Dynamic properties of superconducting weak links

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A comprehensive theoretical picture of the dynamic properties of the order parameter and the quasiparticles in superconducting short weak links is presented. Both diffusion and inelastic scattering are found to be important in relaxing nonequilibrium populations. At low voltages a dissipative current, which is considerably larger than the normal ohmic current, is found and at higher voltages the maximum supercurrent is enhanced. These effects describe quantitatively well the foot structure in the I-V characteristic observed experimentally by Octavio, Skocpol, and Tinkham.

I. INTRODUCTION

Qualitatively, superconducting weak links[1] behave rather similar to Josephson tunnel junctions. For example, in the static case, short weak links obey a simple sinusoidal current-phase relationship. Quantitatively, however, a greater variety of individual properties are found with weak links which shows that they have more internal structure than the oxide layer of a tunnel junction. In particular, the variety of dynamic properties reveals the activation of internal degrees of freedom.

In the following, we want to investigate theoretically the "foot" structure in the current-voltage characteristic of short microbridges. Such structures have been noted by a number of workers, but were studied particularly quantitatively in variable-thickness bridges by Octavio *et al.*[2] It is found that the voltage which first appears above the critical current increases much more slowly than expected (i.e., with differential resistance much less than the normal resistance) up to a certain point, after which the expected steeper rise in voltage (i.e., with differential resistance comparable with the normal resistance) sets in. An effect of this type had been predicted theoretically by Aslamasov and Larkin[3] on one side, and by Golub[4] on the other, in terms of an enhancement of superconductivity by a nonequilibrium distribution of quasiparticles. In contrast to these theories, our work includes as an essential feature, a nonequilibrium contribution to the supercurrent which is the leading-order contribution when the length of the bridge goes to zero. Otherwise, our treatment follows essentially the picture drawn by Tinkham[5] (which, in turn, has been motivated by the papers mentioned above[3,4]), where the motion of the gap in the bridge leads to a deficit of quasiparticles of low energy, and thus, to an effective cooling of the bridge. An essential contribution to the nonequilibrium properties of the bridge results from the dependence of the supercurrent on the quasiparticle population at the gap edge. An important parameter of the theory is the electron-phonon collison time τ_E, and we will show that for small voltages $eV\tau_E \ll 1$, the nonequilibrium increment of the supercurrent is out of phase by $\frac{1}{2}\pi$ and hence, appears to be dissipative, whereas in the case $eV\tau_E \gg 1$, the contribution to the supercurrent is in phase with the static part. We also calculate the correction to the current for finite length of the bridge which allows us to conclude that our theory is applicable when the length of the bridge is smaller than $\xi(0)[(T_c - T)/T_c]^{-1/4}$.

The plan of the paper is as follows. In Sec. II, we present our model as well as the Boltzmann equation for the quasiparticles and the Ginzburg-Landau equation for the order parameter. Next, we will show in Sec. III that quasiparticles of sufficiently small energy are confined to the bridge region, which is a prerequisite of the cooling effect mentioned above. Further, we present arguments showing how the Boltzmann equation can be solved in an adequate approximation. We evaluate this solution in Sec. IV in the local approximation for the generalized densities of states in the limits of small and large voltages. We repeat the same calculation in Sec. V, except for the difference that the generalized densities of states are chosen according to a form obtained recently by Artemenko, Volkov, and Zaitsev,[6] and we obtain qualitative agreement with the results derived previously. The resulting current-voltage characteristic is compared in Sec. VI with experimental data. In the last section, we compare with other theoretical and experimental work.

II. MODEL SYSTEM

We consider the following standard model[7] of a weak link. A narrow link ("bridge") connects two bulk superconductors ("banks") of the same material which we assume to have a short electron mean free path ($\tau_{imp}T_c \ll 1$).[8] The local state of the bridge depends only on the distance from the banks (since its cross section is uniform and small), and, at the end points it attains the equilibrium values characteristic for each of the banks (since diffusion in three dimensions restores equilibrium effectively).

For the sake of simplicity, we restrict our attention to the case of a short bridge where the length $(2a)$ is small compared to the coherence length $\xi(T)$. In such a case, the local state of the bridge is predominantly determined by diffusion processes with the banks acting as reservoirs. There is, however, the following important exception: excitations with energy less than the gap of the banks will be confined to the bridge region, and their number only changes due to inelastic electron-phonon scattering. This is the essence of the arguments made by Aslamasov and Larkin[3] as well as by Tinkham.[5] If it were not for this effect (which has important consequences), the rather elegant theory of Artemenko, Volkov, and Zaitsev[6] would answer most of our questions.

In order to study the quasiparticle dynamics, we start from the Boltzmann equation for the quasiparticles in a form which one obtains in the Green's-function technique.[9–11] In the dirty limit ($\tau_{\text{imp}}T_c \ll 1$), it is only necessary to consider the isotropic part $f_E(\vec{r},t,)$ of the distribution function. Furthermore, it is advantageous to split f_E into two parts $f_E^{(L)}$ and $f_E^{(T)}$, where the longitudinal (L) and the transverse (T) parts are such that $f_E^{(L)}-\frac{1}{2}$ is an odd, and $f_E^{(T)}$ is an even function of the energy E. Correspondingly, we obtain a set of two Boltzmann equations [see, for instance Eq. (44) of Ref. 11; the notation,[12] however, is the same as in Ref. 10]

$$-D\vec{\nabla}M^{(L)}\cdot\vec{\nabla}f^{(L)}-2D\vec{\nabla}N_2R_2\cdot\vec{Q}f^{(T)}+2eDN_2R_2\vec{Q}\cdot\dot{\vec{A}}\partial f^{(L)}/\partial E+N_1\dot{f}^{(L)}+R_2\dot{\Delta}\partial f^{(L)}/\partial E-K^{(L)}(f^{(L)})=0 \quad (1)$$

and

$$-D\vec{\nabla}M^{(T)}\cdot(\vec{\nabla}f^{(T)}-e\dot{\vec{A}}\partial f^{(L)}/\partial E)-2DN_2R_2\vec{Q}\cdot\vec{\nabla}f^{(L)}+(\partial/\partial t)(N_1f^{(T)})+2\Delta N_2f^{(T)}+(N_1e\dot{\phi}+\Delta\dot{\theta}N_2)\partial f^{(L)}/\partial E-K^{(T)}(f^{(T)})=0 \ .$$

In deriving this equation, we have expressed the complex order parameter $\underset{\sim}{\Delta}$ by $\Delta\exp(-i\theta)$ and the electromagnetic fields by $(\phi,\vec{A})$. Furthermore, $\vec{Q}=-(\vec{\nabla}\theta+2e\vec{A})$ is the superfluid momentum and $D=\frac{1}{3}v_F^2\tau_{\text{imp}}$ the diffusion constant. The quantities N_i, R_i, and M may be called generalized densities of states since they are combinations of retarded and advanced Green's functions. In a homogeneous superconductor, for instance, we encounter relations of the form

$$N_1(E)+iR_1(E)=\frac{E+i\Gamma}{[(E+i\Gamma)^2-\Delta^2]^{1/2}} \ ,\qquad N_2(E)+iR_2(E)=\frac{i\Delta}{[(E+i\Gamma)^2-\Delta^2]^{1/2}} \quad (2)$$

$(\mathrm{Re}\Gamma>0)$ where Γ is the pair-breaking energy (level broadening). Furthermore, $M^{(L)}(E)=N_1^2-R_2^2$ and $M^{(T)}(E)=N_1^2+N_2^2$. The electron-phonon collision integrals are denoted by K and do not need to be specified here.

In the limit $\Delta \ll T$, the Ginzburg-Landau equation is of the form

$$\left[\frac{\pi}{8T}\frac{\partial}{\partial t}+\frac{1}{\Delta}\int dE(R_2\delta f^{(L)}+iN_2f^{(T)})\right]\underset{\sim}{\Delta}=-[\alpha+\beta\Delta^2/T^2-\xi(0)^2(\vec{\nabla}-2ie\vec{A})^2]\underset{\sim}{\Delta} \ . \quad (3)$$

We have introduced the notation $\delta f^{(L)}=f^{(L)}-f_{\text{th}}$, where f_{th} is the thermal distribution function (Fermi function), whereas $\alpha=(T-T_c)/T_c$ and $\beta=7\zeta(3)/8\pi^2$. Furthermore, the current and charge densities are given by

$$\vec{j}=\frac{\sigma}{e}\left\{\frac{\pi}{4T}\Delta^2\vec{Q}-\int dE\left[M^{(T)}\left(\vec{\nabla}f^{(T)}-e\dot{\vec{A}}\frac{\partial f^{(L)}}{\partial E}\right)+2N_2R_2\vec{Q}\delta f^{(L)}\right]\right\} \ ,$$

$$\rho=2eN(0)\left[\int dE\,N_1f^{(T)}-e\phi\right] \ , \quad (4)$$

where $\sigma=2e^2N(0)D$. It is interesting to note that in cases where Eq. (2) is valid and the level broadening is small $\Gamma \ll \Delta$ the last term in the integral for $\vec{j}$ which we denote by $\delta\vec{j}_s$ gives a contribution to the supercurrent

$$\delta\vec{j}_s=(\sigma/e)\pi\Delta\vec{Q}(-\delta f_{E=\Delta}^{(L)}) \ . \quad (5)$$

We assert that in short bridges, this contribution, which depends sensitively on the depletion of states at the gap edge, produces the largest nonequilibrium effect.

Furthermore, for short bridges, the terms containing spatial derivatives play the most important role in the Boltzmann and Ginzburg-Landau equations. Thus, in a gauge where $\vec{A}=0$, $\underset{\sim}{\Delta}$ is determined[13] in leading order by the equation $\nabla^2\underset{\sim}{\Delta}=0$. Choosing the x direction parallel to the bridge, such that the banks are located at $x=\pm a$, we have

$$\underset{\sim}{\Delta}(x)=\tfrac{1}{2}\Delta_b[(1-x/a)e^{-i\gamma(t)}+(1+x/a)e^{i\gamma(t)}] \ , \quad (6)$$

where Δ_b is the gap of the banks. The phase difference $\phi=2\gamma$ obeys the Josephson relation $\dot{\phi}=2eV(t)$, where $V(t)$ is the voltage difference of the banks. It follows from Eq. (6), that

$$\Delta^2(x,t)=\Delta_b^2[\cos^2\gamma+(x^2/a^2)\sin^2\gamma] \quad . \tag{7}$$

From this, the static (equilibrium) current density is found to be equal to

$$j_{\text{eq}}=j_0\sin\phi \tag{8}$$

with

$$j_0=\frac{\pi\Delta_b^2}{4eT}\left(\frac{\sigma}{2a}\right) \quad .$$

It is also possible to calculate corrections to this expression due to the finite length of the bridge.[14] The correction $\underset{\sim}{\Delta}^{(1)}$ to the order parameter follows from

$$\xi^2(0)\frac{\partial^2}{\partial x^2}\underset{\sim}{\Delta}^{(1)}=\left(\alpha+\beta\frac{\Delta^{(0)2}}{T^2}\right)\underset{\sim}{\Delta}(0) \tag{9}$$

with the boundary conditions $\underset{\sim}{\Delta}^{(1)}(x=\pm a)=0$, and $\underset{\sim}{\Delta}^{(0)}$ is given by Eq. (6). Hence, one finds

$$\delta j_{\text{eq}}=\tfrac{2}{15}[2a/\xi(T)]^2 j_0\sin\phi\sin^2(\phi/2) \quad . \tag{10}$$

III. CONFINEMENT OF QUASIPARTICLES

In the following, we discuss a solution of the Boltzmann equation (1) in the case where the Josephson frequency is not too large such that $eV \ll \Delta_b \ll T$. For the sake of simplicity, our arguments are based on generalized densities of states of the simple form of Eq. (2) in the limit $\Gamma\rightarrow 0$, though the conclusions we reach are valid rather generally. We emphasize the local nature of Eq. (2) since we understand that $\Delta=\Delta(x,t)$ is a function of space and time as given by Eq. (7). It follows, for instance, that the reduced density of states $N_1(E)$ is finite even for energies $|E|$ below the gap Δ_b of the banks provided that $|E|>\Delta(x,t)$. This is also found for $M^{(L)}$, which in the considered limit is $M^{(L)}=\theta[E^2-\Delta^2(x,t)]$. Therefore, for quasiparticles involved in the longitudinal mode with an energy in the range $\Delta_b|\cos\gamma|<|E|<\Delta_b$, the diffusion is confined to the region $|x|<x_c$, where

$$x_c^2/a^2=[(E/\Delta_b)^2-\cos^2\gamma]/\sin^2\gamma \quad ,$$

whereas quasiparticles of higher energy $|E|>\Delta_b$ can diffuse freely. [In contrast, $M^{(T)}$ is nonzero everywhere and there is no restriction imposed on the diffusion of the quasiparticles involved in the transverse mode. However, due to the conversion term $2\Delta N_2 f^{(T)}$, the transverse part of the distribution function rapidly vanishes in the region where $|E|<\Delta(x,t)$.]

The boundary conditions are imposed by the banks, which are in equilibrium at potential energies $\frac{1}{2}\pm eV$. Consequently, at $x=\pm a$

$$\delta f^{(L)}=0;\ |E|>\Delta_b \quad ,$$

$$f^{(T)}=\mp\frac{\partial f_{\text{th}}}{\partial E}eV/2 \quad . \tag{11}$$

In cases where diffusion is unrestricted, diffusion relaxes the nonequilibrium quasiparticles at a fast rate of order $D/a^2 \gg \Delta_b^2/T$, and in comparison, other processes are irrelevant. For an estimate, we disregard the singular character of the generalized densities of states (which are otherwise of the order of 1) and find, since $\vec{Q}\sim 1/a$, that $(\partial^n/\partial x^n)f\sim eV/(a^nT)$. On the other hand, it will be shown later that the distribution function of the confined ($|E|<\Delta_b$) quasiparticles is of order

$$\delta f^{(L)}\sim\min(\tau_E\Delta eV/T,\Delta/T) \quad ,$$

where the inelastic collision time τ_E is usually large such that $\tau_E\Delta$ is a very large number. These arguments allow us to put for all x

$$\delta f^{(L)}=0,\ |E|>\Delta_b \quad , \tag{12}$$

$$f^{(T)}=0,\ \text{all}\ E \quad .$$

Thus, the problem is reduced to solve for energies $|E|<\Delta_b$, a Boltzmann equation of the relatively simple form

$$-D\nabla M^{(L)}\nabla f^{(L)}+N_1\dot{f}^{(L)}+R_2\dot{\Delta}\frac{\partial f^{(L)}}{\partial E}+\frac{N_1}{\tau_E}(f^{(L)}-f_{\text{th}})=0 \quad , \tag{13}$$

where also, we have replaced the electron-phonon collision operator by a relaxation ansatz.

This equation can be solved in a closed form as follows. Taking into account the steplike dependence of $M^{(L)}$ on x, we conclude that $\partial f^{(L)}/\partial x=0$ at the boundary of the accessible region $|x|\leq x_c$. (This point has been emphasized in Ref. 11.) Thus, the diffusion operation has one eigenvalue zero, whereas the other ones are at least of the order D/a^2. This means that the spatial average $\langle f^{(L)}\rangle$ of $f^{(L)}$ is much larger than any of its Fourier components of finite wave vector. Consequently, in the space average of Eq. (13), we may replace $\langle N_1\dot{f}^{(L)}\rangle$ by $\langle N_1\rangle\langle\dot{f}^{(L)}\rangle$ with very good accuracy. Similar substitutions can be made at other places. A further point deserves attention. If we define

$$\epsilon=\int_0^E dE'N_1(E') \quad , \tag{14}$$

we may, using Eq. (2), convince ourselves that $N_1=(\partial\epsilon/\partial E)_t$ and $R_2\Delta=-(\partial\epsilon/\partial t)_E$ (the subscript denotes the variable which is kept constant). It is even possible to show, that these relations are generally valid. Since the spatial averaging is interchangeable with integration and differentiation, we may write the space average of Eq. (13) in the following form:

$$\left(\frac{\partial\langle\epsilon\rangle}{\partial E}\right)_t\left(\frac{\partial\langle f^{(L)}\rangle}{\partial t}\right)_E-\left(\frac{\partial\langle\epsilon\rangle}{\partial t}\right)_E\left(\frac{\partial\langle f^{(L)}\rangle}{\partial E}\right)_t+\frac{1}{\tau_E}\left(\frac{\partial\langle\epsilon\rangle}{\partial E}\right)_t(\langle f^{(L)}\rangle-f_{\text{th}})=0\ . \quad (15)$$

This equation is equivalent to the simple form

$$\left(\frac{\partial\langle\delta f^{(L)}\rangle}{\partial t}\right)_{\langle\epsilon\rangle}+\frac{1}{\tau_E}\langle\delta f^{(L)}\rangle=-\frac{\partial f_{\text{th}}}{\partial E}\left(\frac{\partial E}{\partial t}\right)_{\langle\epsilon\rangle}\ , \quad (16)$$

which may be integrated to yield

$$\langle\delta f^{(L)}(t)\rangle=\frac{1}{4T}\int_{t_0}^{t}dt' e^{-(t-t')/\tau_E}\left(\frac{\partial E}{\partial t'}\right)_{\langle\epsilon\rangle}\ , \quad (17)$$

where we have put $(-\partial f_{\text{th}}/\partial E)=(1/4T)$. Condition (12) implies, that at a given $\langle\epsilon\rangle$, t_0 is defined to be the latest time (with respect to t), where $|E(\langle\epsilon\rangle;t_0)|=\Delta_b$. Furthermore, if $|E(\langle\epsilon\rangle;t)|>\Delta_b$, we have $\langle\delta f^{(L)}\rangle=0$.

In the case where the Josephson frequency is small such that $eV\tau_E\ll 1$, we obtain

$$\langle\delta f^{(L)}\rangle=\frac{\tau_E}{4T}\left(\frac{\partial E}{\partial t}\right)_{\langle\epsilon\rangle} \quad (18)$$

if $E(\langle\epsilon\rangle,t)<\Delta_b$; and zero otherwise. In the opposite case of fast motion where $eV\tau_E\gg 1$, the limiting form for the distribution function is

$$\langle\delta f^{(L)}\rangle=(1/4T)[E(\langle\epsilon\rangle;t)-\Delta_b\,\text{sgn}E]\ . \quad (19)$$

It is not difficult to write down corrections of the next order to the expressions given above. Furthermore, let us note that these results prove the assertion on $\delta f^{(L)}$ made above.

IV. THE LOCAL APPROXIMATION

We will evaluate here the theory specifically in the local approximation which means the spectral quantities are given by Eq. (2). For instance, we have $N_1(E)=|E|(E^2-\Delta^2)^{-1/2}$ and hence, $\epsilon=(E^2-\Delta^2)$ for $|E|>\Delta$. In the energy range $\Delta_b|\cos\gamma|<|E|<\Delta_b$, the spatial averaging leads to the simple result

$$\langle\epsilon\rangle=\frac{\pi}{4}\frac{E^2-\Delta_b^2\cos^2\gamma}{\Delta_b|\sin\gamma|}\ . \quad (20)$$

Note that the space averaged density of states is just proportional to $|E|$.

a. Slow motion $(eV\tau_E\ll 1)$. Here, we have from Eqs. (18) and (20)

$$\langle\delta f^{(L)}\rangle=-\frac{\tau_E\Delta_b^2}{8TE}\left(1+\sin^2\gamma-\frac{E^2}{\Delta_b^2}\right)\frac{\cos\gamma}{\sin\gamma}\dot{\gamma}\ . \quad (21)$$

The leading nonequilibrium contribution to the current is given by Eq. (5). Taking into account that $N_2R_2=(\pi/2)\Delta E\delta(E^2-\Delta^2)$, we find for the spatial average[15]

$$\langle\delta j_s^{(1)}\rangle=\tau_E\dot{\phi}p_1(\phi/2)j_0\ ,$$

$$p_1(\gamma)=\tfrac{1}{2}[\gamma(1+\sin^2\gamma)\,\text{cotan}\gamma-\cos^2\gamma]\ , \quad (22)$$

where j_0 is defined in Eq. (8). The function, $p_1(\gamma)$ is even in γ; it is of period π; and the expression given above is valid in the range $|\gamma|\leq\pi/2$. Furthermore, $p_1(\gamma)\geq 0$; it vanishes at $\gamma=0,\ \pm\pi/2$; and it has a maximum of 0.40 at $\gamma=\pm 1.05$. We emphasize the fact that $\delta j_s^{(1)}$, though being a pair current is dissipative since it carries the factor $\dot{\phi}=2eV$.

Nonequilibrium corrections to the pair current due to the finite length of the bridge can be calculated in a way similar to the equilibrium case where the contribution given by Eq. (10) has been found. The correction to the order parameter is now found as the solution of

$$\xi(0)^2\frac{\partial^2}{\partial x^2}\underset{\sim}{\Delta}^{(1)}=\frac{1}{\Delta^{(0)}}\int dE\,R_2\langle\delta f^{(L)}\rangle\underset{\sim}{\Delta}^{(0)}\ , \quad (23)$$

with boundary conditions $\underset{\sim}{\Delta}^{(1)}(x=\pm a)=0$. Other contributions originating from the time derivative of the Ginzburg-Landau equation and from the transverse mode $f^{(T)}$ are small in the sense of approximation (12). Thus we obtain

$$\delta j_s^{(2)}=\frac{\Delta_b}{T}\left(\frac{2a}{\xi(0)}\right)^2\tau_E\dot{\phi}p_2(\phi/2)j_0\ ,$$

$$p_2(\gamma)=-\frac{\pi}{64}\frac{\cos^2\gamma}{|\sin\gamma|^3}[(1+\sin^2\gamma)^2\ln|\cos\gamma|+\tfrac{1}{2}\sin^2\gamma+\tfrac{5}{4}\sin^4\gamma]\ . \quad (24)$$

The function $p_2(\gamma)$ has properties similar to $p_1(\gamma)$; furthermore, it has a maximum value of 0.020 at $\gamma=\pm 1.04$. We conclude that even for moderately short bridges, $\delta j_s^{(2)}$ is smaller than $\delta j_s^{(1)}$, say, by one order of magnitude.

The quasiparticle current which results from the integral containing ∇f^T of Eq. (4) can be neglected since it is of the order $(\sigma/2a)V$ which is the current along the bridge in its normal state. This is much smaller than the leading contribution $\delta j_s^{(1)}$ which is of the order $(\tau_E\Delta_b^2/T)(\sigma/2a)V$. This conclusion is in accordance with the estimates of the previous section.

b. Fast motion $(eV\tau_E\gg 1)$. According to the result of Eq. (19), the quasiparticles are almost kept out of the sub-gap region $|E| < \Delta_b$. The consequences of this distribution on the current will be discussed in analogy of the case of slow motion. Thus we obtain from Eq. (5)

$$\langle \delta j_s^{(1)} \rangle = [r_1(\phi/2) - \sin\phi] j_0 \ ,$$
$$r_1(\gamma) = \cos\gamma \ln \frac{1+\sin\gamma}{1-\sin\gamma} \ . \tag{25}$$

This contribution to the pair current is nondissipative and hence, we have written it in a form which allows to add the equilibrium contribution (8) easily. Thus, the supercurrent is equal to $r_1(\phi/2) j_0$. The function $r_1(\gamma)$ is odd in γ; it is of period π; and it is nonnegative in the range $0 \le \gamma \le \pi/2$. Furthermore, it vanishes at $\gamma = 0$ and $\gamma = \pm\pi/2$, and it has a maximum value of 1.325 at $\gamma = 0.99$.

Nonequilibrium corrections due to the finite length of the bridge can be calculated as before. One obtains

$$\delta j_s^{(2)} = \frac{\Delta_b}{T} \left[\frac{2a}{\xi(0)} \right]^2 r_2(\phi/2) j_0 \ ,$$
$$r_2(\gamma) = \frac{2\pi}{48} \left[\tfrac{9}{8}\cos^2\gamma - 2\cos\gamma + \tfrac{3}{8} + \frac{1}{1+\cos\gamma} \right] \times \cos\gamma \ \mathrm{sgn}(\sin\gamma) \ . \tag{26}$$

We remark that r_2 has the same general properties as r_1. It has a maximum value of 0.026 at $\gamma = 1.22$.

Comparing the finite length correction $\delta j_s^{(2)}$ with the leading term $\delta j_s^{(1)}$, we find that, in slow as well as in fast motion, the correction is small provided that

$$2a \lesssim \xi(0) [(T_c - T)/T_c]^{-1/4} \ .$$

We consider this as a criterion for the applicability of our theory.

For sake of completeness, we also consider the limit

$$\xi(0)[(T_c - T)/T_c]^{-1/4} \ll 2a \ll \xi(T)$$

discussed by Aslamasov and Larkin (AL). In this case, the maximal supercurrent is given by[16]

$$j_{1,\mathrm{AL}} = g \frac{2a}{\xi(0)} \left[\frac{\Delta_b}{T} \right]^{1/2} j_0 \ , \tag{27}$$

where the factor $g = 0.280$ ($g = 0.238$) for $\Delta_b/T = 1$ ($\Delta_b/T = 0$). Considering a graph where $j_0 + \delta j_s^{(1)} - \delta j_s^{(2)}$ as well as j_{AL} are plotted as a function of $2a$, we may infer that the crossover between these two limiting forms occurs at $2a \gtrsim \xi(0) \times [(T_c - T)/T_c]^{-1/4}$.

V. ALTERNATIVE DERIVATION

The local approximation used so far has the virtue to allow a straightforward qualitative discussion of the processes in a short weak link, but it may be inappropriate on a quantitative level. In particular the curvature of the order parameter is known to have a pair breaking effect $\Gamma = (4T/\pi)\xi^2(0) \,\mathrm{Re}\,[(\nabla^2 \Delta)/\Delta]$, which can be determined from Eq. (23), and is found to be not always small. We, therefore, will rederive in this section expressions for the generalized densities of states starting directly from the equations of motion for the Green's function. Using approximations appropriate for short weak links these can be solved as was shown by Artemenko, Volkov, and Zaitsev.[6] We will find results similar to those obtained in the local approximations with certain quantitative corrections.

We first summarize the derivation given in Ref. 6. As was shown by Larkin and Ovchinnikov,[11] the equation of motion for the quasiclassical Green's function

$$\hat{G}_{\vec{p}_F}(t,t',\vec{R}) = \frac{i}{\pi} \int d\xi_p \hat{G}(t,t',\vec{R} + \tfrac{1}{2}\vec{r}, \vec{R} - \tfrac{1}{2}\vec{r}) e^{-i\vec{p}\cdot\vec{r}} d^3r$$

can be written in the form

$$\frac{1}{m} \vec{p}_F \vec{\nabla}_{\vec{R}} \hat{G} + \tau_3 \frac{\partial \hat{G}}{\partial t} + \frac{\partial \hat{G}}{\partial t'} \tau_3 + [\hat{H} + i\hat{\Sigma}, \hat{G}]_- = 0 \ ,$$
$$\hat{G}^2 = 1 \ . \tag{28}$$

The notation follows Ref. 11. It is understood that an integration over internal time arguments is performed. The caret expresses the fact that the quantities are matrices as introduced by Keldysh,[17] e.g.,

$$\hat{G} = \begin{bmatrix} G^R & G \\ 0 & G^A \end{bmatrix} \ . \tag{29}$$

It should be noted that G, G^R, and G^A are still the 2×2 matrices characteristic for superconductivity. In a short weak link, of a dirty material ($T_c \tau_{\mathrm{imp}} \ll 1$) the most important terms in Eq. (28) are the space derivative and the part of the self-energy describing elastic scattering at impurities

$$\hat{\Sigma}_{\mathrm{imp}} = -\frac{i}{2\tau_{\mathrm{imp}}} \int \frac{d\Omega_p}{4\pi} \hat{G} \ . \tag{30}$$

Hence, to lowest order it is sufficient to consider

$$\frac{1}{m} \vec{p}_F \vec{\nabla}_{\vec{R}} \hat{G} + \frac{1}{2\tau_{\mathrm{imp}}} \left[\left[\int \frac{d\Omega_p}{4\pi} \hat{G} \right], \hat{G} \right]_- = 0 \tag{31}$$

which can be reduced further by expanding $\hat{G} = \hat{G}_0 + p_x \hat{G}_1$. The normalization yields $[\hat{G}_0, \hat{G}_1]_+ = 0$. By taking angular averages of Eq. (31), one finds that $\hat{G}_1$ is independent of the space coordinate x, and that

$$\frac{\partial \hat{G}_0}{\partial x} - \frac{m}{\tau_{\mathrm{imp}}} \hat{G}_1 \hat{G}_0 = 0 \ . \tag{32}$$

The boundary conditions require that $\hat{G}_0(x)$ attains the equilibrium values $\hat{G}^{\pm}$ at $x=\pm a$. Thus one obtains

$$\hat{G}_1=\frac{\tau_{\rm imp}}{2ma}\ln(\hat{G}^+\hat{G}^-)$$

$$=+\frac{\tau_{\rm imp}}{2ma}\,\text{arcsinh}\,\tfrac{1}{2}[\hat{G}^+,\hat{G}^-] \tag{33}$$

and

$$\langle\hat{G}_0\rangle=(\ln\hat{G}^+\hat{G}^-)^{-1}\cdot(\hat{G}^+-\hat{G}^-)\ , \tag{34}$$

where $\langle\rangle$ denotes the spatial average across the weak link. For completeness we mention that from Eq. (28) the equilibrium Green's function follows to be

$$\hat{G}_{\rm eq}=\hat{\alpha}\tau_3+\hat{\beta}\tau_\theta,\quad \tau_\theta=e^{-i\theta\tau_3}\tau_1\ , \tag{35}$$

where

$$\alpha^{R(A)}=\frac{E}{[(E\pm i0)^2-\Delta^2]^{1/2}}\ ,$$

$$\beta^{R(A)}=\frac{i\Delta}{[(E\pm i0)^2-\Delta^2]^{1/2}}\ . \tag{36}$$

For simplicity we have neglected level broadening ($\Gamma\to 0+$) in the banks, which in the longitudinal mode has no significant effects. It is convenient to choose the cut of the square roots extending from $-\Delta\mp i0$ to $\Delta\mp i0$, and the signs such that $\{[\cdots]^{1/2}\}^{R(A)}\to\pm E$ for $|E|>>\Delta$. In particular $\hat{G}^+$ and $\hat{G}^-$ are of the form given above with $\theta=-\gamma$ and $+\gamma$, respectively, and $\Delta=\Delta_b$.

We now can calculate the generalized densities of states. One quantity of interest is N_2R_2Q. Using the relations (35), (36) and the definition (2) one can show that

$$8N_2R_2Q={\rm Tr}\left[\tau_3\left(G_0^R\frac{\partial}{\partial x}G_0^R-G_0^A\frac{\partial}{\partial x}G_0^A\right)\right]\ . \tag{37}$$

(For simplicity we choose a gauge where $\vec{A}=0$.) This quantity was also introduced in Ref. 11, where it was denoted by j_E. Furthermore, from the relation (32) one finds

$$N_2R_2Q=-\frac{1}{8}\frac{m}{\tau_{\rm imp}}{\rm Tr}\tau_3(G_1^R-G_1^A)\ , \tag{38}$$

where, in turn $G_1^R\,(G_1^A)$ can be expressed by $G^{\pm R}(G^{\pm A})$. We thus find

$$G_1^R=\frac{\tau_{\rm imp}}{2ma}\frac{i\Delta_b\cos\gamma\tau_3-E\tau_1}{\{(E^2-\Delta_b^2\cos^2\gamma)^{1/2}\}^R}\times\ln\frac{\{[(E^2-\Delta_b^2\cos^2\gamma)^{1/2}]^R-\Delta_b\sin\gamma\}^2}{E^2-\Delta_b^2} \tag{39}$$

and a similar relation for G_1^A with R and A interchanged.[18] An equivalent convention for the sign and the cut of $(E^2-\Delta_b^2\cos^2\gamma)^{1/2}$ is chosen as mentioned above. Furthermore, the logarithm, which has a cut for negative real arguments, leads to a cut on the real E axis in the range $\Delta_b|\cos\gamma|\le|E|\le\Delta_b$. In this range the infinitesimal Γ have to be carefully taken into account. It is then straightforward to show that N_2R_2Q is zero for $|E|\le\Delta_b|\cos\gamma|$ and for $|E|\ge\Delta_b$ whereas, for $\Delta_b|\cos\gamma\le|E|\le\Delta_b$ it is given by

$$N_2R_2Q=\frac{\pi}{4a}\frac{\Delta_b\cos\gamma}{(E^2-\Delta_b^2\cos^2\gamma)^{1/2}}{\rm sgn}(E\sin\gamma)\ . \tag{40}$$

Notice that this space independent result is identical to the space averaged value $\langle N_2R_2Q\rangle$ obtained in Sec. IV.

In the limit of fast motion ($eV\tau_E>>1$, but $eV<<\Delta$) the distribution function is again given by Eq. (19). If we take the generalized density of states to be the equilibrium value (40), we obtain exactly the same expression for the extra current contribution as we found in Eq. (25).

In the opposite limit of slow motion ($eV\tau_E<<1$), in order to find the distribution function we have to determine the quantity $(\partial E/\partial t)_{\langle\epsilon\rangle}$. It is convenient to express this by the spatial average of the density of states $\langle N_1\rangle$ as follows

$$\left(\frac{\partial E}{\partial t}\right)_{\langle\epsilon\rangle}=-\frac{1}{\langle N_1(E)\rangle}\left(\frac{\partial\langle\epsilon\rangle}{\partial\gamma}\right)_E\dot{\gamma}\ , \tag{41}$$

where

$$\langle N_1\rangle=\frac{\partial\langle\epsilon\rangle}{\partial E},\quad \langle\epsilon\rangle=\int_0^E dE'\langle N_1(E')\rangle\ . \tag{42}$$

The quantity $\langle N_1\rangle$, on the other hand, can be obtained as usual from $\langle N_1\rangle=\frac{1}{2}(\langle G_0\rangle_{11}^R-\langle G_0\rangle_{11}^A)$, where $\langle G_0\rangle_{11}^{R(A)}$ is found from Eq. (34) by an analysis similar to the one discussed above. In the interesting range $\Delta_b|\cos\gamma|\le|E|\le\Delta_b$ we find

$$\langle N_1\rangle=\frac{|E|\Delta_b|\sin\gamma|}{(\Delta_b^2-E^2)^{1/2}(E^2-\Delta_b^2\cos^2\gamma)^{1/2}}\,2\pi\Big/\left[\pi^2+\left(\ln\frac{\Delta_b\sin\gamma+(E^2-\Delta_b^2\cos^2\gamma)^{1/2}}{\Delta_b\sin\gamma-(E^2-\Delta_b^2\cos^2\gamma)^{1/2}}\right)^2\right]\ , \tag{43}$$

whereas for $|E|<\Delta_b|\cos\gamma|$ this quantity vanishes, and for $|E|>\Delta_b$ and $\gamma=0$, $\langle N_1\rangle$ is equal to the BCS reduced density of states. Using the result Eq. (43) we can calculate the current in the considered limit.[18] Up to corrections of a few percent, we obtain

$$\langle\delta j_s^{(1)}\rangle=\tau_E\dot{\phi}\bar{p}_1(\phi/2)j_0\ , \tag{44}$$

where $\bar{p}_1(\gamma)$ is of period π. In the range $|\gamma| \le \frac{1}{2}\pi$ it is given by

$$\bar{p}_1(\gamma) = (\gamma/2)\sin(2\gamma) \quad . \tag{45}$$

It is nearly of the same form as $p_1(\gamma)$ and is only about 10% larger.

Thus, starting from microscopic equations of motion in a form that applies to the situation in short dirty weak links, we found expressions for the generalized densities of states which in general differ from those obtained in the local approximation. However, the spatially averaged currents (also the equilibrium current) are either identical or only slightly different in both approaches. We conclude that the expression for the order parameter as given in Eq. (6) and the local approximation are a very reasonable starting point for a qualitative discussion of the dynamic processes in short weak links. For the following analysis of the current-voltage characteristics, we will take into account the modifications derived in this section.

VI. CURRENT-VOLTAGE CHARACTERISTIC

In this section we will discuss the relation between the time-averaged voltage and the total current through the weak link. We start with the expression for the total current

$$j = j_0 \sin\phi + \delta j_s + \frac{\sigma}{2a}\frac{\dot{\phi}}{2e} \quad , \tag{46}$$

where, depending on $\tau_E\dot{\phi}$, δj_s reduces to the different results presented above. Since at large voltages the normal current cannot be neglected we have included it in Eq. (46). At small voltages the addition of this term actually is inconsistent. Since, in the spirit of the approximations made already, it has no significant effect, we may keep it in order to obtain a unified picture over a larger voltage range.

It is important to realize that even at low currents $0 \le (j - j_0) \ll j_0$ the time evolution of the phase is not slow in the whole cycle. Rather it is of the form as sketched in Fig. 1. There is an initial fast rise of ϕ for $0 \le \phi \lesssim \phi_1$, followed by a slow motion for $\phi_1 \lesssim \phi \lesssim \phi_2 (< \pi)$, which turns into a fast motion again during the rest of the cycle $\phi_2 \lesssim \phi \le 2\pi$. During these fast parts of the cycle, the normal current term has to be included. On the other hand, for large currents, the motion is always fast. It is clear from Eq. (46) that the characteristic time scale for the fast motion is given by $\tau_J = 2T/\pi\Delta_b^2$ which, in most cases of interest, is much shorter than the inelastic scattering time τ_E. The strong inequality $\tau_J \ll \tau_E$ allows further approximations in deriving simple relations for the current-voltage characteristic.

a. Small currents. For currents below a certain j_1,

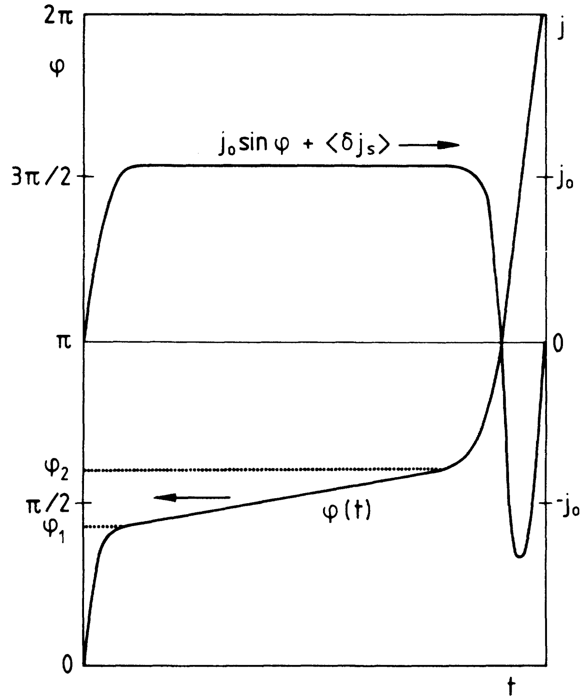

FIG. 1. The phase difference ϕ across the weak link and the supercurrent $j_s = j_0 \sin\phi + \langle\delta j_s\rangle$ is qualitatively plotted as a function of time for one cycle of the periodic process. For typical values of the parameters, say $j/j_0 = 1.1$ and $\tau_J/\tau_E = \frac{1}{100}$ the distinction between slow and fast parts is much more pronounced.

the motion can be divided into the three distinct regimes mentioned above. If we neglect for the moment the finite length corrections, we find

$$j_1 = \max r_1(\phi/2) j_0 = 1.325 j_0 \quad . \tag{47}$$

It is clear that near $\phi = 0$ and $\phi = \pi$, where the supercurrent vanishes, the characteristic time for the evaluation of ϕ is given by τ_J. This is also true for the backward cycle $\pi \le \phi \le 2\pi$, where the supercurrent flow is opposite to the total current. In calculating the time Δt needed for one complete cycle $0 \le \phi \le 2\pi$, we may neglect the times required for the fast parts of the cycle. The corrections are of order $\tau_J/\tau_E \ll 1$. (Actually, for typical values of the parameters, e.g., $\tau_J/\tau_E = \frac{1}{50}$ and $j/j_0 \approx 1.1$ the fast motion is considerably faster than shown in Fig. 1.) However, the fast motion reduces considerably the range of ϕ, where the system moves slowly, $\phi_1 \le \phi \le \phi_2$ and hence it has a large effect on Δt. Thus, we obtain (consistently neglecting the normal current which is also of relative order τ_J/τ_E)

$$\frac{\Delta t}{\tau_E} = \frac{1}{\tau_E}\int_{\phi_1}^{\phi_2} d\phi/\dot{\phi} = \int_{\phi_1}^{\phi_2} \frac{\bar{p}_1(\phi/2)}{j/j_0 - \sin\phi} = \frac{2\pi}{g(j/j_0)} \quad . \tag{48}$$

A good estimate for the lower cutoff ϕ_l is obtained from the limit of the fast motion $j = j_0 r_1(\phi_l/2)$ which means that ϕ_l is the smaller of the two solutions of

$$j/j_0 = \cos(\phi/2) \ln \frac{1+\sin(\phi/2)}{1-\sin(\phi/2)} \quad . \tag{49}$$

On the other hand, the upper cutoff ϕ_2 is not easily determined. The reason is that Eq. (25) is valid only if the system moves fast during the whole cycle, when the distribution function is of the form given in Eq. (19), which means it is determined by the initial conditions at $\phi = 0$. If, however, the motion is intermediately slow, the distribution function relaxes towards a thermal function during this time and hence, in the following fast motion is fixed by a different initial condition. Therefore, j_s will be smaller than $j_0 r_1(\phi/2)$. Nevertheless, we may choose ϕ_2 to be the larger solution of Eq. (49). The error thus introduced is small and vanishes both for $j \approx j_0$ and for $j \approx j_1$.

Thus, the integral defining $g(j/j_0)$ is only a function of j/j_0, independent of other parameters, and the time-averaged normalized voltage $\bar{V}\sigma/(2aj_0) = (\tau_J/\tau_E) g(j/j_0)$ simply scales with τ_J/τ_E. [In the normal state $\bar{V}\sigma/(2aj_0) = j/j_0$.] A plot $g(z)$ is given in Fig. 2. For very small values of $z-1$, one finds $g(z) = 3.6(z-1)^{1/2}$. For larger values the function g becomes approximately linear with slope of order 17. We want to point out again that the strong inequality $\tau_J/\tau_E << 1$ allows us to divide the motion into the distinct regimes with different time scales and to neglect the time spent in the fast regions. If this inequality is not strictly satisfied, our approximation scheme is less reliable, in particular, if the current j is close to j_1. The corrections are of the same order as the effect of the normal current, hence both are neglected consistently.

b. Large currents. If the current is larger than j_1 (and not too close to j_1) the motion is always fast. Hence, we find simply

$$\frac{\Delta t}{\tau_J} = \int_0^{2\pi} d\phi \frac{1}{j/j_0 - r_1(\phi/2)} \equiv 2\pi / k(j/j_0) \tag{50}$$

from which we see that the normalized voltage $\bar{V}\sigma/(2aj_0) = k(j/j_0)$ only depends on j/j_0. A plot of $k(z)$ is given in Fig. 3. It has a square root behavior near $j = j_1$ and approaches the normal-state result $k(z) \to z$ for large arguments. It turns out that in a good numerical approximation we can set $k(z) = [z^2 - (j_1/j_0)^2]^{1/2}$ which would be the result of the simple resistively shunted junction (RSJ) model modified to have a maximum supercurrent j_1 instead of j_0.

We thus find as long as we may neglect the finite length corrections, that the current-voltage characteristic can be very simply described by the two universal functions g and k and the ratio τ_J/τ_E, everywhere except in a narrow region where $j \approx j_1$, the width of which diminishes as τ_J/τ_E becomes small. In Fig. 3 we also included the result obtained above for small currents $j \leq j_1$ for a value of the ratio $\tau_J/\tau_E = \frac{1}{100}$. The average voltage $\bar{V}$ as a function of the current shows clearly the qualitative features observed in the experiments of Octavio *et al.*[2]: a "foot" at low currents followed by a steep rise which, extrapolated back to $\bar{V} = 0$, yields j_1. The slope of the foot in the region where it is approximately linear, yields a normalized differential resistance which is proportional to $(T_c - T)^{-1}$.

On the other hand, for a quantitative comparison with the experiment, we can no longer neglect the finite length corrections. The approximation scheme discussed above can easily be generalized to include them. Their most significant effect is to shift j_1 to a

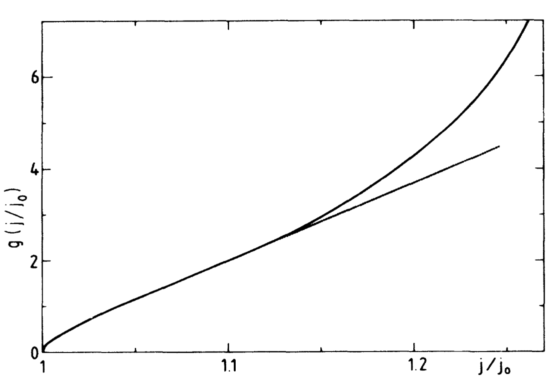

FIG. 2. The function $g(j/j_0)$, defined in Eq. (48) vs j/j_0. From g the time-averaged normalized voltage in the foot region is found as $\bar{V}/(R_n j_0) = (\tau_J/\tau_E) g(j/j_0)$. In the region where g is approximately linear, it has a slope of order 17 (dotted line).

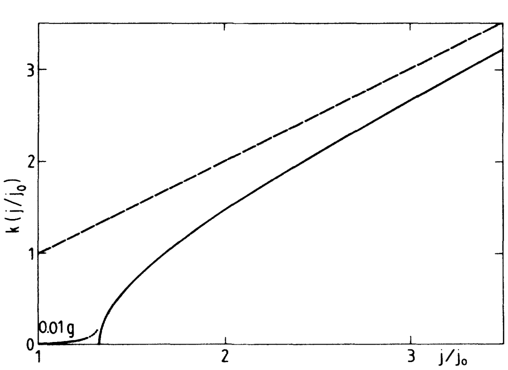

FIG. 3. The function $k(j/j_0)$, defined in Eq. (50) vs j/j_0. From k the high voltage (but $eV << \Delta$) part of the I-V characteristic is obtained as $\bar{V}/(R_n j_0) = k(j/j_0)$. For comparison the low voltage part for $\tau_J/\tau_E = \frac{1}{100}$ is also included.

higher value given by

$$j_1/j_0 = \max\left[r_1(\gamma) + \frac{\Delta_b}{T}\left(\frac{2a}{\xi(0)}\right)^2 r_2(\gamma)\right] . \quad (51)$$

The correction δj_{eq} given in Eq. (10) can be neglected. As long as the second term in Eq. (51) is a small correction we find simply

$$\frac{j_1}{j_0} = 1.325 + 0.019\frac{\Delta_b}{T}\left(\frac{2a}{\xi(0)}\right)^2 . \quad (52)$$

Also, the finite length corrections reduce the slope of the foot since $\dot{\phi}$ in the slow phase becomes smaller and the range $\phi_1 \leq \phi \leq \phi_2$ of the slow motion is increased. As long as this correction is small the normalized differential resistance of the foot becomes[19]

$$\frac{R_{\text{eff}}}{R_n} = 10.8\frac{1}{T_c \tau_E}\left(\frac{T_c}{\Delta_b}\right)^2 \left[1 + 0.052\frac{\Delta_b}{T_c}\left(\frac{2a}{\xi(0)}\right)^2\right]^{-1} , \quad (53)$$

where $R_n = 2a/\sigma$.

In Fig. 4 we plotted the ratio j_1/j_0 as a function of temperature for values of $2a/\xi(0) = 4$, 6, and much smaller than 1. Also experimental data taken from Ref. 2 are included. The lengths of the bridges used in the experiments were found to be 0.5–0.65 μm and $\xi(0) = 0.13$ μm. Considering the uncertainty in determining the precise effective length, and of the extrapolation to find j_1, the agreement between theory and experiment has to be considered reasonable. Also, the length of the bridges are rather large and we are at the limit of the region of applicability of the theory. In Fig. 5 we plotted the normalized differential resistance of the "foot" versus temperature for the same parameters. The inelastic scattering time was chosen $\tau_E = 2.8 \times 10^{-10}$ sec, which is a very reasonable value for tin. The experimental data from Ref. 2 on tin microbridges are also included. Here we find a good quantitative agreement.

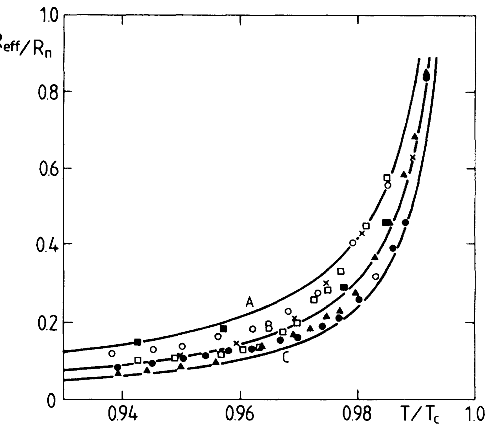

FIG. 5. The normalized differential resistance R_{eff}/R_n in the foot region as given by Eq. (53) vs temperature for the same lengths as in Fig. 4, $T_c = 3.8$ K and $\tau_E = 2.8 \times 10^{-10}$ sec. The experimental data on tin microbridges taken from Ref. 3 are also included. The value of τ_E, which was used to fit the data, is very reasonable for tin.

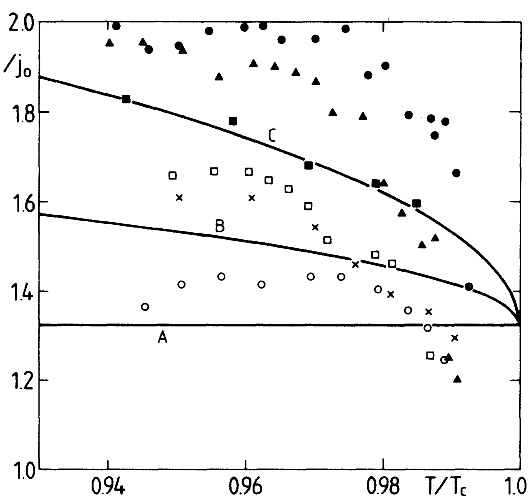

FIG. 4. The reduced maximum supercurrent j_1/j_0 as given by Eq. (51) vs temperature for weak links of length $2a/\xi(0) << 1$ (curve A), $2a/\xi(0) = 4$ (curve B), and $2a/\xi(0) = 6$ (curve C). Also experimental data on six different bridges taken from Ref. 2 are included. The lengths are chosen to agree with the range of the experimental parameters.

We can estimate the voltage limit of the "foot" by combining Eqs. (52) and (53) to compute $V_{\max} \equiv (j_0 - j_0)R_{\text{eff}}$, which gives a measure of the extent of the linear rise before the limiting enhancement of the supercurrent is reached. In the short bridge limit, this product is simply

$$eV_{\max} = 2.76(1/\tau_E) . \quad (54)$$

Thus, for $\tau_E = 2.8 \times 10^{-10}$ sec, $V_{\max}$ would be ~ 7 μV, which is in satisfactory agreement with the I-V curves of Ref. 2.

VII. DISCUSSION

We have presented in this paper a theory describing weak links of short length $2a << \xi(T)$. In this case, the order parameter in the bridge and the generalized densities of states are fixed by the boundary conditions in the superconducting banks. Diffusion processes relax all nonequilibrium quasiparticles at a fast rate except for those which are confined in the bridge because their energy is lower than the energy gap in the banks. Inelastic electron-phonon collisions are responsible for the relaxation of these quasiparticles. Furthermore, the strong diffusion has the effect that the distribution function of the confined quasiparticles is spatially homogeneous in the range which is energetically accessible. The current through the weak link, which for short bridges in equilibrium has the well-known sinusoidal phase dependence, is substantially modified due to nonequilibrium quasiparticles. At low voltages we find a dissipative contribution to the current which is larger than the normal ohmic current and which is responsible for a "foot" in the current-voltage characteristic of weak links (which is not found using a simple RSJ model). At high voltages the supercurrent is increased, hence the high voltage part of the current-voltage characteristic extrapolates to a higher zero voltage current than the equilibrium value. The comparison between experiments[2] on tin bridges and the theory, augmented by finite length corrections, is quantitatively good.

Since the characteristic length scales are so much longer in aluminum than in tin, it is relatively easy to make aluminum bridges for which the zero-length limit should be a good approximation. Data on such bridges have been reported by Klapwijk *et al.*[20] and by Bindslev Hansen *et al.*[21] Taking τ_E for aluminum to be $\approx 1.5\times10^{-8}$ sec, the voltage limit for the "foot" given by Eq. (54) is $\approx 0.1\ \mu$V, where problems of noise rounding and limits of experimental sensitivity make quantitative observation difficult or impossible. In the work of Klapwijk *et al.*, a small initial rounding at the foot of the voltage rise was ignored in defining the critical current as that at which a rapid rise in voltage appeared. With this definition, their *measured* critical current would be the *enhanced* one, $I_{c1}\propto j_1$, given by the maximum of Eq. (25), namely, $1.325\ I_{c0}$. This factor would account for roughly half of the excess of the I_cR products reported by Klapwijk *et al.* above the theoretically expected values. However, one must be very cautious in drawing any conclusion from the numerical magnitude of the I_cR product, since other factors appear also to influence it. For example, data of Octavio[22] on tin bridges show that the unambiguously identified I_{c0} at the beginning of the foot can have dI_{c0}/dT larger than predicted by the Aslamazov-Larkin theory, and Yanson[23] found similar effects with lead and indium. An important further observation of Klapwijk *et al.* was that the measured critical currents scaled linearly with (T_c-T), as expected from the present model in the zero-length limit for both I_{c0} and I_{c1}. By contrast, the Aslamazov-Larkin model predicts an enhanced I_{c1} scaling as $(T_c-T)^{5/4}$. In the work of Bindslev Hansen *et al.*, the published I-V curves (Fig. 1 of Ref. 21) are shown at roughly five times higher voltage resolution than in the published data of Klapwijk *et al.*, and a submicrovolt foot is discerned. In summary, although no definite conclusion can be drawn from the results published for aluminum bridges, they certainly are not in contradiction with the present theory.

To conclude, we want to draw a comparison with previous theoretical investigations. The qualitative features of the current-voltage characteristic have been discussed before by Aslamasov and Larkin.[3] However, since they consider only time-averaged quantities, their approach is restricted to high voltages, where a large number of oscillations take place in the time τ_E. Also, their result for the maximum supercurrent j_1, (which we quoted at the end of Sec. IV) was derived under restrictions which can hardly be satisfied. Indeed, if one takes the numerical coefficients seriously and also the experimentally determined lengths of the bridges, this result does not fit the experimental data.[2] Furthermore, since they did not consider the extra contribution δj_s to the current (the importance of which we have shown) they arrived erroneously at the conclusion that nonequilibrium quasiparticles would not modify the current in very short bridges $2a<\xi(0)[(T_c-T)/T_c]^{-1/4}$. Finally, we have shown that their approach of considering only time and spatially averaged distribution functions, which leads to the concept of "energy diffusion," can be avoided in favor of a more straightforward method presented in this paper.

The reduction of the effective resistance at low currents—the "foot"—was also discussed by Golub.[4] His starting point is similar to ours and he also realized that the diffusion of quasiparticles is the most important process in short weak links. However, his conclusion that the distribution function would be spatially inhomogeneous with strong gradients proportional to $1/a$, and that also for this reason, inelastic scattering can be neglected is incorrect as we showed in the present paper. In fact, the experimental results[2] show no indication of the strong length dependence of the resistance in the foot region following from his assumptions.

Finally, we want to mention that the work of Artemenko, Volkov, and Zaitsev[6] is complementary to the present paper in the respect that the limit $eV \gg \Delta$ is discussed. There they find an "excess current"—or "insufficient voltage"—correction to the normal ohmic current, which is in agreement with experimental observations of the current-voltage characteristic of weak links.[24] The physical origin of this effect seems to be a property of the boundaries between banks and bridge in the weak link since the same effect is found if a single normal-metal-superconductor boundary is considered.[25] In contrast to this "excess current," the "excess current" apparent in Fig. 3 vanishes for larger currents. The latter is simply due to the fact that the time evolution of the phase difference across the weak link is slower when the supercurrent and total current flow are parallel than if they are antiparallel. This results in a net time average of the supercurrent. A generalization of the approach of Ref. 6 to describe also the dynamics at low voltages ($eV << \Delta$) is not easy since effects which are important in this case, as for example the inelastic scattering of confined quasiparticles, are no longer included in their approximate equation of motion [Eq. (31)].

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[19]For values of $\tau_J/\tau_E \approx \frac{1}{10}$, Eq. (53) yields an effective resistance R_{eff} larger than R_n. It is not clear whether this result is correct or whether the condition $\tau_J/\tau_E << 1$ for the validity of our approximation is already violated. The terms neglected in the derivation would reduce R_{eff}/R_n. Further investigation of this point is planned.

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