the order parameter) there remains disagreement as to whether analytic continuation *around* the transition point in the complex H (z or ρ) plane can correctly describe the metastable state. Two approaches which reach opposite conclusions on this point are the droplet-model approximation pursued originally by Langer[2] and Fisher[3] and the renormalization-group (RG) work of Klein, Wallace, and Zia[4] (KWZ). This difference is somewhat surprising since many of the conclusions in KWZ are based on considerations similar to those employed in the droplet model. Moreover, both approaches have been criticized[7,8]. The droplet model for ignoring possibly important classes of fluctuations[7] and the RG approach for certain apparent inconsistencies.[8]

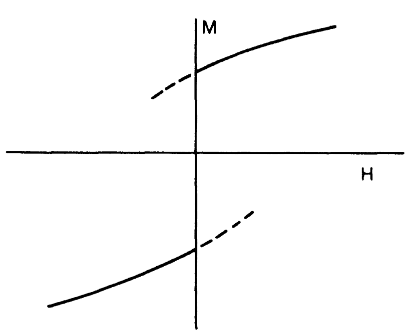

FIG. 1. Magnetization as a function of magnetic field for $T < T_c$. Dotted lines represent analytic continuation of the isotherm to represent the metastable state.

The purpose of this paper is twofold. First, we modify the standard droplet model in a way which makes it (we believe) physically more reasonable. With this modification the model is still solvable exactly on two dimensions. In the second part of our program we analyze the model in two dimensions with the RG technique employed by Klein, Wallace, and Zia and clarify certain misconceptions about this approach. We also investigate the possible analytic continuation of our modified droplet model and find that it cannot describe a metastable state.

This paper is structured as follows. In Sec. II we briefly review the droplet model and the RG approach of Klein, Wallace, and Zia. In Sec. III we introduce the modified droplet model and discuss its applicability to Ising models in two dimensions, and we solve the model exactly. In Sec. IV we present the RG solution and in Sec. V we discuss higher dimensions. In the final section we discuss the implication of this work for theories of metastability and for recent series analysis of the singular structure at the first-order phase transition.

II. DROPLET MODEL AND RG APPROACH

In this section we review briefly the properties of the droplet model and the RG approach used by Klein, Wallace, and Zia.

The droplet model approximates the free energy per site of an Ising like model by assuming that the only important fluctuations are spherical droplets, i.e.,

$$F(H,K)=\sum_{l=0}^{\infty}\exp(-Hl^d-kl^{d-1}) \quad . \tag{2.1}$$

If we restrict ourselves to two dimensions and define $t=(H/K)l$ we can convert Eq. (2.1) into[2]

$$F(H,K)=\frac{K}{H}\int_0^\infty e^{-(K^2/H)(t^2+t)}\,dt \tag{2.2}$$

in the limit $H\rightarrow 0$. This integral can be done by completing the square and we find

$$F(H,K)=(\sqrt{H})^{-1}\exp\left[\frac{K^2}{4H}\right]\operatorname{erfc}\left[\frac{K}{2\sqrt{H}}\right] \quad . \tag{2.3}$$

In the limit $H\rightarrow 0$ the singular structure of Eq. (2.3) is the same as that of Eq. (2.1). $F(H,K)$ has an infinitely differentiable singularity[9] at $H=0$. In addition, there is a branch cut along the negative H axis which introduces a discontinuity in the imaginary part of

$$2[(|H|)^{1/2}]^{-1}\exp(-K^2/4|H|) \quad . \tag{2.4}$$

The interpretation of this result is that in the droplet model one cannot analytically continue through the point of the first-order transition (i.e., $H=0$) to obtain a description of the metastable phase, however one can analytically continue around $H=0$ in the complex H plane (Fig. 2) and obtain such a description. The real part of the continued free energy is the free energy of the metastable phase, and the discontinuity in the imaginary part is inversely related to the lifetime of that phase. This model, if applicable to real systems, would mean that the metastable phase is capable of being described by analytic continuation of the stable phase functions although in a different and more subtle way than envisioned in the mean-field theories. Evidence for a weak singularity at $H=0$ for $T<T_c$ has been found[5,6,8] in the Ising model. The droplet model, however, has ignored a large class of fluctuations the inclusion of which may be important.

Klein, Wallace, and Zia investigated the possibility of finding such a weak singularity in the Ising model with the aid of the renormalization group. The basis of the approach is the following assumption: For large enough magnetic field $\tilde{H}$ the dropletlike fluctuations which cause the singularity will be sufficiently damped to allow truncation of the low-temperature expansion for the free energy per spin. That is

$$M(\tilde{H},\tilde{K})=1+e^{-2\tilde{H}-2C\tilde{K}}+\cdots \tag{2.5}$$

is a valid description of the magnetization per spin for large enough $\tilde{H}$. Here $\tilde{K}$ is the coupling constant and c the coordination number. From considerations similar to those employed in the droplet model, $\tilde{H}$ should be large enough so that the damping of the droplet fluctuations should be dominated by the volume term; i.e.,

$$\alpha\tilde{K}/\tilde{H}=1 \quad . \tag{2.6}$$

Here α is a constant (dependent on the dimension d) which would be determined in a complete theory. Since the singularity occurs for values of the coupling constant K and magnetic field H such that $H\rightarrow 0$ and $K/H\rightarrow\infty$ we must find a relation between $\tilde{H}$, $\tilde{K}$ and H, K. This is accomplished via the RG. For H sufficiently small and $K>K_c$ (the critical value of the coupling constant) RG flow lines approach the neighborhood of the $H=0$, $K=\infty$ fixed point (Fig. 3). In this neighborhood the recursion relations are (in $d=2$)

$$H'=b^2H \quad , \quad K'=bK \quad . \tag{2.7}$$

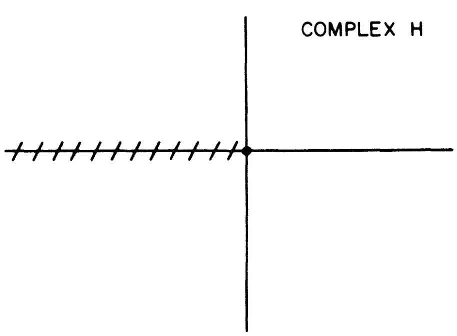

FIG. 2. Location of singularities of the free energy of the droplet model in the complex H plane. There is an infinitely differentiable singularity at $H=0$ and a branch cut along the negative real axis.

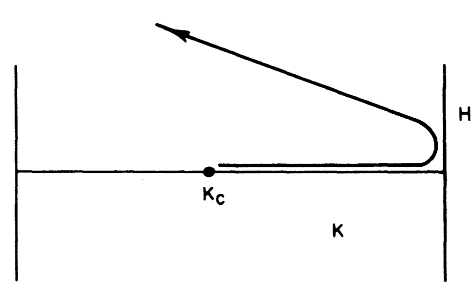

FIG. 3. Schematic representation of the renormalization-group flow lines for the Ising model in $d=2$. Restriction to two scaling fields was imposed for simplicity.

We now renormalize J times until

$$\tilde{H} = b^{2J} H \ , \tag{2.8}$$

$$\tilde{K} = b^J K \ , \quad \alpha \tilde{K} = \tilde{H} \ . \tag{2.9}$$

From Eqs. (2.5), (2.8), and (2.9) we have that the singular part of $M(H,K)$ is given by

$$M(H,K) = \exp[-2(\alpha^2 + c\alpha)K^2/h] \ . \tag{2.10}$$

There are many unanswered questions about this result which, if correct, strongly suggests that no analytic continuation to a metastable state is possible in $d=2$. This is rather puzzling as the RG method depends heavily on droplet-model ideas and the droplet model does allow such an analytic continuation. Another confusing feature of this method is that instead of renormalizing until $\alpha\tilde{K}/\tilde{H}=1$ one could just as plausibly have renormalized until

$$\tilde{H} = A \ , \tag{2.11}$$

where the equation

$$M(\tilde{H},\tilde{K}) = 1 + e^{-2A-2C\tilde{K}} + \cdots \tag{2.12}$$

in a valid description. Using Eqs. (2.8), (2.11), and (2.12) gives the singular function

$$M_S(H,K) = \exp(-2cK/H^{d-1/d}) \ .$$

In the next section we modify the droplet model in order to begin to take into account fluctuations which are not dropletlike. We shall see that this modified droplet model has a singularity which is a linear combination of a usual droplet-model singularity [i.e., Eq. (2.3)] and a term similar to Eq. (2.10). We shall also see that the RG approach of KWZ correctly obtains the latter but misses the former. Moreover many of the puzzling aspects of the KWZ approach mentioned above will be clarified.

III. MODIFIED DROPLET MODEL

The main assumption of the droplet-model approximation is that the dominant fluctuations are dropletlike; i.e., the surface to volume ratio is l^{d-1}/l^d. The basis of this hypothesis is that for very low-temperature entropy considerations should be secondary and the structure of important fluctuations should be determined by energy considerations. That is, dominant fluctuations are those for which the *energy* difference resulting from a fluctuation, e.g., of overturned spins is minimized.

This minimal energy difference is clearly realized by droplet configurations.

The basis of our modification is the observation that for a fluctuation of size l such that

$$Hl^d >> Kl^{d-1} \tag{3.1}$$

the energy difference generated by a fluctuation is dominated by the volume contribution and the surface contribution is negligible. Since the energy contribution from the surface is negligible the fluctuations can lower the free energy difference by assuming more ramified shapes to increase entropy. However, droplets are not uniform. There also exists internal surfaces which can increase entropy by becoming more numerous and ramified. This mechanism might allow the fluctuations to remain as compact droplets. The lower the dimension, however, the less likely this will occur. (We will discuss higher dimensions in Sec. VI.) We find support for this point of view from global flow diagrams in RG calculations.

Consider the RG flow for the Ising model in two dimensions (Fig. 3). For unrenormalized H small but nonzero and unrenormalized $K > K_c$ the stable fixed point is at infinite temperature.[10] We interpret this in the following way. The renormalized free energy includes only the effects of fluctuations with a linear dimension greater than b^n where b is the rescaling length and n is the number of times we have renormalized. For large enough n the renormalized temperature is high. We can then ask what kind of fluctuations occur for systems in high magnetic fields and high temperatures. Although (as far as we know) no calculations or simulations have been performed to answer this question, qualitatively we would conclude that due to the high temperature the dominant fluctuations would not be spherical droplets but more ramified shapes.

Conversely, in the percolation problem where we know[11] that large clusters are droplets independent of the value of the magnetic field, like variable h (ghost field), the RG flows indicate that the stable fixed point is $h=\infty$, $p=1$, where p is a temperaturelike variable (Fig. 4) and $p=1$ corresponds to $T=0$. Returning to the RG flow for the $d=2$ Ising model we can see from approximate calculations that in the neighborhood of the $H=0$, $K=\infty$ the renormalized coupling constant K grows upon renormalization and

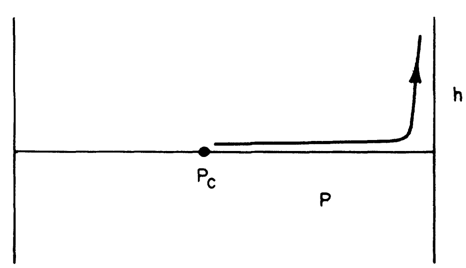

FIG. 4. Schematic representation of the renormalization-group flow lines for percolation in $d=2$. The probability p that a site is occupied is the temperaturelike variable and h, the ghost field, is analogous to the magnetic field.

continues to grow until $K^{(n)} \sim H^{(n)}$. For $H^{(n)} \gtrsim K^{(n)}$ the renormalized K decreases. (For the sake of simplicity we are restricting ourselves to two scaling fields.)

The character of these RG flows and the physical arguments discussed above lead us to propose that a more realistic approximation to Ising behavior for small T in $d=2$ is

$$F(K,H) = \sum_{l=0}^{l_{\max}} e^{-Hl^2-Kl} + \sum_{l=l_{\max}}^{\infty} e^{-Hl^{\tau}-Kl^{\sigma}} , \quad (3.2)$$

where

$$l_{\max} = \alpha K/H, \quad \alpha > 1 \quad (3.3)$$

and $\sigma > 1, \tau \leq 2$. The second sum in Eq. (3.2) is clearly not a representation of the system for droplet sizes up to infinity but is only meant to represent a possible first-order correction to the standard droplet model. We will not concern ourselves very much with this term. We will however, argue (Appendix B) that it's singular behavior is different from the first term, and hence, cannot cancel singularities obtained from the first term in Eq. (3.2).

Consider then only the first term in Eq. (3.2):

$$\bar{F}(K,H) = \sum_{l=0}^{\alpha K/H} e^{-Hl^2-Kl} . \quad (3.4)$$

As in Sec. II we can convert this sum into an integral in the limit $H \to 0$

$$\begin{aligned}\bar{F}(K,H) &= \frac{K}{H}\int_0^{\alpha} e^{-(K^2/H)(t^2+t)}\,dt \\ &= \frac{K}{H}\int_0^{\infty} e^{-(K^2/H)(t^2+t)}\,dt \\ &\quad - \frac{K}{H}\int_{\alpha}^{\infty} e^{-(K^2/H)(t^2+t)}\,dt . \end{aligned} \quad (3.5)$$

Completing the square we have

$$\begin{aligned}\bar{F}(K,H) &= \frac{1}{\sqrt{H}} \exp\left[\frac{K^2}{4H}\right] \operatorname{erfc}\left[\frac{K}{2\sqrt{H}}\right] \\ &\quad - \frac{1}{\sqrt{H}} \exp\left[\frac{K^2}{4H}\right] \operatorname{erfc}\left[(\alpha+\tfrac{1}{2})\frac{K}{\sqrt{H}}\right] . \end{aligned} \quad (3.6)$$

The first term in Eq. (3.6) is identical to the free energy for the usual droplet model [Eq. (2.3)]. The second term however behaves asymptotically as

$$[(\alpha+\tfrac{1}{2})K]^{-1}\exp[-(\alpha^2+\alpha)K^2/H] . \quad (3.7)$$

The effect of the cutoff at $l_{\max} = \alpha K/H$ is to generate an essential singularity. The primary importance of this term is that it cannot be analytically continued[9] to describe the metastable state. The difficulty arises from the fact that the analytic continuation of Eq. (3.6) diverges as $H \to 0$ along the negative real axis. Consequently, if this modified droplet model correctly describes the physics in two dimensions then analytic continuation around $H=0$ in the complex H plane is not a correct method for describing the metastable state.

We have also investigated the effect of smoothing the cutoff in Eq. (3.2). We have considered

$$\bar{F}'(K,H) = \sum_{l=0}^{\infty} \frac{e^{-Hl^2-KL}}{1+\exp[(Hl^n - Kl^{n-1} - \delta K^n/H^{n-1})p]} , \quad (3.8)$$

where $\delta = \alpha^n - \alpha^{n-1}$. If we were to take $\lim p \to \infty$ we would retrieve Eq. (3.4). It can be shown (see Appendix A and Sec. IV) that $\bar{F}'(K,H)$ given by Eq. (3.8) has the same asymptotic behavior as $\bar{F}(K,H)$ [given by Eq. (3.4)] if $n > 2$. That is, if the damping for large droplets (i.e., $l > \alpha K/H$) is stronger than that of the standard droplet model an essential singularity of the form given in Eq. (3.7) is introduced.

IV. RENORMALIZATION GROUP

In this section we apply the RG technique employed by KWZ to the modified droplet model. We use the block-spin ideas of Neimeijer and Van Leeuwen[12] and Kadanoff.[13]

Consider Eq. (3.4):

$$\bar{F}(K,H) = \sum_{l=0}^{\alpha K/H} e^{-Hl^2-Kl} .$$

The summand in this series, gives the probability of finding a cluster of size l in the droplet-model approximation.[2] Ne^{-Hl^2-Kl} is the number of such clusters. After renormalization clusters with $l > b$ (where b is the rescaling length) renormalize into clusters of size $l' = l/b$. Clearly the number of clusters of size l' must be equal to the number of clusters of size $l > b$. This implies that in terms of renormalized variables H', K', and $N' = N/b^2$, the number of clusters of size $l(n_l)$ is given by

$$n_l = N'b^2e^{-H'l'^2-K'l'} , \quad (4.1)$$

where

$$H' = b^2H, \quad K' = bK . \quad (4.2)$$

From the equation[12]

$$\bar{F}(K,H) = g(K,H) + (1/b^2)\bar{F}(K',H') , \quad (4.3)$$

which is a fundamental equation of position space RG we see that after renormalization

$$\sum_{l=0}^{\alpha K/H} e^{-Hl^2-Kl} = g(K,H) + \sum_{l'=1}^{\alpha K'/H'} e^{-H'l'^2-K'l'} , \quad (4.4)$$

where H' and K' are given above and $g(K,H)$ (the inhomogeneous term) has the information about droplets with $l' < 1$. As we can see from Eq. (4.4) as we renormalize, the cutoff $\alpha K'/H' = \alpha K/bH$ is reduced by a factor b. We now renormalize until

$$\alpha K^{(J)}/H^{(J)} = 1 \quad , \tag{4.5}$$

where

$$K^{(J)} = b^J K \quad , \tag{4.6a}$$

$$H^{(J)} = b^{2J} H \quad . \tag{4.6b}$$

For these values of the renormalized coupling constants the renormalized free energy in Eq. (4.3) is reduced to (after J renormalizations)

$$\bar{F}(H^{(J)}, K^{(J)}) = e^{-H^{(J)} - K^{(J)}} \quad . \tag{4.7}$$

From Eqs. (4.5) and (4.6) we have that

$$J = \frac{\ln(\alpha K/H)}{\ln b} \quad . \tag{4.8}$$

So that from Eqs. (4.5), (4.7), and (4.8) we have a singularity of the form

$$\bar{F}_\delta(H,K) = e^{-(\alpha^2+\alpha)K^2/H} \quad . \tag{4.9}$$

Comparing with Eq. (3.7) we find that apart from an amplitude the RG procedure correctly produces the essential singularity of the droplet model modified with a cutoff.

It is also clear from Eq. (3.6) that this RG procedure has completely missed a singularity identical to that of the standard droplet model. From this we are forced to conclude that KWZ in their treatment of the Ising model may also have missed such a singularity. We discuss this point again in Sec. VI. However, we will point out here that the presence of a term similar to that in Eq. (4.9) makes the real part of the analytic continuation of the stable state free energy diverge as $H \to 0$ from the negative real side. Hence, this continuation is not a proper description of the metastable state. The RG also allows us to investigate in a rather simple way the behavior of the smoothed cutoff model [Eq. (3.8)]. From Eqs. (3.8) and (4.2) we have

$$\bar{F}(H',K') = \sum_{l'=1}^{\infty} \frac{e^{-H'l'^2 - K'l'}}{1+\exp[(H'l'^n - K'l'^{n-1} - \delta' K'^n/H'^{n-1})p']} \quad , \tag{4.10}$$

where p' and δ' will be determined below.

Expressing the summand in terms of the unrenormalized variables we have

$$\frac{e^{-Hl^2 - Kl}}{1+\exp[(Hl^n - Kl^{n-1} - \delta' K^n/H^{n-1})p'/b^{n-2}]} \quad . \tag{4.11}$$

In order to have the number of droplets of size $l' - l/b$ equal to the number of size l we must have the summand invariant under renormalization. This imposes the choice $\delta' = \delta$ and

$$p' = b^{n-2} p \quad . \tag{4.12}$$

From Eq. (4.12) we see that for $n > 2$ (since as $H \to 0$ we renormalize an infinite number of times) the renormalized p will diverge and we recover the sharp cutoff model as far as the singular behavior is concerned. This will be shown rigorously in Appendix A.

One important question concerning the RG procedure is the justification for stopping the RG flow at $\alpha K^{(J)}/H^{(J)} = 1$ [Eq. (4.5)]. Clearly one cannot apply the above method if one renormalizes until $\alpha K^{(J)}/H^{(J)} < 1$. Since at this point the renormalized free energy vanishes and all of the information resides in the inhomogeneous term in Eq. (4.4).

In order to understand why we cannot stop at $\alpha K^{(J)}/H^{(J)} = 2$ (for example) we must reexamine Eq. (4.4) in more detail.

In order for Eq. (4.4) to be correct the sum

$$\sum_{l'=1}^{\alpha K'/H'} e^{-H'l'^2 - K'l'} \tag{4.13}$$

cannot be restricted to just interger values of l' but must be over all $1 \leq l' \leq \alpha K'/H' (l' = l/b)$. If not then we would leave droplets of various sizes out of the sum. This would not be dangerous if $\alpha K^{(J)}/H^{(J)} \to \infty$ since the singular behavior would not be affected. However, the method we have used requires that we renormalize until $\alpha K^{(J)}/H^{(J)}$ is finite. In the limit $H \to 0$ a finite $\alpha K^{(J)}/H^{(J)}$ requires an infinite number of renormalizations. Consequently if we stop our renormalization at $\alpha K^{(J)}/H^{(J)} = B > 1$ then we have (in the limit $H \to 0$) an infinite number of terms in the series for the renormalized free energy and we could not apply our method. We can only stop renormalizing where there is finite number of terms in the renormalized free energy series [Eq. (4.13)], i.e., when $\alpha K^{(J)}/H^{(J)} = 1$.

V. DIMENSION GREATER THAN TWO

The main evidence we have used to argue for a droplet model with a cutoff rather than the standard droplet model is the RG flows. These flows are well studied in $d = 2$ but little information is available for $d > 2$.

Preliminary studies with two cell clusters indicate that the flows in $d = 3$ are substantially the same as for the two-dimensional case. If this result is substantiated by more thorough studies then we would conclude that in $d = 3$ the droplet model with a cutoff retains the essential physics of the $d = 3$ Ising model below T_c. The method of KWZ would again predict an essential singularity of the form

$$e^{-K^3/H^2} \tag{5.1}$$

at the phase boundary rather than the weaker singularity predicted by the standard droplet model. A two cell calculation in four dimensions is presently being carried out.[14]

VI. CONCLUSION

We have shown that within the context of a strong cutoff droplet model that the result of Klein, Wallace, and Zia is correct in that it obtains exactly (within a constant) a part of the singularity. However the KWZ method does miss a part of the singularity identical to that of the standard droplet model. This is an important point to consider when designing a test to determine which of the two models (if either) properly describes the physics near the first-order transition. An investigation of the derivatives of the free energy with respect to H has (in $d=2$) been carried out by Nienhuis.[8] He showed that $dF^n/dH^n \rightarrow \infty$ as $n \rightarrow \infty$ which leads him to rule out the singularity proposed by KWZ in favor of one similar to the standard droplet model. In light of the work presented in this paper we must conclude that the results of Nienhuis do not rule out the existence of a singularity such as that proposed in the present work.

We have also argued that the droplet model with a cutoff is a more physical model than the standard droplet model in low dimensions. This was based primarily on the form of the RG flows. However, other tests can be employed to investigate the validity of this model. One possibility is to simulate Ising systems in the spirit of Binder[5] in order to ascertain whether the dominant fluctuations for $l > \alpha K/H$ are indeed nondropletlike.

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APPENDIX A

In this appendix we prove that the dominant singular behavior of

$$\bar{F}'(H,K)=\sum_{l=0}^{\infty}\frac{e^{-Hl^2-Kl}}{1+\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]} \tag{A1}$$

is identical to that of

$$\bar{F}(H,K)=\sum_{l=0}^{\alpha K/H} e^{-Hl^2-Kl} \tag{A2}$$

for $n>2$ and K,H real and positive.

We convert Eq. (A1) to an integral which has the same singular behavior as $H\rightarrow 0$

$$\bar{F}'(H,K)=\int_0^{\infty}\frac{e^{-Hl^2-Kl}}{1+\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]} \tag{A3}$$

Equation (A3) can be written as

$$\bar{F}'(H,K)=\int_0^{\alpha K/H}e^{-Hl^2-Kl}dl-\int_0^{\infty}e^{-Hl^2-Kl}\left[\frac{\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]}{1+\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]}-O(l)\right], \tag{A4}$$

where

$$O(l)=\begin{cases}0, & l>\alpha K/H\ ,\\ 1, & l\leq \alpha K/H\ .\end{cases}$$

The first term in Eq. (A4) is identical to the sharp cutoff free energy [Eq. (3.5)].

We now show that the second integral in Eq. (A4), namely,

$$\int_0^{\infty}e^{-Hl^2-Kl}\left[\frac{\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]}{1+\exp[(Hl^n-Kl^{n-1}-\delta K^nH^{n-1})p]}-O(l)\right]dl \tag{A5}$$

is bounded in the limit $H\rightarrow 0$ by $\epsilon A e^{-(K^2/H)(\alpha^2+\alpha)}$ where A is a constant and ϵ is arbitrarily small but finite. We begin by dividing the range of integration of Eq. (A5) into three parts $(0,\alpha K/H,-\epsilon)$, $(\alpha K/H-\epsilon,\alpha K/H+\epsilon)$, and $(\alpha K/H+\epsilon,\infty)$.

We consider the range $(\alpha K/H+\epsilon,\infty)$ first. In this range Eq. (A5) becomes

$$\int_{\alpha K/H+\epsilon}^{\infty}\frac{e^{-Hl^2-Kl}dl}{1+\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]} \quad . \tag{A6}$$

Equation (A6) is bounded by

$$e^{-H(\alpha K/H+\epsilon)^2-K(\alpha K/H+\epsilon)}\int_{\alpha K/H+\epsilon}^{\infty}\frac{dl}{1+\exp[(Hl^n-Kl^{n-1}-\delta K^n H^{n-1})p]} \quad . \tag{A7}$$

In the integral in Eq. (A7) we make the transformation $w=l-\alpha K/H$ and obtain

$$\int_{\epsilon}^{\infty}\frac{dw}{1+\exp\{[H(w+\alpha K/H)^n-K(w+\alpha K/H)^{n-1}-\delta K^n/H^{n-1}]p\}} \quad . \tag{A8}$$

The integral is uniformly convergent in H in the neighborhood of $H=0$ so that we can take the limit $H\to 0$ inside the integral. Since the integrand vanishes as $H\to 0$ we have that Eq. (A7) goes to zero faster than $e^{-(\alpha^2+\alpha)K^2/H}$.

Next we consider the interval $(\alpha K/H-\epsilon,\alpha K/H+\epsilon)$. Equation (A5) becomes

$$\int_{\alpha K/H-\epsilon}^{\alpha K/H+\epsilon}\frac{e^{-Hl^2-Kl}\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]dl}{1+\exp[(Hl^n-Kl^{n-1}-\delta K^n/H^{n-1})p]}-\int_0^{\alpha K/H+\epsilon}e^{-Hl^2-Kl}dl \quad . \tag{A9}$$

Defining $w=l-\alpha K/H$ Eq. (A9) becomes

$$e^{-(K^2/H)(\alpha^2+\alpha)}\int_{-\epsilon}^{\epsilon}\frac{e^{-H[w^2+2\alpha(K/H)w)-Kw}\exp\{[H(w+\alpha K/H)^n-K(w+\alpha K/H)^{n-1}-\delta K^n/H^{n-1}]p\}dw}{1+\exp\{[H(w+\alpha K/H)^n-K(w+\alpha K/H)^{n-1}-\delta K^n/H^{n-1}]p\}}$$
$$-e^{-(K^2/H)(\alpha^2+\alpha)}\int_0^{\epsilon}e^{-H[w^2+2\alpha(K/H)w]-Kw}dw \quad . \tag{A10}$$

Both integrands in Eq. (A10) are bounded over the range of integration so that the contribution from Eq. (A10) is bounded by $C\epsilon e^{-(K^2/H)(\alpha^2+\alpha)}$, where C is independent of ϵ and ϵ is arbitrarily small.

Finally, we consider the range $(0,\alpha K/H-\epsilon)$. In this range Eq. (A5) becomes

$$\frac{K}{H}\int_0^{\alpha-\epsilon' H/K}\frac{e^{-(K^2/H)(t^2+t)}\exp\{[(K^n/H^{n-1})(t^n-t^{n-1}-\delta)p]\}}{1+\exp\{[(K^n/H^{n-1})(t^n-t^{n-1}-\delta)p]\}} \quad , \tag{A11}$$

where $t=lK/H$.

Equation (A11) is bounded by

$$\frac{K}{H}\int_0^{\alpha-\epsilon H/K}e^{-(K^2/H)(t^2+t)}e^{(K^n/H^{n-1})(t^n-t^{n-1}-\delta)p}dt \quad . \tag{A12}$$

Since $n>2$, as $H\to 0$ the maximum value of this integrand will be that value of t for which $t^n-t^{n-1}-\delta$ is closest to zero. That value is clearly $t=\alpha-\epsilon H/K$. For this value the integrand becomes (in the limit $H\to 0$)

$$e^{-(K^2/H)(\alpha^2+\alpha)}e^{(2\alpha\epsilon K+\epsilon K)}\exp\{-(K^{n-1}/H^{n-2})[n\alpha^{n-1}-(n-1)\alpha^{n-2}]\} \quad .$$

Therefore the integral (A11) goes to zero faster than $e^{-(K^2/H)(\alpha^2+\alpha)}$ as $H\to 0$. Collecting the contributions of the three ranges of integration we find that as $H\to 0$ Eq. (A5) is bounded by $A\epsilon e^{-(\alpha^2+\alpha)K^2/H}$, where A is independent of ϵ and ϵ is arbitrarily small but finite. This implies that the free energy given by Eq. (A1) in the limit $H\to 0$ is virtually the same as the free energy given by Eq. (3.6). The only possible modification is a small change to the constant $[(\alpha+\frac{1}{2})k]^{-1}$ in Eq. (3.6). Since ϵ is arbitrarily small the new constant cannot be zero.

APPENDIX B

In this appendix we argue that the term

$$\sum_{l=\alpha K/H}^{\infty}e^{-Hl^{\tau}-Kl^{\sigma}} \tag{B1}$$

in Eq. (3.2) will not have singular terms which cancel those from the first term in Eq. (3.2). To see this we rewrite Eq. (B1) as

$$\sum_{l=0}^{\infty}e^{-Hl^{\tau}-Kl^{\sigma}}-\sum_{l=0}^{\alpha K/H}e^{-Hl^{\tau}-Kl^{\sigma}} \quad . \tag{B2}$$

The second term in Eq. (B2) can be handled using the KWZ method with RG recursion relations given by

$$H' = B^{\tau} H \quad , \quad K' = b^{\sigma} K \quad . \tag{B3}$$

This produces a singularity of the form

$$\exp[-(\alpha+1)(\alpha K/H)^{\sigma/(\tau-\sigma)} K] \tag{B4}$$

plus a term which cancels the contribution of

$$\sum_{l=0}^{\infty} e^{-Hl^{\tau} - Kl^{\sigma}} \quad . \tag{B5}$$

Equation (B5) is the droplet model as modified by Fisher[3] and solved by him.

The above result holds as long as $\tau > \sigma$. When $\tau = \sigma$ the singular behavior can be obtained by conversion to an integral and performing the integral directly.

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