# Quantum statistical mechanics of extended objects. IV. Correlation functions in the one-dimensional kink-bearing systems

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The technique of semiclassical approach is extended to deal with the correlation functions in a sine-Gordon system and a $\phi^4$ system in the 1 + 1 dimension. The present analysis justifies, to a large extent, the results obtained within the ideal-gas phenomenology. The role of breathers in the sine-Gordon system is clarified.

## I. INTRODUCTION

In a series of papers[1-3] (referred to hereafter as I, II, and III), we have developed the quantum statistical mechanics of the one-dimensional system with discrete symmetry.[4] In particular, making use of the method of functional integral, we have calculated the expression of the free energy and the soliton (or kink) density within the semiclassical approximation.

The purpose of this paper is to extend the above analysis to other thermodynamical quantities as well as to the dynamical correlation functions. The latters reveal most clearly the important role played by kinks (or solitons) in these systems. Furthermore, the dynamical correlation functions are of particular interest, since they are accessible by inelastic neutron scattering experiments, for example.

## II. GENERAL FORMULATION

As emphasized before,[1-3] the sine-Gordon $\phi^4$, and double quartic systems have degenerate ground states. Then a soliton (or kink) is a Bloch wall separating two different ground states. In calculating the thermodynamic quantities, we can divide the functional space into sectors, which are orthogonal to each other (i.e., the super selection rule). In a sine-Gordon system, the sectors are enumerated by the topological charge $N$ associated with a sector

$$N = N_S - N_{\bar{S}} \quad , \tag{1}$$

where $N_S$ and $N_{\bar{S}}$ are the total numbers of solitons and antisolitons in the sector, respectively; i.e., the sector can be characterized by the soliton number. The partition function of the sine-Gordon systems is then written

$$Z = \int \mathfrak{D}(\phi) \exp\left[-\int_0^\beta H\, d\tau\right] = \sum_{N=-\infty}^{\infty} Z_N \quad , \tag{2}$$

where

$$Z = \int \mathfrak{D}_N(\phi) \exp\left[-\int H\, d\tau\right] \tag{3}$$

and $D_N(\phi)$ is the functional integral over $\phi$ in the $N$th sector.

Then in the low-temperature limit [i.e., $\beta E_S >> 1$, where $\beta = (k_B T)^{-1}$ is the inverse of the temperature and $E_S$ is the soliton energy (at $T = 0$ K)], we can neglect the interaction between solitons and between soliton and antisoliton (i.e., the ideal-gas approximation).[4,5] In this limit, it suffices to calculate $Z_0$ and $Z_1$, since

$$Z_{-1} = Z_1 \tag{4}$$

and $Z_N$'s are well approximated by

$$Z_N \cong \sum_{n=0}^{\infty} \frac{1}{n!(N+n)!} Z_1^{N+n} (Z_{-1})^n \quad . \tag{5}$$

With this approximation the thermodynamic potential is given, for example, by[1,3]

$$\Omega = \Omega_0 - 2\beta^{-1} \sum_p n_S(p) \quad , \tag{6}$$

where

$$n_S(p) = Z_1(p) Z_0^{-1} = e^{-\beta[\Omega_1(p) - \Omega_0]} \quad , \tag{7}$$

$\Omega_0 = -\beta^{-1} \ln Z_0$, $\Omega_1(p) = -\beta^{-1} \ln Z_1(p)$, and $p$ is the momentum of the classical soliton. In deriving Eq. (6), we have made use of Eq. (4); the soliton and the antisoliton are identical as far as their thermodynamic properties are concerned. The thermodynamic potential $\Omega_0$ and $\Omega_1(p)$ are calculated within the semiclassical approximation in I, II, and III. Here it is worth noting that by dealing with the difference $\Omega_1(p) - \Omega_0$ as the soliton contribution to $\Omega$, the present method automatically includes the quantum correction of the soliton energy at $T = 0$ K,[6] and the proper "free-energy sharing" at finite temperatures.[4]

 

More generally, the ensemble average of an observable $Q(\phi)$ is calculated as

$$\langle Q \rangle = Z^{-1} \sum_{n=-\infty}^{\infty} \int \mathfrak{D}_N(\phi) Q(\phi) \exp\left[-\int_0^\beta H\, d\tau\right] . \quad (8)$$

Again in the low-temperature limit ($\beta E_S >> 1$), Eq. (8) is further simplified as

$$\langle Q \rangle = \langle Q \rangle_0 + 2 \sum_p [\langle Q \rangle_1(p) - \langle Q \rangle_0] n_S(p) , \quad (9)$$

where $\langle Q \rangle_0$ is the thermal average of $Q$ in the soliton free sector ($N=0$), and $\langle Q \rangle_1(p)$ is the one in the sector with a single soliton with the momentum $p$. More explicitly, $\langle Q \rangle_1(p)$ is given by

$$\langle Q \rangle_1(p) \equiv Z_1(p)^{-1} \frac{1}{L} \int_{-L/2}^{L/2} dx \int \mathfrak{D}(\phi_S(x,p) + \hat{\phi}) \left[ Q(\phi_S(x,p) + \hat{\phi}) \exp\left[-\int_0^\beta H(\phi_S(x,p) + \hat{\phi})\, d\tau\right]\right] , \quad (10)$$

where $\phi_S(x,p)$ is the classical solution of a soliton with the velocity $v(=\partial E_S(p)/\partial p)$ at the position $x$ and the functional integral over $\hat{\phi}$ is carried out within the Gaussian approximation. In the above transformation we have assumed implicitly

$$\langle Q \rangle_1(p) = \langle Q \rangle_{-1}(p) . \quad (11)$$

This is true for most of observables. An important exception is, however, the soliton charge which vanishes identically in our model.

Perhaps it is important to point out, that if $Q$ is a local observable, $\langle Q \rangle_1(p) - \langle Q \rangle_0$ is of the order of $L^{-1}$. This cancels a factor $L$ arising from the $p$ summation

$$\sum_p = \frac{L}{2\pi} \int dp . \quad (12)$$

Expression (9) is therefore finite in the limit that $L$ tends to infinity.

In the following, limiting ourselves to a sine-Gordon system and a $\phi^4$ system, we shall calculate some thermal averages as well as the dynamical correlation functions.

## III. SINE-GORDON SYSTEM

As in I and II, we shall study the system described by the following Hamiltonian:

$$H = \frac{1}{2} \int dx \left[ \pi^2(x) + \left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{2m^2}{g^2} \mathfrak{N}(\cos g\phi) + \frac{2m_0^2}{g^2} \right] , \quad (13)$$

where $\pi(x) = \partial\phi/\partial t$ and $\mathfrak{N}$ is the generalized normal product as defined in I. Before going into the calculation of correlation functions, we shall examine the thermal average of $\cos g\phi$. Substituting $Q = \cos g\phi$ into Eq. (10), we obtain

$$\langle \cos g \rangle_1(p) = \frac{1}{L} \int_{-L/2}^{L/2} dx \cos g\phi_S(x,p) \langle \cos g\hat{\phi} \rangle_1^p$$
$$= (1 - 4/m\gamma L) \langle \cos g\phi \rangle_0 , \quad (14)$$

where

$$\gamma = (1-v^2)^{-1/2} = [1 + (p/E_S)^2]^{1/2} , \quad (15)$$

and we have made use of a classical solution of a soliton with velocity $v$;

$$\cos[g\phi_S(x,p)] = 1 - 2\,\mathrm{sech}^2[m\gamma(x - vt)] . \quad (16)$$

In Eq. (4), $\langle Q(\hat{\phi}) \rangle_1^p$ is the average with respect to the fluctuation $\hat{\phi}$ in the $N=1$ sector, and is approximated with that in the $N=0$ sector;

$$\langle A(\hat{\phi}) \rangle_1^p \cong \langle A(\phi) \rangle_0 , \quad (17)$$

which is appropriate in the weak-coupling limit (i.e., up to the first order in $g^2$).[1] Substituting Eq. (14) into Eq. (9), we obtain

$$\langle \cos g\phi \rangle = \left(\frac{2\Lambda}{m_0}\right)^{-g^2/4\pi} \exp\left(-\frac{g'^2}{2} f_0(\beta m)\right)\left(1 - \frac{8}{m}\bar{n}_S\right) , \quad (18)$$

where $\Lambda$ is the cutoff momentum, $g'^2[\equiv g^2(1-g^2/8\pi)^{-1}]$ is the renormalized coupling constant, and the total soliton density $\bar{n}_S$ and the function $f_0(z)$ have been defined and evaluated in I.

In evaluating $\langle Q \rangle_0$, we have assumed that the functional space of the $N=0$ sector is covered in terms of phonons (i.e., small fluctuations around the ground state $\phi=0$) as in I. Alternatively the $N=0$ sector is described in terms of breathers. We believe that the second approach is more appropriate and exact in the low-temperature limit. For example, Dashen *et al.*[6] have shown that the phonons disappear completely from the quantum field theory of a sine-Gordon system. Rather the lowest breather mode can be considered as the renormalized phonons. Furthermore, we have shown in II that we cannot count the phonons and the breathers as independent degrees of freedom, since in the weak-coupling limit ($g^2 << 1$) two alternative approaches result in the same $\Omega_0$, which is consistent with the exact result of the classical statistics of a sine-Gordon system[7] valid in the high-temperature region (i.e., $T >> m$). Inclusion of these two degrees of freedom destroys completely the above consistency.[2,4] This suggests strongly that in the correct theory of the sine-Gordon system no degrees of freedom should be associated with phonons.

We shall remark here only that, within the breather approach the function $\frac{1}{2}g'^2 f_0(\beta m)$ in the exponent of Eq. (18) has to be replaced by

$$\frac{g'^2}{2}f_0(\beta m) \to 8\sum_{n=1}^{n_0}\left[\frac{\beta E_n}{2\pi}\right]^{1/2}\tan\left[\frac{ng'^2}{16}\right] \times\left[1+\sin\left[\frac{ng'^2}{16}\right]\right]e^{-\beta E_n} , \quad (19)$$

where

$$E_n = 2E_S \sin(\tfrac{1}{16}ng'^2) . \quad (20)$$

We shall give a derivation of Eq. (19) in the Appendix.

Now we shall consider the following correlation functions:

$$\chi_{cc}(x,t) = \langle \cos[g\phi(x,t)]\cos[g\phi(0,0)]\rangle , \quad (21)$$

$$\chi_{ss}(x,t) = \langle \sin[g\phi(x,t)]\sin[g\phi(0,0)]\rangle , \quad (22)$$

$$\chi_{c/2,c/2}(x,t) = \langle \cos[\tfrac{1}{2}g\phi(x,t)]\cos[\tfrac{1}{2}g\phi(0,0)]\rangle . \quad (23)$$

For example, substituting

$$Q = \cos[g\phi(x,t)]\cos[g\phi(0,0)] \quad (24)$$

into Eq. (10), we obtain

$$\langle Q\rangle_1(p) = \chi_{cc}^p(\tilde{x})\langle \cos[g\hat{\phi}(x,t)]\cos[g\hat{\phi}(0,0)]\rangle_1^p + \chi_{ss}^p(\tilde{x})\langle \sin[g\hat{\phi}(x,t)]\sin[g\hat{\phi}(0,0)]\rangle_1^p , \quad (25)$$

where

$$\chi_{cc}^p(\tilde{x}) = \frac{1}{L}\int_{-L/2}^{L/2} dx_0 \cos[g\phi_S(x-x_0,p)] \times\cos[g\phi_S(-x_0,p)] = 1+(8/m\gamma L)[-1+2f(\tilde{x})] , \quad (26)$$

$$\chi_{ss}^p(x) = \frac{1}{L}\int_{-L/2}^{L/2} dx_0 \sin[g\phi_S(x-x_0,p)] \times\sin[g\phi_S(-x_0,p)] = -(8/m\gamma L)g(\tilde{x}) , \quad (27)$$

$$f(x) = \operatorname{cosech}^2 x\,(x\coth x - 1) , \quad (28)$$

$$g(x) = x\operatorname{cosech} x\,[x - 2\coth x + 2\operatorname{cosech}^2 x] , \quad (29)$$

and

$$\tilde{x} = m\gamma(x - vt) . \quad (30)$$

Making use of approximation (17) for the averages $\langle\ \rangle_1^p$ in Eq. (25), and then substituting Eq. (25) into Eq. (9), we obtain

$$\chi_{cc}(x,t) = \left[1+\frac{16}{mL}\sum_p \gamma^{-1}n_S(p)[-1+2f(\tilde{x})]\right]\chi_{cc}^{(0)}(x,t) - \frac{16}{mL}\sum_p \gamma^{-1}n_S(p)g(\tilde{x})\chi_{ss}^{(0)}(x,t) , \quad (31)$$

where

$$\chi_{cc}^{(0)}(x,t) \cong \langle \cos g\phi\rangle_0^2 , \quad (32)$$

$$\chi_{ss}^{(0)}(x,t) \simeq g^2 D(x,t) = g^2\langle \phi(x,t)\phi(0,0)\rangle_0 . \quad (33)$$

The Fourier transform of the first term in Eq. (31) gives rise to the dynamical structure factor $S(q,\omega)$ derived before.[8] The second term of Eq. (31) yields a weak structure (of the order of $g^2$) at $\omega = (m^2 + q^2)^{1/2}$.

We can calculate similarly the sine-sine correlation function

$$\chi_{ss}(x,t) = \left[1+\frac{16}{mL}\sum_p \gamma^{-1}n_S(p)[-1+2f(\tilde{x})]\right]\chi_{ss}^{(0)}(x,t) - \frac{16}{mL}\sum_p \gamma^{-1}n_S(p)g(\tilde{x})\chi_{cc}^{(0)}(x,t) . \quad (34)$$

Of particular interest is $\chi_{c/2,c/2}(x,t)$, which is given by

$$\chi_{c/2,c/2}(x,t) \cong \langle \cos(\tfrac{1}{2}g\phi)\rangle_0^2 \times\exp\left[-\frac{4}{mL}\sum_p \gamma^{-1}n_S(p)h(\tilde{x})\right] , \quad (35)$$

where

$$h(x) = x\coth x . \quad (36)$$

In Eq. (35) we have summed over all sectors, which is a simple generalization of Eq. (9). Contrary to the case of other correlation functions, in the case of $\chi_{c/2,c/2}$ the second term of Eq. (9) is no longer small for $|\tilde{x}| \gg 1$. Therefore, we have to include the contribution from the sectors with large $N$ as well. In other words, in $\chi_{c/2,c/2}(x,t)$, the multisoliton term plays an essential role, while in $\chi_{cc}$ and $\chi_{ss}$ the single-soliton term suffices to describe the soliton contribution.

In fact in the limit of large $|x|$, $h(x)$ becomes

$$h(x) \cong |x| . \quad (37)$$

Then Eq. (35) is very similar to $\chi_{\phi\phi}$ in the $\phi^4$ system discussed earlier by Krumhansl and Schrieffer.[5] For example, the static correlation length $\xi_{c/2}$ is given in terms of the total soliton density $\bar{n}_S[=L^{-1}\sum_p n_S(p)]$;

$$\chi_{c/2,c/2}(x,0)\propto \exp(-4\bar{n}_S|x|) \quad (38)$$

or

$$\xi_{c/2}^{-1}=4\bar{n}_S \ .$$

More generally, Eq. (35) may be approximated by

$$\chi_{c/2,c/2}(x,t)=\langle\cos(\tfrac{1}{2}g\phi)\rangle_0^2 \exp[-4\bar{n}_S(x^2+v_0^2t^2)^{1/2}] \ , \quad (39)$$

where

$$v_0=(2/\pi\beta E_S)^{1/2} \ . \quad (40)$$

Equation (39) describes correctly the asymptotic behavior of Eq. (35) both for $|x| >> v_0|t|$ and for $|x| << v_0|t|$. Then the dynamical structure factor for $\chi_{c/2,c/2}$ is given by[9]

$$S_{c/2,c/2}(q,\omega)\equiv\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int dt\,dx\,e^{i(qx-\omega t)}\chi_{c/2,c/2}(x,t)=\langle\cos(\tfrac{1}{2}g\phi)\rangle_0^2\frac{K}{2\pi v_0}\left[q^2+\left(\frac{\omega}{v_0}\right)^2+K^2\right]^{-3/2} \ , \quad (41)$$

where $K\equiv 4\bar{n}_S$.

It should be pointed out that $K$ in Eq. (41) is larger than $K'$ in the $\phi^4$ theory[5] [see also Eq. (47)] by a factor 2. This reflects the fact that in the SG theory both soliton and antisoliton contribute to $K$ as their sequence is completely arbitrary, while in the $\phi^4$ theory only the soliton contributes. It appears that the above correlation function has been recently observed by inelastic-neutron-scattering experiment on quasilinear antiferromagnetic compound $(CH_3)_4NMnCl_3$ known as TMMC in a magnetic field at low temperatures.[10]

## IV. $\phi^4$ SYSTEM

We consider the system described by the Hamiltonian[3]

$$H=\frac{1}{2}\int dx\left[\pi^2(x)+\left(\frac{\partial\phi}{\partial x}\right)^2-\frac{(m^*)^2}{2}\phi^2+\frac{\lambda}{4}\phi^4\right] \ , \quad (42)$$

where $m^*$ is the bare mass of the field $\phi$. In this system the correlation function of particular interest is $\chi_{\phi\phi}$ defined by[5]

$$\chi_{\phi\phi}(x,t)=\langle\phi(x,t)\phi(0,0)\rangle \ . \quad (43)$$

This correlation function is evaluated in a way similar to that outlined above for $\chi_{c/2,c/2}$ in the sine-Gordon system and we obtain

$$\chi_{\phi\phi}(x,t)=D(x,t)+\phi_0^2 \times\exp\left[-\frac{2}{mL}\sum_p\gamma^{-1}n_S(p)h'(\tilde{x})\right] \ , \quad (44)$$

where $D(x,t)=\langle\phi(x,t)\phi(0,0)\rangle_0$,

$$h'(x)=x\coth(\tfrac{1}{2}x) \ , \quad (45)$$

and $\phi_0=\langle\phi\rangle_0$. In the limit $|x|>>1$, $h'(x)$ is simplified

$$h'(x)\cong|x| \ . \quad (46)$$

The second term of Eq. (44) with $h'(x)$ given by Eq. (46) has been obtained previously by Krumhansl and Schrieffer.[5] The dynamical structure factor is then given by

$$S_{\phi\phi}(q,\omega)\cong D(q,\omega)+\phi_0^2\frac{K'}{4\pi v_0}\left[q^2+\left(\frac{\omega}{v_0}\right)^2+K'^2\right]^{-3/2} \ , \quad (47)$$

where $K'=2\bar{n}_S$ and $v_0$ is given by Eq. (40). Here we have made use of the same approximation as in Eq. (39).

## V. CONCLUDING REMARKS

We have extended our analysis[1-3] to the correlation functions in the sine-Gordon system and the $\phi^4$ system. In the intermediate temperature region ($E_S >> T >> m$), the present analysis confirms the results obtained with the ideal-gas phenomenology for solitons,[4,5,8] except that the classical soliton energy has to be replaced by the one with quantum correction.[3] In the low-temperature region ($T\lesssim m$) the soliton density is different from that obtained in the classical theory,[3] though this effect may be difficult to observe experimentally. The more important conclusion of this analysis is that in the sine-Gordon system the phonon has to be excluded from the theory; solitons, antisolitons, and breathers give a complete description of the theory. This is completely consistent with the conclusion reached in field theory.[6]

In the $\phi^4$ system, we have not attempted an analysis of the two soliton bound states. However, it is quite possible that again in this case the phonon may disappear in the final theory; the degrees of freedom associated with phonons are completely absorbed in those of the soliton bound states. Clearly this is one of the most interesting questions.

*Note added in proof.* Making use of the present technique, we calculate the correlation function $\langle[\phi(x,t)-\phi(0,0)]^2\rangle$ in the sine-Gordon system as

$$\langle[\phi(x,t)-\phi(0,0)]^2\rangle=2\left(\frac{2\pi}{g}\right)^2\int\frac{dp}{2\pi}|x-vt|n_s(p) \cong 2\bar{n}_s\left(\frac{2\pi}{g}\right)^2(x^2+v_0^2t^2)^{1/2} \ .$$

A similar space-time dependence is also obtained by Gunther and Imry,[11] who made use of analogy to the random walk process.

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## APPENDIX: THERMAL AVERAGE OF $\cos g\phi$ IN TERMS OF THE BREATHER FORMULATION

We shall here limit our consideration to the $N=0$ sector. For the purpose of calculation we assume that the functional space of the $N=0$ sector is decomposed into the breather-free part and the parts each with one $n$th breather as in II. Then following the similar procedure as described in Sec. II, we can write

$$\langle Q\rangle_0^* = \langle Q\rangle_0 + \sum_{p,n}[\langle Q\rangle_n(p) - \langle Q\rangle_0]n_B(n,p) \quad , \quad (\text{A1})$$

where * means the functional average in the functional space extended to include the breather mode, while $\langle Q\rangle_0$ is the same as in Eq. (9), and $n_B(n,p)$ is the average density of the $n$th breather with momentum $p$.[2] Making use of the similar approximation as Eq. (17), we obtain

$$\langle Q\rangle_n(p) = \langle Q\rangle_0 \frac{1}{L}\int_{-L/2}^{L/2} dx\, Q(\phi_B(n;x,p)) \quad ,$$

where $\phi_B(n;x,p)$ is the classical solution corresponding to the $n$th breather. Since $\phi_B(n;x,p)$ is time dependent (even for $p=0$), the definition of $\langle Q\rangle_n(p)$ requires also an average over the period of the breather.

In the case of $Q=\cos g\phi$, we obtain

$$\langle \cos g\phi\rangle_n(p) = \langle Q\rangle_0 \frac{1}{T_n L}\int_{-L/2}^{L/2} dx \int_0^{T_n} dt \left[1 - \frac{8S_nC_n}{(C_n+S_n)^2}\right] \quad , \quad (\text{A2})$$

where

$$T_n = 2\pi/mw_n\gamma \quad ,$$

$$S_n = (w_n^{-2}-1)\sin^2[mw_n\gamma(t-vx)] \quad ,$$

$$C_n = \cosh^2[m(1-w_n^2)^{1/2}\gamma(x-vt)] \quad ,$$

and

$$w_n = \cos(\tfrac{1}{16}ng'^2) \quad . \quad (\text{A3})$$

The space-time integral in Eq. (A2) is easily done after Lorentz transformation, and we obtain

$$\langle \cos g\phi\rangle_n(p) = \langle Q\rangle_0\left[1 - \frac{8}{mL}\sin\left[\frac{n}{16}g'^2\right]\right] \quad . \quad (\text{A4})$$

On the other hand, $n_B(n,p)$ has been already obtained in II:

$$n_B(n,p) = \frac{m[1+\gamma^{-1}\sin(\frac{1}{16}ng'^2)]}{T\cos(\frac{1}{16}ng'^2)} \times \gamma\exp[-\beta(E_n^2+p^2)^{1/2}] \quad , \quad (\text{A5})$$

with $E_n = 2E_S\sin(\frac{1}{16}ng'^2)$. Putting these together, we obtain

$$\langle \cos g\phi\rangle_0^* = \langle \cos g\phi\rangle_0[1 - I_B(\beta E_S, g'^2)] \quad , \quad (\text{A6})$$

where

$$I_B(\beta E_S, g'^2) = 8\sum_{n=1}^{n_0}\left[\frac{\beta E_n}{2}\right]^{1/2}\tan\left[\frac{ng'^2}{16}\right] \times \left[1+\sin\left[\frac{ng'^2}{16}\right]\right]^2 e^{-\beta E_n} \quad . \quad (\text{A7})$$

So far we have considered up to the one breather term in the functional space. As long as we neglect the interaction between breathers (again in the ideal-gas approximation), the multibreather terms are easily included. The effect of the multibreather term is easily obtained from Eq. (A6) as

$$\langle \cos g\phi\rangle_0^* = \langle \cos g\phi\rangle_0\exp[-I_B(\beta E_S, g'^2)] \quad . \quad (\text{A8})$$

Finally, we have to eliminate the contribution associated with the phonon degrees of freedom, which should be exhausted by the breathers. Then the average of $\cos g\phi$ in the breather picture is given by

$$\langle \cos g\phi\rangle_0^B = \exp[-I_B(\beta E_S, g'^2)] \quad , \quad (\text{A9})$$

which is nothing but the replacement indicated by Eq. (19). Actually in the limit $g'^2 \to 0$, we can show in the low-temperature region ($T \lesssim m$)

$$\langle \cos g\phi\rangle_0^B = \exp[-\tfrac{1}{2}g'^2 f_0(\beta m)] \quad . \quad (\text{A10})$$

On the other hand, in the temperature region ($m \ll T \ll E_S$), Eq. (A9) does not reduce to Eq. (A10) even in the limit $g'^2 \to 0$, of which origin is not clear for the moment.

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