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Equilibrium order parameters and chemical potentials in rotating superfluids

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Elementary microscopic arguments are given showing that in equilibrium the order parameter of a superfluid rotates with the container. We show that this rotating equilibrium is also a consequence of the phenomenological equations of superfluid hydrodynamics, provided considerable care is taken in specifying which variable in those equations is to be identified with the chemical potential. Our results clarify the physics underlying the so-called gauge wheel effects in superfluid ^{3}He and ^{4}He, demonstrate that recently raised doubts on the validity of the proposed hydrodynamics of ^{3}He are misconceived, and make some novel points about the behavior of the superfluid chemical potential under Galilean transformations and its role in the equations of Landau's two-fluid model.

I. INTRODUCTION

Consider a superfluid in a uniformly rotating container. We shall show that in the equilibrium state, for quite elementary reasons, the superfluid order parameter rotates with the container. We characterize this behavior by the term "rotating equilibrium."

We shall then examine how rotating equilibrium is embodied in the standard equations of superfluid hydrodynamics. This is a somewhat more subtle question. In the case of a scalar superfluid (^{4}He) we shall show that the rotation of the equilibrium scalar field characterizing the order parameter is driven by terms in the hydrodynamic equations that can easily be overlooked, if insufficient care is taken in specifying the variables in those equations that is to be identified with the chemical potential.

In the case of a vector superfluid (^{3}He) these same terms (which we shall refer to as terms of the first type) are present in the hydrodynamics, supplemented by the additional terms (terms of the second type) required to drive a uniform rotation when the order parameter is characterized by vector fields.

The terms of the second type are explicitly present in the proposed hydrodynamic equations for superfluid ^{3}He, and some of the interesting effects they drive in nonequilibrium configurations have recently been explored by Liu and Cross.[1] Comparable attention has not been paid to the similar and equally important role played by the terms of the first type of either ^{4}He or ^{3}He.

There are several reasons for publishing the quite elementary observations that make up the bulk of this paper:

(a) The rotation of the equilibrium order parameter in ^{4}He is relevant to experiments, both actual[2] and *Gedanken*,[3] but the way in which this rotation is built into the two-fluid hydrodynamics has never been made explicit.

(b) For a complete understanding of effects such as those proposed by Liu and Cross it is essential to take into account the implicit terms of the first type that drive the rotation of the order parameter as well as the explicit terms of the second type peculiar to ^{3}He. Spectacular but completely spurious effects can be predicted if the terms of the first type are overlooked.

(c) Arguments have been put forward disputing the form of the proposed hydrodynamic equations for ^{3}He. It has been asserted that the terms of the second type are absent[4] or that they are present, but with a coefficient that vanishes as the temperature approaches zero.[5] Such revised hydrodynamic theories can be ruled out for the simple reason that they do not have rotating equilibrium as a solution.

(d) Certain comparatively obscure expressions in the standard two-fluid description of ^{4}He (and analogous terms in the description of ^{3}He) acquire a strikingly simple and natural form, when the subtleties associated with the proper identification of the chemical potential are made explicit.

In Sec. II we give the elementary argument that if superfluid ^{3}He or ^{4}He is in equilibrium in a uniformly rotating container, then the order parameter rotates with the container. The proof consists of nothing more than an explicit working out of the consequences of the following fact: If there exists a frame of reference in which the Hamiltonian is time independent, then a time-independent thermal equilibrium state can be attained in that frame. In the case at hand the frame rotating with the container is such a frame, and the rotation of the equilibrium order parameter in the lab frame then follows. Readers who already find this convincing should skip all of Sec. II except for the transformation at the end of the section of the simple statements (2.19) and (2.20) of rotating equilibrium into the rather more intricate differential equations suitable for direct comparison with the equations of hydrodynamics.

In Sec. III we show how rotating equilibrium is indeed built into the existing hydrodynamic theories of superfluid ^{4}He and ^{3}He; i.e., we show that when applied to helium in a uniformly rotating container these equations have solutions that describe a superfluid characterized by an order parameter undergoing rigid body rotation at the angular velocity of the container. We note that proposed modifications in the hydrodynamics of ^{3}He do not have this property.

Before this can be done it is necessary to explore some important features of the chemical potential that bear on rotating superfluids but have not, as far as we know, received sufficient emphasis in the existing literature on superfluidity. In Sec. III A we show how the chemical potential transforms under a Galilean transformation, and emphasize that the quantity commonly identified as the chemical potential in most treatments of superfluidity is, in fact, the chemical potential in the local $v_s=0$ frame, and not the laboratory-frame chemical potential. In Sec. III B we show that the equilibrium conditions for a superfluid in a container that rotates uniformly about a stationary axis, require the laboratory-frame chemical potential to be constant. In Sec. III C we show how the conclusions of the preceding sections immediately yield rotating equilibrium when applied to the existing hydrodynamic equations of either helium superfluid. Any proposed modification in the hydrodynamics of ^{3}He must (unlike the proposals in Refs. 4 and 5) continue to pass this simple test.

In Appendix A we emphasize that the hydrodynamic terms of the first kind are as important as the terms of the second kind in understanding what Liu and Cross have described as "gauge wheel" effects in ^{3}He-*A*. In Appendix B we show the simple forms assumed by the Gibbs-Duhem equation and the energy current in ^{4}He, when the different forms assumed by the chemical potential in different frames of reference are explicitly taken into account.

We conclude this introductory section with some cautionary remarks on two common misconceptions:

(a) Rotating equilibrium in ^{4}He is *not* inconsistent with the fact that the superfluid velocity remains zero in a slowly rotating cylindrical container. The superfluid velocity being zero is equivalent to the order parameter being spatially uniform, and the rotation of a uniform order parameter leaves it completely unaltered. To test rotating equilibrium in ^{4}He requires configurations in which the equilibrium order parameter is nonuniform. This happens when the rotation rate of the container is large enough to stabilize a vortex lattice. Rotating equilibrium then requires the vortex lattice to rotate with the container, and that is what it does.[2,6]

(b) Rotating equilibrium is driven by terms in the hydrodynamic equations containing the normal fluid velocity $\vec{v}_n$. Although the normal fluid *density* vanishes at zero temperature in the helium superfluids, it is not correct to conclude from this that the normal fluid *velocity* must drop out of the zero temperature hydrodynamics. The velocity $\vec{v}_n$ is the velocity of the frame of reference in which the helium is locally in equilibrium with external systems with which it can exchange momentum. It follows that $\vec{v}_n$ is the velocity of the frame in which the excitations are in local equilibrium, but $\vec{v}_n$ is not deprived of a hydrodynamical role if there are no excitations present to exhibit this velocity. Moving walls can exchange momentum with the helium even in the absence of excitations, and the terms in the zero temperature hydrodynamics involving $\vec{v}_n$ are essential for the description of this process.

II. ROTATING EQUILIBRIUM: MICROSCOPICS

We inquire into the equilibrium behavior of superfluid ^{3}He or ^{4}He in a uniformly rotating container. The argument is simplest if the container does not have perfect cylindrical symmetry about the rotation axis. It is sufficient to consider this case since it includes the important case of a *macroscopically* symmetric container because of its residual surface roughness or, ultimately, because of its atomic structure.

We can then represent the interaction between the helium and the container by a term in the Hamiltonian of the form

$$V_t=e^{-i\vec{\omega}\cdot\vec{L}t}V_0e^{i\vec{\omega}\cdot\vec{L}t} , \tag{2.1}$$

where V_0 is independent of time. The unitary time evolution operator obeys the equation

$$i(\partial/\partial t)U_t=(H+V_t)U_t , \tag{2.2}$$

where H is the rotationally invariant[7] Hamiltonian of the helium alone. Because of the simple time dependence of V_t one easily verifies that the explicit solution to Eq. (2.2) is

$$U_t=e^{-i\vec{\omega}\cdot\vec{L}t}e^{-i\tilde{H}t} , \tag{2.3}$$

where

$$\tilde{H}=H+V_0-\vec{\omega}\cdot\vec{L} . \tag{2.4}$$

In the Heisenberg picture the time evolution of the field operator $\Psi(\vec{r},t)$ is given by

$$\Psi(\vec{r},t)=U_t^{\dagger}\Psi(\vec{r},0)U_t . \tag{2.5}$$

Using the explicit form [Eq. (2.3)] of U we can write this as

$$\Psi(\vec{r},t) = e^{i\tilde{H}t}e^{i\vec{\omega}\cdot\vec{L}t}\Psi(\vec{r},0)e^{-i\vec{\omega}\cdot\vec{L}t}e^{-i\tilde{H}t}$$

$$= e^{i\tilde{H}t}\Psi(R_t^{-1}\vec{r},0)e^{-i\tilde{H}t} \ , \qquad (2.6)$$

where R_t acts on the coordinate r to rotate it through an angle ωt about the cylinder axis. If we define a rotated field operator $\tilde{\Psi}(\vec{r},t)$ by

$$\tilde{\Psi}(\vec{r},t) = \Psi(R_t\vec{r},t) \ , \qquad (2.7)$$

then Eq. (2.6) gives

$$\tilde{\Psi}(\vec{r},t) = e^{i\tilde{H}t}\tilde{\Psi}(\vec{r},0)e^{-i\tilde{H}t} \ ; \qquad (2.8)$$

i.e., the time evolution of $\tilde{\Psi}$ is governed by the time-dependent Hamiltonian $\tilde{H}$.

Because $\tilde{H}$ is time independent it follows that $\tilde{\Psi}$ has an equilibrium mean value given by the Maxwell-Boltzmann-Gibbs distribution appropriate to $\tilde{H}$; i.e., there exist constants β and μ such that

$$\langle\tilde{\Psi}(\vec{r},t)\rangle = \mathrm{Tr}\, e^{-\beta(\tilde{H}-\mu N)}\tilde{\Psi}(\vec{r},t)/\mathrm{Tr}\, e^{-\beta(\tilde{H}-\mu N)} \qquad (2.9)$$

or[8]

$$\langle\tilde{\Psi}(\vec{r},t)\rangle = e^{-i\mu t}\langle\tilde{\Psi}(\vec{r},0)\rangle \ . \qquad (2.10)$$

In ^{4}He the equilibrium order parameter $\psi(\vec{r},t)$ is just the equilibrium average of the field operator $\Psi(\vec{r},t)$. It follows that when equilibrium is attained in a rotating container, the time evolution of the order parameter is given by

$$\psi(\vec{r},t) = \langle\Psi(\vec{r},t)\rangle = \langle\tilde{\Psi}(R_t^{-1}\vec{r},t)\rangle$$

$$= e^{-i\mu t}\langle\tilde{\Psi}(R_t^{-1}\vec{r},0)\rangle$$

$$= e^{-i\mu t}\langle\Psi(R_t^{-1}\vec{r},0)\rangle$$

$$= e^{-i\mu t}\psi(R_t^{-1}\vec{r},0) \ . \qquad (2.11)$$

In ^{3}He the equilibrium order parameter $\psi(\vec{r}_1s_1, \vec{r}_2s_2,t)$ is the equilibrium average of the product of two fields operators, $\Psi(\vec{r}_1s_1)\Psi(\vec{r}_2s_2)$. Trivial modifications of the argument leading to Eq. (2.10) now imply that when equilibrium is attained in a rotating container, the time evolution of the order parameter in ^{3}He is given by

$$\psi(\vec{r}_1s_1, \vec{r}_2s_2,t) = e^{-2i\mu t}\psi(R_t^{-1}\vec{r}_1s_1, R_t^{-1}\vec{r}_2,s_2,0) \ . \qquad (2.12)$$

In the London limit the ^{4}He order parameter has the form

$$\psi(\vec{r},t) = \psi_0 e^{i\theta(\vec{r},t)} \qquad (2.13)$$

and Eq. (2.10) then requires that the time evolution of the phase is given by

$$\Theta(\vec{r},t) = \Theta(R_t^{-1}\vec{r},0) - \mu t \ . \qquad (2.14)$$

The London limit form of the order parameter for ^{3}He-*A* (the only ^{3}He superfluid phase we consider here in explicit detail) is

$$\psi(\vec{r}_1s_1, \vec{r}_2s_2,t) = \vec{\phi}\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)\cdot(\vec{r}_1-\vec{r}_2)\chi_{s_1s_2}\left[|\vec{r}_1-\vec{r}_2|,\ \vec{l}\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)\cdot(\vec{r}_1-\vec{r}_2)\right] \ , \qquad (2.15)$$

where

$$\vec{\phi} = \hat{\phi}^{(1)} + i\hat{\phi}^{(2)}, \quad \vec{l} = \hat{\phi}^{(1)} \times \hat{\phi}^{(2)} \qquad (2.16)$$

and the $\hat{\phi}^{(i)}$ are a pair of orthonormal vectors. Equation (2.11) then requires that the time evolution of $\vec{\phi}$ is given by

$$\vec{\phi}(\vec{r},t) = R_t\vec{\phi}(R_t^{-1}\vec{r},0)e^{-2i\mu t} \ . \qquad (2.17)$$

The superfluid velocity fields are given by, for ^{4}He and ^{3}He, respectively,

$$\vec{v}_s = (\hbar/M)\vec{\nabla}\Theta \ , \quad \vec{v}_s = (\hbar/2M)\phi_i^{(1)}\vec{\nabla}\phi_i^{(2)} \ . \qquad (2.18)$$

Since μ is independent of $\vec{r}$, it follows from Eqs. (2.14) and (2.17) that in either superfluid the time dependence of $\vec{v}_s$ is just that of a rigid body rotation with angular frequency ω about the cylinder axis

$$\vec{v}_s(\vec{r},t) = R_t\vec{v}_s(R_t^{-1},\ \vec{r},0) \ . \qquad (2.19)$$

In ^{3}He-*A* the anisotropy axis $\vec{l}$ evolves in precisely the same way

$$\vec{l}(\vec{r},t) = R_t\vec{l}(R_t^{-1}\vec{r},0) \ . \qquad (2.20)$$

To facilitate the comparison to be made in Sec. III between the rotating equilibrium forms (2.19) and (2.20) and the equations of hydrodynamics, we rewrite Eqs. (2.19) and (2.20) in the equivalent differential forms

$$(\partial/\partial t)\vec{v}_s = -(\vec{\omega}\times\vec{r})\cdot\vec{\nabla}\vec{v}_s + \vec{\omega}\times\vec{v}_s \qquad (2.21)$$

and

$$(\partial/\partial t)\vec{l} = -(\vec{\omega}\times\vec{r})\cdot\vec{\nabla}\vec{l} + \vec{\omega}\times\vec{l} \ . \qquad (2.22)$$

Elementary vector identities permit us to rewrite Eq. (2.21) as

$$(\partial/\partial t)\vec{v}_s = -\vec{\nabla}(\vec{v}_s\cdot\vec{\omega}\times\vec{r}) + (\vec{\omega}\times\vec{r})\times(\vec{\nabla}\times\vec{v}_s) \ . \qquad (2.23)$$

The rotating equilibrium configuration in ^{4}He ($\vec{\nabla}\times\vec{v}_s = 0$) therefore satisfies the simple differential equation

$$(\partial/\partial t)\vec{v}_s = -\vec{\nabla}(\vec{v}_s\cdot\vec{\omega}\times\vec{r}) \ . \qquad (2.24)$$

In ^{3}He-*A* the curl of v_s does not vanish, but is given by[9]

$$\vec{\nabla} \times \vec{v}_s = \frac{\hbar}{4M} \epsilon_{ijk} l_i \vec{\nabla} l_j \times \vec{\nabla} l_k \ . \qquad (2.25)$$

The last term in Eq. (2.23) then becomes

$$(\vec{\omega} \times \vec{r})(\vec{\nabla} \times \vec{v}_s) = (\hbar/2M) \epsilon_{ijk} l_i \vec{\nabla} l_j (\vec{\omega} \times \vec{r} \cdot \vec{\nabla}) l_k \ . \qquad (2.26)$$

If the term $(\vec{\omega} \times \vec{r} \cdot \vec{\nabla})\vec{l}$ in Eq. (2.26) is replaced by $\vec{\omega} \times \vec{l} - \partial \vec{l}/\partial t$ [see Eq. (2.22)], then the rotating equilibrium configuration in ^{3}He-A satisfies the differential equation

$$\frac{\partial}{\partial t}\vec{v}_s = -\vec{\nabla}\left(\vec{v}_s \cdot \vec{\omega} \times \vec{r} + \frac{\hbar}{2M}\vec{\omega} \cdot \vec{l}\right) + \frac{\hbar}{2\mu} \epsilon_{ijk} l_i \frac{\partial}{\partial t} l_j \vec{\nabla} l_k \ . \qquad (2.27)$$

Equation (2.27) is satisfied if and only if $\vec{v}_s$ and $\vec{l}$ [related by Eq. (2.25)] undergo rigid body precession. We have tortured Eqs. (2.19) and (2.20) to yield this grotesque expression of rotating equilibrium, because in Sec. III we shall see it emerging more naturally from the hydrodynamic equations of ^{3}He-A. We have produced it in the present context to emphasize that it is nothing but a statement of rotating equilibrium, which must therefore continue to be valid in any attempted revision of the ^{3}He-A hydrodynamics.

III. ROTATING EQUILIBRIUM: MACROSCOPICS

The discovery of anisotropic superfluidity in ^{3}He has revived interest in the foundations of two-fluid *hydrodynamics*, as a part of the effort to develop the hydrodynamics of the more exotic ^{3}He superfluids. Less attention has been paid to the much simpler but at least as important question of the foundations of two-fluid *thermodynamics*. In this section we examine with some care certain features of the normal fluid velocity and especially of the chemical potential, in a superfluid (^{3}He or ^{4}He) which is in equilibrium with a specified total momentum and/or orbital angular momentum.[10] Having derived the equilibrium conditions for $\vec{v}_n$ and μ, we will then show how the Eqs. (2.24) and (2.27) characterizing rotating equilibrium are contained in the hydrodynamic equations of superfluid ^{4}He and ^{3}He-A.

A. Normal fluid velocity and the transformation law for the chemical potential

We consider a homogeneous superfluid, characterized by an entropy density $s(\epsilon, \rho, \vec{g}, \vec{v}_s)$ where ϵ, ρ, and $\vec{g}$ are the energy, mass, and momentum densities, and $\vec{v}_s$ is the superfluid velocity field. We suppress explicit reference to the additional Galilean invariant intensive fields that are required to specify the configuration of the more intricate types of superfluids produced by anisotropic pairing (e.g., the $\vec{l}$ field in ^{3}He-A). Such fields are understood to be held constant in all the differentiations performed below.

Two configurations of the superfluid that differ only by a Galilean boost must have the same degree of disorder: i.e., the entropy density must be a Galilean invariant. We can therefore relate the entropy density for any value of $\vec{g}$ to the entropy density s_0 measured by an observer in the $\vec{g} = 0$ frame

$$s(\epsilon, \rho, \vec{g}, \vec{v}_s) = s_0(\epsilon - g^2/2\rho, \rho, 0, \vec{v}_s - \vec{g}/\rho) \ . \qquad (3.1)$$

The Galilean transformation properties of the various thermodynamic functions follow from taking partial derivatives of Eq. (3.2). Thus the ϵ derivative shows that the temperature is a Galilean invariant

$$\frac{1}{T} = \frac{\partial s}{\partial \epsilon} = \left(\frac{\partial s}{\partial \epsilon}\right)_0 \ . \qquad (3.2)$$

The derivative with respect to $\vec{v}_s$ gives a vector invariant

$$\left(\frac{\partial s}{\partial \vec{v}_s}\right) = \left(\frac{\partial s}{\partial \vec{v}_s}\right)_0 \qquad (3.3)$$

and the derivative with respect to $\vec{g}$ relates $\partial s/\partial \vec{g}$ to this invariant

$$\frac{\partial s}{\partial \vec{g}} = -\frac{\vec{g}}{\rho T} - \frac{1}{\rho}\frac{\partial s}{\partial \vec{v}_s} \ , \qquad (3.4)$$

thereby demonstrating that $-T\partial s/\partial \vec{g}$ transforms like $\vec{g}/\rho$, i.e., like a velocity. This velocity is given the name $\vec{v}_n$:

$$\vec{v}_n = -T(\partial s/\partial \vec{g}) \ . \qquad (3.5)$$

Note that $\vec{v}_n$ is to momentum exchange as T is to energy exchange: if a superfluid is in equilibrium with respect to the exchange of energy with any other system, then the two must have the same temperature; if the two are in equilibrium with respect to the exchange of energy *and* momentum, then they must have the same $\vec{v}_n$. We shall be interested in this fact primarily in the case in which the other system is the containing vessel. Since the entropy of the container is not a function of an additional velocitylike field $\vec{v}_s$, a similar application of Galilean invariance to the entropy density of the container shows that $\vec{v}_n$ of the container is simply $\vec{g}/\rho$, the local mean velocity. Thus a superfluid in local equilibrium with a container (which may be moving) has a velocity field $\vec{v}_n$ at the walls of the container equal to the local wall velocity. We shall make use of this familiar boundary condition below.

The transformation properties of the chemical potential follow from taking a density derivative of Eq.

(3.1). With the aid of Eqs. (3.2), (3.4), and (3.5) the result can be cast in the form

$$\frac{\partial s}{\partial \rho}-\frac{\vec{v}_n^2}{2T}=\left(\frac{\partial s}{\partial \rho}\right)_0-\frac{1}{2T}\left(\vec{v}_n-\frac{\vec{g}}{\rho}\right)^2 . \tag{3.6}$$

If the chemical potential (divided by the ^{3}He mass) is defined by

$$\mu=-T(\partial s/\partial \rho) , \tag{3.7}$$

then the content of Eq. (3.6) is that the quantity

$$\mu+\tfrac{1}{2}v_n^2 \tag{3.8}$$

is a Galilean invariant.

It is useful to relate the chemical potential μ to the values of the chemical potential in the local frames in which either $\vec{v}_n(\vec{r})$ or $\vec{v}_s(\vec{r})$ vanishes. Calling these μ_n and μ_s, we have from the invariance of Eq. (3.8)

$$\mu+\tfrac{1}{2}\vec{v}_n^2=\mu_n=\mu_s+\tfrac{1}{2}(\vec{v}_n-\vec{v}_s)^2 . \tag{3.9}$$

Since the chemical potential is Galilean invariant to first order in the velocities the distinctions made in Eq. (3.9) are often of no practical importance.[11] However the discovery of the superfluid phases of ^{3}He has stimulated interest in effects that are nonlinear in the velocities, and in considering these it can be very important to bear in mind the distinctions implied by Eq. (3.9). In particular it is important to note that the quantity called the chemical potential and denoted by the symbol "μ" in most treatments of superfluidity in texts and published papers is, from our point of view, not the true chemical potential but μ_s, the chemical potential in the local $\vec{v}_s=0$ frame.

Since the hydrodynamic equations themselves determine the functional form of μ provided the correct Gibbs-Duhem equation is used, it might appear that the question of which μ is dignified with the title of chemical potential is of no practical interest. This would be true if the hydrodynamic equations could indeed be rigorously solved, but in most applications one applies a blend of mathematical analysis and simplifying physical intuition. One can then overlook important effects by being vague on the question of which chemical potential one expects to be uniform in equilibrium. We show in part B of this section that when ^{3}He or ^{4}He is in equilibrium in a container that rotates uniformly about a fixed axis then it is the true chemical potential μ that is uniform. The extraction of rotating equilibrium from the hydrodynamic equations in part C then depends critically on the fact that Khalatnikov's chemical potential[12] (our μ_s) is *not* uniform in rotating equilibrium.

B. The equilibrium conditions on $\vec{v}_n$ and μ in a uniformly rotating container

Consider superfluid ^{3}He or ^{4}He in a container which we take, for simplicity, to have negligible mass and heat capacity, so that the entropy and other extensive quantities are dominated by the contribution from the helium.[13] A container rotating uniformly about a stationary axis is then characterized by the vanishing of the total momentum of the helium and a fixed nonzero angular momentum. Equilibrium is determined by maximizing the entropy subject to these constraints and the constraints of constant total energy and mass. The equilibrium condition is thus

$$\begin{aligned}0&=\int dV(\delta s-a\,\delta\epsilon-b\,\delta\rho-\vec{c}\cdot\delta\vec{g}-\vec{d}\cdot\vec{r}\times\delta\vec{g})\\&=\int dV\Bigg[\left(\frac{1}{T}-a\right)\delta\epsilon-\left(\frac{\mu}{T}+b\right)\delta\rho\\&\qquad-\left(\frac{\vec{v}_n}{T}+\vec{c}+\vec{d}\times\vec{r}\right)\cdot\delta\vec{g}-\frac{(\vec{g}-\rho\vec{v}_n)}{T}\cdot\delta\vec{v}_s\Bigg] ,\end{aligned} \tag{3.10}$$

where a, b, c, and d are constant Lagrange multipliers.

Equilibrium is thus attained at a uniform temperature *and* a uniform chemical potential μ [with, we emphasize, μ defined by Eq. (3.7)]. The equilibrium $\vec{v}_n$ has the form

$$\vec{v}_n=-T\vec{c}-T\vec{d}\times\vec{r} , \tag{3.11}$$

where the constants $\vec{c}$ and $\vec{d}$ are determined by the following argument.

In ^{4}He we can write $\delta\vec{v}_s$ as $\vec{\nabla}\delta\Theta$. In ^{3}He-*A* there is an additional piece to $\delta\vec{v}_s$ involving $\delta\vec{l}$, but this only contributes to the stationary condition with respect to variations in the $\vec{l}$ field, which we do not consider here. Thus in either superfluid we can replace δv_s by $\vec{\nabla}\delta\Theta$, and integrate by parts. Vanishing of the surface term requires the normal component of $\vec{g}-\rho\vec{v}_n$ to vanish at the surface, and the remaining condition requires the divergence of $\vec{g}-\rho\vec{v}_n$ to vanish in the bulk. The two conditions together then give

$$0=\int dV\vec{r}[\vec{\nabla}\cdot(\vec{g}-\rho\vec{v}_n)]=\int dV\rho\vec{v}_n-\int dV\vec{g} , \tag{3.12}$$

but since the total momentum vanishes, the form [Eq. (3.11)] of $\vec{v}_n$ must reduce to the form

$$\vec{v}_n=\vec{\omega}\times(\vec{r}-\vec{r}_0) , \tag{3.13}$$

where

$$M\vec{r}_0=\int dV\rho\vec{r},\quad \vec{\omega}=-T\vec{d} . \tag{3.14}$$

For the container to be in equilibrium with the helium at the surface [see the remarks following Eq. (3.5)] it must therefore rotate with angular velocity

$\vec{\omega}$ about an axis passing through $\vec{r}_0$. (The surface condition on $\vec{g}-\rho\vec{v}_n$ is then just the condition that there should be no mass flux through the wall of the container in the frame moving with each surface element.)

The two equilibrium conditions of major interest in what follows, for ^{3}He or ^{4}He in a container rotating uniformly about a fixed axis (somewhere along which we place the origin of coordinates) are

$$\vec{\nabla}\mu=0 \ , \quad \vec{v}_n=\vec{\omega}\times\vec{r} \ . \tag{3.15}$$

C. Rotating equilibrium solutions to the hydrodynamic equations

All dissipative terms vanish in equilibrium. The time evolution of v_s in ^{4}He is therefore given by the hydrodynamic equation[14]

$$\partial\vec{v}_s/\partial t=-\vec{\nabla}(\mu_s+\tfrac{1}{2}v_s^2)=-\vec{\nabla}(\mu+\vec{v}_n\cdot\vec{v}_s) \ . \tag{3.16}$$

When the equilibrium conditions (3.15) hold, this reduces precisely to the Eq. (2.24) describing a velocity field $\vec{v}_s$ rotating rigidly with the container.

The nondissipative part of the equation of motion for $\vec{v}_s$ in ^{3}He-*A* is now generally agreed to be

$$\partial\vec{v}_s/\partial t=-\vec{\nabla}[\mu+\vec{v}_n\cdot\vec{v}_s+(\hbar/2M)\gamma\,\vec{l}\cdot\vec{\nabla}\times\vec{v}_n] + \frac{\hbar}{2M}\epsilon_{ijk}l_i\frac{\partial}{\partial t}l_j\vec{\nabla}l_k \ , \tag{3.17}$$

where purely hydrodynamical reasoning requires the coefficient γ to be precisely $\frac{1}{2}$, if angular momentum is to be conserved.[15,16]

The result $\gamma=\frac{1}{2}$ has recently been disputed in papers that purport to derive the hydrodynamics on microscopic grounds. Combescot[4] claims that γ vanishes at all temperatures, while Nagai[5] claims that γ depends on temperature, vanishing at $T\to 0$.

It is therefore instructive to note that if $\gamma=\frac{1}{2}$ (and only if $\gamma=\frac{1}{2}$) then the equilibrium conditions (3.15) reduce Eq. (3.17) to the correct form [Eq. (2.27)] that holds when $\vec{v}_s$ and $\vec{l}$ rotate rigidly with the container. Thus any modification of the *A*-phase hydrodynamics setting γ equal to a value other than $\frac{1}{2}$, must contain additional reactive terms which reduce numerically to $(\hbar/2M)(1-2\gamma)(\vec{l}\cdot\vec{\omega})$, when the equilibrium conditions (3.15) hold. We are not aware of (nor do we see any basis for) such a proposed modification.

Note added in proof: R. Combescot (private communication) tells us that we misinterpreted his theory by characterizing it as giving $\gamma=0$ in Eq. (3.17). His theory has no terms of the second type, but he regards their absence as arising from a non-zero value of $\gamma=1/2$ *and* a precisely cancelling term in the chemical potential. Viewing his theory in this way does not alter our opinion of it.

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APPENDIX A: GAUGE WHEEL EFFECTS

We consider here the time evolution of $\vec{v}_s$ in ^{3}He-*A* under conditions in which the $\vec{l}$ field is independent of time. We stress that this is not necessarily the case in real experiments; we do it here in order to isolate the effects of the terms not explicitly involving $\partial\vec{l}/\partial t$ in the equation of motion (3.17).

When $\vec{l}$ does not depend on time one can regard Eq. (3.17) as asserting that $\vec{v}_s$ is driven by the effective chemical potential

$$\bar{\mu}=\mu+\vec{v}_n\cdot\vec{v}_s+(1/4M)\vec{l}\cdot\vec{\nabla}\times\vec{v}_n \ . \tag{A1}$$

The work of Liu and Cross[1] focuses attention on the term in $\vec{l}\cdot\vec{\nabla}\times\vec{v}_n$ (which we have characterized in Sec. I as a term of the second type). We wish to emphasize here that the term in $\vec{v}_n\cdot\vec{v}_s$ (a term of the first type), which is the only source of gauge wheel effects in ^{4}He, can play a role quite as important as the term of the second type in the gauge wheel effects of ^{3}He-*A*. [We also emphasize that this term can be entirely overlooked if one replaces $\mu+\vec{v}_n\cdot\vec{v}_s$ by $\mu_s+\frac{1}{2}v_s^2$ in Eq. (A1) and disregards the fact that it is μ and not μ_s that is uniform in rotating equilibrium.]

For simplicity we consider configurations of cylindrical symmetry in which $\vec{v}_n$ is, at least initially, entirely along the $\hat{\phi}$ direction (as it would be in equilibrium in a uniformly rotating container). The solution to the curl $\vec{v}_s$ Eq. (2.25) requires that[9] for n an integer

$$\hat{\phi}\cdot\vec{v}_s=(n-l_z)/2Mr \ . \tag{A2}$$

If we then represent $\vec{v}_n$ in the form

$$\vec{v}_n=r\omega(r,z)\hat{\phi} \tag{A3}$$

then the term of the first type in the effective chemical potential becomes

$$\bar{\mu}_{\text{I}}=\vec{v}_n\cdot\vec{v}_s=(1/2M)(n-l_z)\omega(r,z) \ . \tag{A4}$$

On the other hand the term of the second type is

$$\bar{\mu}_{\text{II}}=\frac{1}{4M}\vec{l}\cdot\vec{\nabla}\times\vec{v}_n = \frac{1}{2M}\left[l_z+\frac{1}{2}r\left(l_z\frac{\partial}{\partial r}-l_r\frac{\partial}{\partial z}\right)\right]\omega(r,z) \ . \tag{A5}$$

If ω is essentially uniform for large positive and negative values of z [i.e., we consider two containers above and below the $z=0$ plane, each in a state of local rotating equilibrium (though not necessarily at the same rotation rate) connected by a cylindrically symmetric tube in which ω will in general be nonuniform] then Eq. (A5) gives an effective chemical potential drop between the two regions

$$\Delta\bar{\mu}_{\rm II} = (1/2M)\Delta(l_z\omega) \ . \tag{A6}$$

This suggests, for example, that a supercurrent would be driven between the two regions if ω were the same but $\vec{l}$ reversed between the two regions, or if $\vec{l}$ were the same and ω reversed. The driving term (A6) is, however, precisely canceled by the contribution of the term of the first type [Eq. (A4)] when $n=0$, and when $n\neq 0$ we are left only with a term

$$\Delta\bar{\mu}_{\rm I} + \Delta\bar{\mu}_{\rm II} = (1/2M)n\,\Delta\omega \tag{A7}$$

which is common to both ^{4}He and ^{3}He-A.

Note that such a cancellation is required if rotating equilibrium is not to be violated, since the terms of the second type by themselves would lead to an accelerating supercurrent along the z direction in a uniformly rotating container, provided the direction of $\vec{l}$ reversed between the bottom and the top.

We reemphasize that this cancellation can easily be overlooked in a hydrodynamic analysis if the chemical potential is incorrectly identified *or* if the constraint [Eq. (2.25)] relating the curl of $\vec{v}_s$ to the spatial variation of $\vec{l}$ is not taken explicitly into account.

APPENDIX B: THE TWO-FLUID MODEL IN TERMS OF μ_s AND μ_n

We show here that by exploiting the various possible forms [Eq. (3.9)] in which the chemical potential can appear in the equations of the two-fluid model, one can considerably simplify the two expressions that do not have an entirely transparent structure.

The Gibbs-Duhem equation

$$dP = s\,dT + \rho\,d\mu + \vec{g}\cdot d\vec{v}_n - (\vec{g}-\rho\vec{v}_n)\cdot d\vec{v}_s \tag{B1}$$

follows from defining the pressure to be

$$P = T(\partial S/\partial V) \ . \tag{B2}$$

[To derive Eq. (B1) one uses the extensivity of the total entropy, $S(E,M,\vec{P},\vec{v}_s,V) = Vs(E/V,M/V,\vec{P}/V,\vec{v}_s)$ and the identification of the derivatives of s made in Sec. III A to deduce from Eq. (B2) that

$$P = Ts + \mu\rho + \vec{g}\cdot\vec{v}_n - \epsilon \ . \tag{B3}$$

This immediately gives Eq. (B2), when taken together with the thermodynamic identity

$$T\,ds = d\epsilon - \mu\,d\rho - \vec{v}_n\cdot d\vec{g} - (\vec{g}-\rho\vec{v}_n)\cdot d\vec{v}_s \tag{B4}$$

(which also follows directly from the results in III A).]

By eliminating μ in favor of $\mu_n = \mu + \frac{1}{2}v_n^2$ and introducing ρ_n and ρ_s through the conventional representation of $\vec{g}$ as

$$\vec{g} = \rho_s \underset{\sim}{v}_s + \rho_n \underset{\sim}{v}_n, \quad (\rho = \rho_s + \rho_n) \ ,$$

one can rewrite Eq. (B1) as

$$dP = s\,dT + \rho\,d\mu_n - \tfrac{1}{2}\rho_s\,d(\vec{v}_s - \vec{v}_n)^2 \ , \tag{B5}$$

in which the independent variables are T, μ_n, and $(\vec{v}_s - \vec{v}_n)^2$. Noting, however, that Eq. (3.9) gives

$$\tfrac{1}{2}(\vec{v}_s - \vec{v}_n)^2 = \mu_n - \mu_s \tag{B6}$$

we can replace the last variable by μ_s. Equation (B6) then becomes simply

$$dP = s\,dT + \rho_n\,d\mu_n + \rho_s\,d\mu_s \ . \tag{B7}$$

This has precisely the structure of the Gibbs-Duhem equation for a classical two-component system, although the interpretation is, of course, quite different. The independent variables μ_n and μ_s are the forms assumed by the single chemical potential μ, evaluated in the $\vec{v}_n = 0$ or the $\vec{v}_s = 0$ frames. Thus a given chemical potential difference is directly related [via Eq. (B6)] to a relative velocity between the superfluid and normal components.

This simplicity of Eq. (B7) [compared with the conventional form Eq. (B1)] is echoed in a very similar simplification in the energy current. This is usually eliminated in favor of the much simpler entropy current, but the form it takes (in the absence of dissipative terms) is[12]:

$$\vec{Q} = (\mu_s + \tfrac{1}{2}v_s^2)g + sT\vec{v}_n + (\vec{g}\cdot\vec{v}_n - \rho\vec{v}_s\cdot\vec{v}_n)\vec{v}_n \ . \tag{B8}$$

Using the definitions [Eq. (3.9)] one easily verifies that this is equivalent to

$$\vec{Q} = [sT + \rho_n(\mu_n + \tfrac{1}{2}v_n^2)]\vec{v}_n + \rho_s(\mu_s + \tfrac{1}{2}v_s^2)\vec{v}_s \ . \tag{B9}$$

The term in Eq. (B9) proportional to $\vec{v}_n$ is simply the energy current for a classical fluid with mass density ρ_n and momentum density $\rho_n\vec{v}_n$ provided μ_n is taken to be the chemical potential in the local rest frame of the fluid. Thus the full energy current consists of a term of precisely the classical structure for the normal fluid, and a second term for the superfluid of the same structure except for the omission of the entropy term.

We believe that on esthetic grounds along these simplifications justify paying more attention to the distinctions summarized in Eq. (3.9).

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[1] M. Liu and M. C. Cross, Phys. Rev. Lett. 41, 250 (1978); 43, 296 (1979).

[2] E. J. Yarmchuk, M. J. V. Gordon, and R. E. Packard, Phys. Rev. Lett. 43, 214 (1979).

[3] T. L. Ho and N. D. Mermin, Phys. Rev. Lett. 44, 330 (1980).

[4] R. Combescot (unpublished); R. Combescot and T. Dombre (unpublished).

[5] K. Nagai (unpublished).

[6] Another nontrivial and less familiar effect of rotating equilibrium in ^{4}He is discussed in Ref. 3.

[7] We have neglected the ^{3}He nuclear magnetic dipolar interactions. If these are retained then $\vec{L}$ must be replaced by the total angular momentum $\vec{J}$. The conclusions about the orbital structure of the order parameter, which are all that concern us here, are unaffected by this change.

[8] The equilibrium time dependence extracted here is nothing but the familiar Josephson relation. It follows unambiguously from Eq. (2.9) provided one notes that the Hamiltonian H contains a small non-number-conserving source term, the amplitude of which is allowed to vanish only after all equilibrium averages have been computed. Note also that the quantity μ that appears here is identical (except for a factor of the helium mass) to the hydrodynamically defined chemical potential introduced in Sec. III A. This connection is not exploited in our subsequent results, but it can be demonstrated by the following argument. First note that integrating the thermodynamic identity (B4) over the entire container and applying the equilibrium conditions of Sec. III B shows that the hydrodynamically defined chemical potential is given by $\mu = -T(\partial S/\partial M)_{E-\vec{\omega}\cdot\vec{L}}$, where S is the entropy of a rotating equilibrium state as a function of its total energy E, mass M, and angular momentum $\vec{L}$. Now the entropy and total mass are independent of the frame of reference, while $E - \vec{\omega}\cdot\vec{L}$ is just the total energy in the rotating frame. Therefore the hydrodynamically defined chemical potential is just $-(T/M)(\partial S/\partial N)_E$, where $(\partial S/\partial N)_E$ is to be evaluated in the rotating frame. But in the rotating frame the density matrix is just $W = e^{-\beta(\tilde{H}-\mu N)}/e^{-\beta(\tilde{H}-\mu N)}$, and the entropy is $S = -k_B \mathrm{Tr} N W \ln W$. By comparing the entropy of two states with the same values of $E\,(E = \mathrm{Tr}\tilde{H}W)$ and slightly different values of $\mathrm{Tr}NW$, one establishes that the μ appearing in the density matrix W is indeed $-T(\partial S/\partial N)_E$.

[9] N. D. Mermin and T. L. Ho, Phys. Rev. Lett. 36, 594 (1976).

[10] Again we neglect the nuclear dipolar interactions, because the more complicated analysis necessary if $\vec{L}$ is replaced by $\vec{J}$ leaves our conclusions about the orbital structure unchanged.

[11] There are situations where the quadratic contributions in velocities in Eq. (3.9) are of crucial importance, such as in describing the shape of the meniscus of a rotating superfluid. [See R. Meservey, Phys. Rev. 133, A1471 (1964)]. We thank A. L. Fetter for calling our attention to this reference.

[12] I. M. Khalatnikov, *Theory of Superfluidity* (Benjamin, New York, 1965), pp. 58–59. That Khalatnikov's chemical potential μ is our μ_s can be seen by comparing his Eq. (8.12) with our Eq. (B4).

[13] One can reach the same equilibrium conditions somewhat less efficiently by choosing the opposite extreme of a reservior container, whose energy and angular momentum are large compared with that of the helium.

[14] Ref. 12, p. 59.

[15] T. L. Ho, in *Quantum Fluids and Solids*, edited by S. B. Trickey, E. D. Adams, and J. W. Duffy (Plenum, New York, 1977), p. 97.

[16] C. R. Hu and W. M. Saslow, Phys. Rev. Lett. 38, 605 (1977).